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Are There Any Natural Physical Interpretations for Some Elementary Inequalities?

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Dedicated to the memory of Professor Constantin Popa

Abstract

We inquire whether there are some fundamental interpretations of elementary inequalities in terms of curvature of a three-dimensional smooth hypersurface in the four-dimensional real ambient space. The main outcome of our exploration is a perspective of regarding the natural substance of some mathematical inequalities, which represent important physical quantities.

1 Presenting the Challenge

We encounter elementary inequalities in various mathematical contexts, from the problems proposed in journal of elementary interest, e.g. the *American Mathematical Monthly*, *Gazeta matematică*, or the *Mathematics Magazine*, to analytic inequalities serving as arguments in various proofs. Some of them seem at the first sight rather intricate and their proof could present a challenge, reminding us of Harald Bohr's saying, cited in [2], p.12: "All analysts spend

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half heir time hunting through the literature for inequalities which they want to use but cannot prove." Definitely the inequalities serving as arguments in a proof are *natural* assertions, since they serve the just cause.

Besides the use of certain inequalities in real analysis, there is another way of regarding certain inequalities as *natural*, and this interpretation is not entirely obvious. The key point is that some symmetric functions (whose detailed study dates back to L.-A. Cauchy at the beginning of the 19th century) admit an interpretation as geometric quantities representing curvature. In several recent works, e.g. [4, 5, 18] certain symmetric polynomials are interpreted as curvature invariants and explorations of their geometric meaning are pursued. The present work aims to expand at a more profound level this reflection.

The strategy of our presentation is the following. *We choose an example of an elementary inequality that is not at all obvious, then we comment on its interpretation in terms of curvature.* In consequence, the main result in this note is not a theorem, but rather a way of regarding the nature of a fundamental algebraic statement. Basically, we show that in an algebraic statement involving three real numbers a, b, c, the sum $a+b+c$ represents the idea of *tension*, $a^2+b^2+c^2$ the Casorati curvature (or the square of the Hilbert-Schmidt norm), $ab+bc+ca$ represents the scalar curvature (more will be said about it in section 5), while abc represents pointwise the Gauss-Kronecker curvature of a smooth hypersurface lying in the four-dimensional Euclidean ambient space. As a consequence, any elementary statement relating these quantities can be envisioned in the spirit of these geometric quantities. And some of them do have important physical meaning.

As it is well-known, in 1905, Albert Einstein published a series of works that established the special theory of relativity, which was an experimentally well-confirmed physical theory regarding the relationship between space and time. A very important feature of this theory is the four-dimensional space endowed with a special metric. It is quite natural to consider three-dimensional objects as subspaces of a four-dimensional real space. For our exploration, we discuss a few fundamental properties of the four-dimensional Euclidean space.

2 An Example

We could deepen our discussion by using an example from any of the aforementioned journals, e.g. the *American Mathematical Monthly*, *Gazeta matematică*, or the *Mathematics Magazine*, as this thought could be conveyed in many ways. For the sake of a precise illustration, we pick up a sample package of fundamental inequalities from the interesting reference [10], where we find many elementary examples that could serve our goal. Take e.g. Problem 73, p. 190, stating the following.

Exercise 1. *Let x, y, z be positive real numbers such that $x^2+y^2+z^2 = xyz$. Prove the following inequalities: (i) $xyz \geq 27$; (ii) $xy+yz+zx \geq 27$; (iii) $x+y+z \geq 9$; (iv) $xy+yz+zx \geq 2(x+y+z)+9$.*

Does this exercise have any hidden *meaning*? Is there anything we don't see here at the first sight? We restate the same assertion in a different way.

Proposition 1. *Let $M^3 \subset \mathbb{R}^4$ by a smooth strictly convex hypersurface with the property that the Casorati curvature equals the Gauss-Kronecker curvature, i.e. at every point the principal curvatures satisfy $x^2+y^2+z^2 = xyz$. Then between the scalar curvature scal and the mean curvature H the following inequality holds:*

$$scal \geq 6H + 9.$$

The equality holds at umbilical points where all the principal curvatures take value 3.

It is the same assertion, just the second time formulated as a differential geometry statement. From this intermediate stage, we can further provide

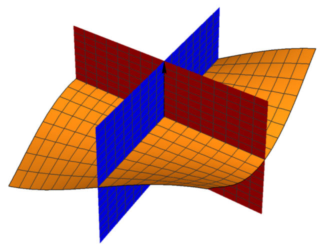
The Physical interpretation: For smooth strictly convex hypersurfaces in $\mathbb{R}^4$ with Casorati curvature equal to the Gauss-Kronecker curvature, the Lagrangian density for the Einstein-Hilbert action exceeds by much the tension H (this is the interpretation of the term $6H+9$).

In the next sections we justify this interpretation. Furthermore, there is an additional interpretation. The Minkowski spheres either in the de Sitter or in the Anti de Sitter spaces admit parametrizations in which the Weingarten matrix has $\frac{1}{a}$ on the diagonal and 0 in all other entries. These Minkowski spheres are examples of universes in which the absence of the ordinary matter, we shall describe below.

For the case $n = 3$, the equation $\frac{1}{a^2} + \frac{1}{a^2} + \frac{1}{a^2} = \frac{1}{a^3}$ admits a solution, which means that in the case $n = 3$ we do have a de Sitter sphere as well as an Anti de Sitter sphere, which match the situation hereby described, i.e. $a = \frac{1}{3}$, thus $x = y = z = 3$.

3 Classical curvature quantities

We are ready now to define mathematically the Gaussian curvature and Sophie Germain's mean curvature. Consider a smooth surface S lying in $\mathbb{R}^3$, and an arbitrary point $P \in S$. Consider N_P the normal to the surface at P.



Consider the family of all planes passing through P that contain the line through P with the same direction as N_P. These planes yield a family of curves on S called *normal sections*. Consider now the curvature $\kappa(P)$ of the normal sections, viewed as planar curves. Then $\kappa(P)$ has a maximum, denoted κ_1, and a minimum, denoted κ_2. The curvatures κ_1 and κ_2 are called the *principal curvatures*. The Gaussian curvature [15] is defined as $K(P) = \kappa_1(P) \cdot \kappa_2(P)$, and Sophie Germain's mean curvature is defined [16] by the arithmetic mean $H(P) = \frac{1}{2}\left[\kappa_1(P) + \kappa_2(P)\right]$.

Complementing over a century of investigations on what the correct definition for curvature should be (if we count the timeline since Leonhard Euler's first investigations on the geometry of surfaces), Casorati introduced in 1890 [8] what is today called the Casorati curvature. In his paper, Casorati argues that there are important geometric reasons why one should investigate $C(P) = \frac{1}{2}\left[\kappa_1^2(P) + \kappa_2^2(P)\right]$. Some authors choose to refer as Casorati curvature to the sum of the squares of the principal curvatures. In the recent year there is a growing interest in the study of Casorati curvature and its applications (see e.g. [6, 11, 12, 19, 20]).

4 Principal Curvatures of a Hypersurface

We discuss a more general case, the context of a smooth hypersurface embedded in an Euclidean ambient space. Namely, for an open set U in the usual topology of $\mathbb{R}^n$, let $\sigma : U \subset \mathbb{R}^n \to \mathbb{R}^{n+1}$ be a hypersurface given by the smooth map σ. Let p be a point on the hypersurface. Denote by $\sigma_k(p) = \frac{\partial \sigma}{\partial x_k}$, for all k from 1 to n. Consider $\{\sigma_1(p), \sigma_2(p), ..., \sigma_n(p), N(p)\}$, the Gauss frame of the hypersurface, where N denotes the normal vector field.

We denote by $g_{ij}(p)$ the coefficients of the first fundamental form and by $h_{ij}(p)$ the coefficients of the second fundamental form:

$$g_{ij}(p) =< \sigma_i(p), \sigma_j(p) >, \qquad h_{ij}(p) =< N(p), \sigma_{ij}(p) > .$$

The Weingarten map (defined as the derivative of the Gauss map) is $L_p = -dN_p \circ d\sigma_p^{-1} : T_{\sigma(p)}\sigma \to T_{\sigma(p)}\sigma$. Weingarten's map is linear, which allows us to denote by $(h^i_j(p))_{1\leq i,j\leq n}$ the matrix associated to Weingarten's map, that is at every point p:

$$L_p(\sigma_i(p)) = \sum_{k=1}^{n} h^k_i(p)\sigma_k(p).$$

It is well-known that Weingerten's operator is self-adjoint, which implies that the roots of the algebraic equation

$$\det(h^i_j(p) - \lambda(p)\delta^i_j) = 0$$

are real. The eigenvalues of Weingarten's linear map are called *principal curvatures of the hypersurface*. They are the roots $k_1(p), k_2(p), ..., k_n(p)$ of this algebraic equation. The mean curvature at the point p is $H(p) = \frac{1}{n}[k_1(p) + ... + k_n(p)]$, and the Gauss-Kronecker curvature is $K(p) = k_1(p)k_2(p)...k_n(p)$.

If M is a submanifold of a Riemannian manifold $\overline{M}$, and if their sectional curvatures are sec and $\overline{\text{sec}}$, respectively, then from the Gauss equation (see e.g. [13], pg.131) we have:

$$\sec(e_i \wedge e_j) - \overline{\sec}(e_i \wedge e_j) = k_i k_j.$$

If the ambient space is Euclidean, then $\overline{sec}(e_i \wedge e_j) = 0$ at every point, in every planar direction. This means that the scalar curvature of a hypersurface in Euclidean ambient space at point $p \in M^n$ is

$$scal(p) = \sum_{i<j} \sec(e_i \wedge e_j) = \sum_{i<j} k_i k_j.$$

Bearing this in mind, we can return to the thought presented in our introduction, and interpret three real numbers $a, b, c \in \mathbb{R}$ as the principal curvatures $k_1 = a, k_2 = b, k_3 = c$ at a point of a smooth hypersurface lying in the four-dimensional real space endowed with the canonical Euclidean metric. Hence, the sum $a + b + c$ represents three times the mean curvature, $\sum_{1\leq i<j\leq 3} k_i k_j = ab + bc + ca$ represents the scalar curvature, as a direct consequence of Gauss' equation, while abc represents pointwise the Gauss-Kronecker curvature.

Denote now by $M^{(q,p)}$ the real Minkowski $(q + p)$-dimensional space with coordinates $(x_0, x_1, ..., x_{q-1}, x_q, x_{q+1}, ..., x_{q+p-1})$ endowed with the Minkowski product

$$\langle a, b\rangle_M := \sum_{\alpha=0}^{q-1} a_\alpha b_\alpha - \sum_{\alpha=q}^{q+p-1} a_\alpha b_\alpha \tag{2.1}$$

Therefore, we choose to work with signature $(+ + ... + - - ...-)$, where $+$ appears q times and $-$ appears p times. A vector $x = (x_0, x_1, ..., x_{q+p-1})$ is a space-like vector if $\langle x, x\rangle_M < 0$.

Consider a hypersurface f of this real Minkowski $(q+p)$-dimensional space. The difference between this ambient space and the case when the hypersurface is embedded in a real Euclidean $(q+p)$ space appears when we identify the coefficients of the Gauss formulas. To see this, we have to consider the case when the normal to the hypersurface vector N satisfies $\langle N, N\rangle_M = -1$, that is when N is a space-like vector. These class of surfaces, called space-like surfaces, are important in physical theories since they can contain time-like geodesics, which describe the phenomenon of motion.

We investigate the case when the normal to the surface is a space-like vector. It is natural to inquire whether there are the same Gauss formulas as in the Euclidean case, namely

$$\frac{\partial^2 f}{\partial x^i \partial x^k}(x) = \Gamma^s_{ik}(x) \cdot \frac{\partial f}{\partial x^s}(x) + N(x) \cdot h_{ik}(x).$$

In this ambient space we have $\langle N, N\rangle_M = -1$, therefore the expressions of the coefficients of the second fundamental form has to change. It turns into

$$h^M_{ij}(x) := -\left\langle N(x), \frac{\partial^2 f}{\partial x^i \partial x^j}(x)\right\rangle_M = \left\langle \frac{\partial N}{\partial x^i}(x), \frac{\partial f}{\partial x^j}(x)\right\rangle_M .$$

This means that, in order to preserve the formula which relates the coefficient of the second fundamental form to the coefficients of the first fundamental form through the coefficients of the *Minkowski-Weingarten matrix*, $h^M_{ij} = h^s_i\, g^M_{sj}$, we need to consider a modified formula for Minkowski-Weingarten coefficients, more specifically

$$\frac{\partial N}{\partial x^i} = h^s_i\, \frac{\partial f}{\partial x^s}.$$

By using Gauss' formulas and Weingarten's formulas, we obtain *Minkowski-Gauss' equations* in the modified form

$$R_{ijkl} = -\left(h^M_{ik} h^M_{jl} - h^M_{il} h^M_{jk}\right).$$

Finally, by considering

$$K_M = \frac{R_{1212}}{\det g^M_{ij}},$$

we define the Minkowski-Gauss curvature by the formula

$$K_M := -\frac{\det h^M_{ij}}{\det g^M_{ij}} = -\det h^j_i.$$

The rest stands the same as in the Euclidean case.

To further see an example where this theory works, we pursue the computations of the Minkowski-Gauss curvature of the affine sphere

$$X_0^2 - X_1^2 - X_2^2 = -a^2.$$

By the previous notations, we are working in a 3-dimensional real Minkowski space $M^{(1,2)}$ with signature is $(+,-,-)$. The parametrization of this space-like Minkowski sphere is $f : \mathbb{R} \times (-\pi, \pi) \longrightarrow \mathbb{M}^{(1,2)}$,

$$f(t, x_1) = (a \sinh t, a \cosh t \cos x_1, a \cosh t \sin x_1).$$

After a straightforward computation, we derive the metric

$$ds^2 = a^2 dt^2 - a^2 \cosh^2 t \; dx_1^2.$$

The non-zero Christoffel symbols are

$$\Gamma^1_{01} = \Gamma^1_{10} = \tanh t,\ \Gamma^0_{11} = \cosh t \sinh t$$

and

$$R^0_{101} = \frac{\partial \Gamma^0_{11}}{\partial t} - \frac{\partial \Gamma^0_{10}}{\partial x_1} + \Gamma^0_{s0}\Gamma^s_{11} - \Gamma^0_{s1}\Gamma^s_{10} = \cosh^2 t.$$

It follows that $R_{0101} = g_{00} R^0_{101} = a^2 \cosh^2 t$, therefore we obtained the Minkowski-Gauss curvature as $K_f^M = -\dfrac{1}{a^2}$.

Since

$$\frac{\partial f}{\partial t}(t, x_1) = (a \cosh t, a \sinh t \cos x_1, a \sinh t \sin x_1)$$

and

$$\frac{\partial f}{\partial x_1}(t, x_1) = (0, -a \cosh t \sin x_1, a \cosh t \cos x_1),$$

in this case we see that the normal vector is

$$N(t, x_1) = (\sinh t, \cosh t \cos x_1, \cosh t \sin x_1) = \frac{1}{a} f(t, x_1).$$

The second fundamental form has the coefficients $h_{ij} = \dfrac{1}{a} g_{ij}$, therefore the Minkowski-Weingarten matrix is $W = (h^i_j) = \begin{pmatrix} \dfrac{1}{a} & 0 \\ 0 & \dfrac{1}{a} \end{pmatrix}$.

Without considering the formulas with the modifications suggested by the space-like nature of the Minkowski-Gauss normal vector, the Minkowski-Gauss curvature would have been obtained with different sign, instead of the value $-\dfrac{1}{a^2}$. Therefore,

$$K_M := -\frac{\det h_{ij}^M}{\det g_{ij}^M} = -\det h_i^j = -\frac{1}{a^2}$$

We call this 2-surface the 2-de Sitter spacetime and we denote it by $dS(2,3)$. It can be observed that $dS(2,3) \subset M^{(1,2)}$. The eigenvalues of the Minkowski-Weingarten map W can be called Minkowski principal curvatures of the surface; the definition can furthermore be extended to space-like hypersurfaces.

Now consider the 2-surface

$$X_0^2 + X_1^2 - X_2^2 = -a^2$$

which is a space-like Minkowski sphere in the form $f : \mathbb{R} \times (-\pi, \pi) \longrightarrow M^{(2,1)}$,

$$f(t, x_1) = (a \sinh t \cos x_1, a \sinh t \sin x_1, a \cosh t).$$

We call this surface the anti de Sitter spacetime and we denote it by $AdS(2,3)$. Some computations leads to the Riemannian metric

$$ds_2^2 = a^2 dt^2 + a^2 \sinh^2 t \; dx_1^2.$$

The non-vanishing Christoffel symbols are

$$\Gamma_{01}^1 = \Gamma_{10}^1 = \coth t,\ \Gamma_{11}^0 = -\sinh t \cosh t$$

and

$$R_{101}^0 = \frac{\partial \Gamma_{11}^0}{\partial t} - \frac{\partial \Gamma_{10}^0}{\partial x_1} + \Gamma_{s0}^0 \Gamma_{11}^s - \Gamma_{s1}^0 \Gamma_{10}^s = \frac{\partial \Gamma_{11}^0}{\partial t} - \Gamma_{11}^0 \Gamma_{10}^1 = -\sinh^2 t.$$

It results $R_{0101} = g_{00} R_{101}^0 = -a^2 \sinh^2 t$, that is $K_f^M = -\dfrac{1}{a^2}$.

The Minkowski normal is in fact, as in the previous example,

$$N(t, x_1) = \frac{1}{a} f(t, x_1)$$

and this can be directly derived from the null Minkowski products $\left\langle f, \dfrac{\partial f}{\partial t} \right\rangle_M ; \left\langle f, \dfrac{\partial f}{\partial x_1} \right\rangle_M$.

By the theory of surfaces in Minkowski spaces when the normal is a space-like vector, the second fundamental form coefficients are

$$h_{ij} = \left\langle \frac{\partial N}{\partial x^i}, \frac{\partial f}{\partial x^j} \right\rangle_M = \frac{1}{a} g_{ij};$$

and the Minkowski-Gauss curvature is calculated by the formula $K_f^M := -\dfrac{\det h_{ij}}{\det g_{ij}} = -\dfrac{1}{a^2}$.

In $M^{(2,1)}$ we consider both the space-like unitary sphere

$$X_0^2 + X_1^2 - X_2^2 = -1$$

and the unit disk described by all $(u, v, 1)$ such that $u^2 + v^2 \leq 1$.
A line

$$\frac{X_0}{u} = \frac{X_1}{v} = \frac{X_2}{1}$$

intersects the unit space-like sphere at the point

$$\left(\frac{u}{\sqrt{1-(u^2+v^2)}}, \frac{v}{\sqrt{1-(u^2+v^2)}}, \frac{1}{\sqrt{1-(u^2+v^2)}} \right).$$

The transformation of coordinates

$$\begin{cases} u = \tanh t \cos x_1 \\ v = \tanh t \sin x_1 \end{cases}$$

provides inside the disk the metric

$$ds^2 = dt^2 + \sinh^2 t \; dx_1^2,$$

the same as for the unitary $AdS(2,3)$ spacetime.

We remark that this is a Riemannian metric with constant negative Gaussian curvature, $K = -1$. This metric was naturally induced by a Minkowski metric. If we choose the geometric transformations

$$\begin{cases} x = \tanh t/2 \cos x_1 \\ y = \tanh t/2 \sin x_1 \end{cases}$$

we obtain the following formulas

$$dx = \frac{\cos x_1}{2\cosh^2 t/2} dt - \tanh t/2 \sin x_1 dx_1,$$

$$dy = \frac{\sin x_1}{2\cosh^2 t/2} dt + \tanh t/2 \cos x_1 dx_1,$$

that is the old metric becomes

$$ds^2 = \frac{4}{(1-x^2-y^2)^2}(dx^2+dy^2),$$

which is the Poincaré metric of the disk. The Anti de Sitter $AdS(2,3)$ space represents an example of a non-Euclidean geometry described starting from a Minkowski metric.

We reached the point when we can increase the dimension, by considering the description of the Anti de Sitter $AdS(4,5)$ spacetime as

$$X_0^2 + X_1^2 - X_2^2 - X_3^2 - X_4^2 = -a^2,$$

that is a hypersurface in the 5-dimensional Minkowski $M^{(2,3)}$ space. We pursue the ideas presented above and compute all geometric quantities related to a given parametrization. We have a two-fold phenomenon here, as both geometry and physics are involved. We find out that Einstein's field equations are satisfied. This is a nice exercise proposed to the reader; in the following presentation we display similar computations for the situation of a de Sitter spacetime. A very good question is: can light travel in this $AdS(4,5)$ spacetime? It is enough to investigate a de Sitter spacetime included into $AdS(4,5)$ because in such spaces there is a light cone at each point (see [3]). If we consider $X_1 = 0$ and the parametrization

$$\begin{cases} X_0 = a\sinh t \\ X_2 = a\cosh t \cos x_2 \\ X_3 = a\cosh t \ \sin x_2 \cos x_3 \\ X_4 = a\cosh t \sin x_2 \sin x_3 \end{cases} \tag{7.17}$$

for the surface

$$X_0^2 - X_2^2 - X_3^2 - X_4^2 = -a^2,$$

which is a de Sitter spacetime, we can denote $dS(3,4)$ whose ambient space is a Minkowski $M^{(1,3)}$ space, whose signature is $(+ - --)$.
The metric attached to this parametrization is

$$ds^2 = a^2dt^2 - a^2\cosh^2 t dx_2^2 - a^2\cosh^2 t \sin^2 x_2 \ dx_3^2.$$

Some extra computations lead to

$$R_{ij} + \frac{2}{a^2}\ g_{ij} = 0,$$

that is the de Sitter spacetime presented above satisfies the Einstein field equations in geometric coordinates

$$R_{ij} - \frac{1}{2}R\ g_{ij} + \Lambda\ g_{ij} = 8\pi G T_{ik}$$

for $R = -\frac{6}{a^2}$, $\Lambda = -\frac{1}{a^2}$ and $T_{ij} = 0$. Therefore we are looking at an example of Universe without any ordinary matter inside it.

Finally, we can guess that the ideas presented above work at each dimension both for the de Sitter spacetime as well as for the Anti de Sitter spacetime. Therefore there exist parametrizations f such that the Minkowski normal to the hypersurface has the same property as in the above examples, that is

$$N = \frac{1}{a}f.$$

All the coefficients of the second fundamental form are claculated with the formula established for the case $n = 2$,

$$h_{ij} = \left\langle \frac{\partial N}{\partial x_i}, \frac{\partial f}{\partial x_j} \right\rangle_M,$$

therefore

$$h_{ij} = \frac{1}{a}g_{ij}.$$

Since $\langle N, N\rangle_M = -1 < 0$, we have

$$R_{ijkl} = -\left(h_{ik}h_{jl} - h_{il}h_{jk}\right),$$

i.e.

$$R_{ijkl} = -\frac{1}{a^2}\left(g_{ik}g_{jl} - g_{il}g_{jk}\right),\ i,j,k,l \in \{0,1,...,n-2\}.$$

Therefore each Minkowski sectional curvature is $-\frac{1}{a^2}$. We are naturally lead to

$$R_{ij} = -\frac{n-2}{a^2}\ g_{ij}$$

, i.e.

$$R = -(n-1)(n-2)\frac{1}{a^2}.$$

Since

$$R_{ij} + \frac{1}{2}(n-1)(n-2)\frac{1}{a^2}\ g_{ij} - \frac{(n-2)(n-3)}{2}\frac{1}{a^2}\ g_{ij} = R_{ij} + \frac{n-2}{a^2}\ g_{ij} = 0$$

it results that, if we choose

$$\Lambda = -\frac{(n-2)(n-3)}{2}\frac{1}{a^2},$$

the previous metric satisfies the Einstein field equations

$$R_{ij} - \frac{1}{2}R\ g_{ij} + \Lambda\ g_{ij} = 8\pi G T_{ij}$$

in the absence of matter, that is with $T_{ij} = 0$. For such parametrizations, we have a diagonal Minkowski-Weingarten's map, therefore we have Minkowski principal curvatures. The main difference is the formula of Gauss-Kronecker curvature: $K(p) = -k_1(p)k_2(p)...k_n(p)$.

Bearing all these examples in mind, we can think at a similar (to the Euclidean case) theory of general space-like surfaces in Minkowski type spaces.

5 Physical interpretation of curvature quantities

Bearing in mind the facts described in the previous section, we are ready to discuss the *meaning* of the curvature invariants, as we aimed to do. The mean curvature H is proportional to the tension field see e.g. [14], or [1], p.71; both references explain that the tension field, usually denoted in literature as $\tau(\phi)(x)$, is exactly the unnormalized mean curvature vector nH (where $n =$ dimension).

In what concerns the scalar curvature, the meaning is very interesting, and could be summarized as follows.

- The Ricci tensor in relativity theory is related to the matter content of the universe via Einstein's field equation. (The scalar curvature is the trace of the Ricci tensor.)
- It is the part of the curvature of spacetime that determines the degree to which matter will tend to converge or diverge in time.
- The scalar curvature is the Lagrangian density for the Einstein-Hilbert action, proposed originally in [17], that yields the Einstein field equations through the principle of least action.

With these arguments, we summarized the physical interpretation mentioned in Section 2.

6 The Elementary Proof of the Exercise from [10]

We could not conclude our discussion without presenting the proof of the example we stated. Remark that the whole discussion is pointwise, at a point on the hypersurface, all the analysis reduces to relations between real numbers satisfying some constraints. The assertions are, in the following order.

For (i), by the AM-GM inequality:

$$xyz = x^2 + y^2 + z^2 \geq 3\sqrt[3]{(xyz)^2}$$

Hence $(xyz)^3 \geq 27(xyz)^2$, and we are done. Equality holds for $x = y = z$.

For (ii), also by AM-GM:

$$xy + yz + zx \geq 3\sqrt[3]{(xyz)^2} \geq 3\sqrt[3]{27^2} = 3 \cdot 3^2 = 27.$$

For (iii), the argument is again AM-GM,

$$x + y + z \geq 3\sqrt[3]{xyz} \geq 9.$$

In all these inequalities the equality case holds whenever $x = y = z$.

Finally, for (iv), from $x^2 + y^2 + z^2 = xyz$ we derive that $x^2 < xyz$, which means $x < yz$. Similarly, $y < xz$, and $z < xy$. Thus: $xy < yz \cdot zx$ yields $z^2 > 1$, and since z is nonnegative, $z > 1$. Similarly, $x > 1$, $y > 1$. Now we use a substitution: $a = x - 1$, $b = y - 1$, $c = z - 1$. By the previous estimate: $a, b, c > 0$. The initial assumption $x^2 + y^2 + z^2 = xyz$ turns into

$$a^2 + b^2 + c^2 + a + b + c + 2 = abc + ab + bc + ca.$$

Denote by $q = ab+bc+ca$ and we get immediately $a^2+b^2+c^2 \geq q$, $a+b+c \geq \sqrt{3q}$, $abc \leq \left(\frac{q}{3}\right)^{3/2} = \frac{(3q)^{3/2}}{27}$. By $a^2+b^2+c^2+a+b+c+2 = abc+ab+bc+ca$. and these last three inequalities, we have $q+\sqrt{3q}+2 \leq a^2+b^2+c^2+a+b+c+2 = = abc + ab + bc + ca \leq \frac{(3q)^{3/2}}{27} + q$. That is: $\sqrt{3q} + 2 \leq \frac{(3q)^{3/2}}{27}$.

The ingenious argument in [10] starts where we denote $\sqrt{3q} = A$. Then the last inequality we obtained, $\sqrt{3q} + 2 \leq \frac{(3q)^{3/2}}{27}$, is equivalent to $A + 2 \leq \frac{A^3}{27}$, or, furthermore, $A^3 - 27A - 2 \cdot 27 \geq 0$. This last inequality factors out as $(A-6)(A+3)^2 \geq 0$, i.e. $\sqrt{3q} = A \geq 6$. This means $q \geq 12$. We obtained that $ab+bc+ca \geq 12$. Henceforth, $(x-1)(y-1)+(y-1)(z-1)+(z-1)(x-1) \geq 12$, from which we obtain the original claim $xy + yz + zx \geq 2(x + y + z) + 9$. □

7 Conclusion and other references.

In conclusion, some elementary inequalities do "hide" a physical meaning, which could be more or less *natural* in function of the physical context of a given problem. This physical meaning is supported by the geometric interpretation of symmetric functions, as shown above. We felt that this discussion on *natural* statements could be useful to anyone who would like to regard beyond the elementary appearance of some exercises in fundamental mathematics. For a similar interpretation, see the final paragraphs in [7].

For a recent thorough explanation of the mathematical foundations of special and general relativity, see [3]. For a comprehensive vision of the idea of curvature and its various interpretations, see the highly useful [9]. As a consequence of our reflection, some elementary statements, even if they deal apparently just with symmetric polynomials of n real numbers, could actually represent something more, in the spirit of the geometric and physical interpretations discussed above.

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