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# A new class of unsaturated mappings: Ćirić-Reich-Rus contractions

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*Dedicated to the memory of Professor Constantin Popa*

**Abstract**

Using the technique of enriching contractive type mappings, we introduce a more general concept of enriched Ćirić-Reich-Rus contraction than the one studied in [Berinde, V.; Păcurar, M. Fixed point theorems for enriched Ćirić-Reich-Rus contractions in Banach spaces and convex metric spaces. *Carpathian J. Math.* **37** (2021), no. 2, 173–184.] and provide convergence results for the Krasnoselskij iterative algorithm used to approximate their fixed points. Examples to illustrate the effectiveness of the new results as well as comparison to other classes of contractive type mappings existing in literature are also presented. In this context, we also conclude that Ćirić-Reich-Rus contractions form a class of unsaturated mappings.

## 1 Introduction and preliminaries

Let $(X, d)$ be a metric space. A mapping $T : X \to X$ is called

- *contraction* if, for some $a \in [0, 1)$, we have:

$$d(Tx, Ty) \leq ad(x, y), \forall x, y \in X; \tag{1}$$

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- *nonexpansive* if
$$d(Tx,Ty) \le d(x,y), \forall x,y \in X; \tag{2}$$
- *Kannan mapping* if, for some $b \in [0,1/2)$, we have:
$$d(Tx,Ty) \le b(d(x,Tx)+d(y,Ty)), \forall x,y \in X. \tag{3}$$

The above classes of mappings, which are very important in fixed point theory, have been generalized, for the case of a linear normed space, by means of the so called technique of enriching contractive type mappings in [6], [7], [10] and [11], as follows.

Let $(X,\|\cdot\|)$ be a linear normed space. A mapping $T : X \to X$ is called

- $(k,a)$-*enriched contraction* ([10]) if, for some $k \in [0,+\infty)$ and $a \in [0,k+1)$ we have
$$\|k(x-y)+Tx-Ty\| \le a\|x-y\|, \forall x,y \in X; \tag{4}$$
- $k$-*enriched nonexpansive* ([6]) if, for some $k \in [0,\infty)$ we have
$$\|k(x-y)+Tx-Ty\| \le (k+1)\|x-y\|, \forall x,y \in X; \tag{5}$$
- $(k,b)$-*enriched Kannan mapping* ([11]) if, for $k \in [0,\infty)$ and $b \in [0,1/2)$ we have
$$\|k(x-y)+Tx-Ty\| \le b\left(\|x-Tx\|+\|y-Ty\|\right), \text{ for all } x,y \in X. \tag{6}$$

**Remark 1.1.** *1) If $T$ is a contraction, then it satisfies* (4) *with $k=0$, i.e., $T$ is a $(0,a)$-enriched contraction;*

*2) If $T$ is nonexpansive, then it verifies* (5) *with $k=0$, that is, $T$ is a $0$-enriched nonexpansive mapping;*

*3) Any Kannan mapping ([32], [33]) satisfies* (6) *with $k=0$, which means that $T$ is a $(0,b)$-enriched Kannan mapping.*

*The above inclusions are strict, as shown by the corresponding examples in [10], [6] and [11], respectively.*

For a mapping $T : X \to X$, let us denote by $Fix\,(T)=\{x \in X : Tx = x\}$ the set of fixed points of $T$.

It was shown in [10] that if $(X,\|\cdot\|)$ is a Banach space and $T : X \to X$ is an enriched contraction, then $Fix\,(T)=\{x^*\}$ and the sequence defined by Krasnoselskij iteration (but not by Picard iteration) converges to $x^*$, for any starting point $x_0$.

Similarly, fixed point results for the class of enriched Kannan mappings in Banach spaces were obtained in [11] and applications of these results for the split feasibility problem and the variational inequality problem were given.

Let now $C$ be a bounded closed and convex subset of a Hilbert space $H$. The main result in [6] (Theorem 6) states that, for any $b$-enriched nonexpansive $T : C \to C$, the set $Fix\,(T)$ is nonempty and that the Krasnoselskij iteration $\{x_n\}_{n=0}^{\infty}$, given by

$$x_{n+1} = (1-\lambda)x_n + \lambda T x_n,\ n \geq 0, \tag{7}$$

converges weakly to a fixed point of $T$, for some $\lambda \in (0,1)$ and for any $x_0 \in C$.

Strong convergence (Theorem 2, [6]) has been obtained with the price of an additional assumption, i.e., if $T$ is also demicompact.

The following result is a generalization of Banach's contraction mapping principle.

**Theorem 1** (Theorem 2.1, [10]). *Let $(X, \|\cdot\|)$ be a Banach space and $T : X \to X$ a $(k, a)$-enriched contraction. Then*

*(i) $Fix\,(T) = \{p\}$, for some $p \in X$;*

*(ii) For a certain $\lambda \in (0,1]$, the Krasnoselskij iteration $\{x_n\}_{n=0}^{\infty}$, given by (7) converges to $p$, for any $x_0 \in X$.*

The contraction conditions (1) and (3) were unified by Ćirić [22], Reich [41] and Rus [42], who considered the class of mappings $T : X \to X$ satisfying the following condition:

$$d(Tx, Ty) \leq a d(x, y) + b\left(d(x, Tx) + d(y, Ty)\right),\ \text{for all } x, y \in X, \tag{8}$$

where $a, b \geq 0$ and $a + 2b < 1$.

Indeed, it is easily seen that, if $b = 0$, condition (8) reduces to (1) while, for $a = 0$, condition (8) reduces to (3).

The mappings $T$ satisfying (8) are commonly called *Ćirić-Reich-Rus contractions* and were intensively studied in the last 10-15 years or so, see [1], [2], [3], [21] [24]–[30], [34], [35], [36]-[39], [40], [45], for a selective list.

By combining the two directions presented above for extending the classical Banach and Kannan fixed point theorems, the following generalization of Ćirić-Reich-Rus contractions has been introduced.

**Definition 1** (Definition 2.3, [12]). *Let $(X, \|\cdot\|)$ be a linear normed space. A mapping $T : X \to X$ is a $(k, a, b)$-enriched Ćirić-Reich-Rus contraction if, for some $k \in [0, \infty)$ and $a, b \geq 0$, satisfying $a + 2b < 1$, one has*

$$\|k(x-y) + Tx - Ty\| \leq a\|x-y\| + b\left(\|x - Tx\| + \|y - Ty\|\right),\ \text{for all } x, y \in X. \tag{9}$$

**Remark 1.2.** *1) If $T$ is a Ćirić-Reich-Rus contraction, then it satisfies* (9) *with $k = 0$;*

*2) For $b = 0$, condition* (9) *reduces to condition* (4) *satisfied by an enriched contraction, while, for $k = 0$ and $a = 0$, from* (9) *we obtain the Kannan contraction condition [32] in the setting of a linear normed space;*

*3) By taking $a = 0$ in* (9)*, we obtain* (6)*;*

*4) By taking $b = 0$ in* (9)*, one obtains*

$$\|k(x-y)+Tx-Ty\| \leq a\|x-y\|, \forall x,y \in X. \tag{10}$$

*with $k \in [0,+\infty)$ and $a \in [0,1)$.*

*One can see that here we have $a \in [0,1)$, while in the definition of the enriched contraction it is required that $a \in [0,k+1)$.*

Starting from the last remark, we consider a more general class of enriched Ćirić-Reich-Rus contractions – but for which we keep the same name – by weakening the condition $a + 2b < 1$, thus removing the aforementioned drawback.

In this way, we should be able to fully recover the class of enriched contractions (4) under the complete assumptions $k \in [0,+\infty)$ and $a \in [0,k+1)$.

For this general class of enriched Ćirić-Reich-Rus contractions, we prove an appropriate fixed point theorem (Theorem 2) which extends Theorem 1, Theorem 2.3 in [12] and Theorem 2.1 in [11].

We also provide a non trivial example that shows that all the new results presented in this paper are effective generalizations of the existing ones in literature.

## 2 Main result

**Definition 2.** *Let $(X, \|\cdot\|)$ be a linear normed space and $T : X \to X$ a self mapping. $T$ is a $(k,a,b)$-enriched Ćirić-Reich-Rus contraction if, for some $k \in [0,\infty)$ and $a, b \geq 0$, satisfying $\dfrac{a}{k+1} + 2b < 1$, the following condition holds.*

$$\|k(x-y)+Tx-Ty\| \leq a\|x-y\| + b\left(\|x-Tx\| + \|y-Ty\|\right), \text{ for all } x,y \in X. \tag{11}$$

**Remarks.**

1) Obviously, any Ćirić-Reich-Rus contraction satisfies (11) with $k = 0$.

2) One can also see that, if $b = 0$, then from (11) we obtain the contraction condition (4) satisfied by an enriched contraction with $k \in [0,+\infty)$ and $a \in [0,k+1)$.

We note that, if $T$ is a self-mapping of a linear normed space $(X, \|\cdot\|)$, then, by defining for any $\lambda \in (0,1]$, the so-called *averaged mapping* $T_\lambda$ as

$$T_\lambda x = (1-\lambda)x + \lambda T x, \text{ for all } x \in X, \tag{12}$$

one has the property $Fix(T_\lambda) = Fix\,(T)$.

**Theorem 2.** *Let $(X, \|\cdot\|)$ be a Banach space and $T : X \to X$ a $(k, a, b)$-enriched Ćirić-Reich-Rus contraction in the sense of Definition 2. Then*

*(i) $Fix\,(T) = \{p\}$;*

*(ii) There exists $\lambda \in (0,1]$ such that the iterative method $\{y_n\}_{n=0}^{\infty}$, given by*

$$y_{n+1} = (1-\lambda)y_n + \lambda T y_n,\ n \geq 0, \tag{13}$$

*converges to $p$, for any $y_0 \in X$;*

*(iii) The following estimates hold*

$$\|y_n - p\| \leq \begin{cases} \alpha^n \cdot \|y_0 - p\|, n \geq 0 \\ \dfrac{\alpha}{1-\alpha} \cdot \|y_n - y_{n-1}\|, n \geq 1 \end{cases} \tag{14}$$

*where $\alpha = \dfrac{a + (k+1)b}{(k+1)(1-b)}$.*

*Proof.* Consider $k > 0$ (if $k = 0$, then we are in the case of Ćirić-Reich-Rus fixed point theorem). Let $T_\lambda$ be the averaged mapping defined by (12) for $\lambda = \dfrac{1}{k+1} < 1$.

The contractive condition (11) yields

$$\left\| \left(\frac{1}{\lambda} - 1\right)(x-y) + Tx - Ty \right\| \leq a\|x-y\| + b\left(\|x - Tx\| + \|y - Ty\|\right),$$

for all $x, y \in X$, which can be written as

$$\|T_\lambda x - T_\lambda y\| \leq a\lambda \|x-y\| + b\left(\|x - T_\lambda x\| + \|y - T_\lambda y\|\right), \text{ for all } x, y \in X.$$

If we denote $a_1 = a\lambda$, by the previous inequality we deduce that $T_\lambda$ satisfies a Ćirić-Reich-Rus contraction condition of the form

$$\|T_\lambda x - T_\lambda y\| \leq a_1 \|x-y\| + b\left(\|x - T_\lambda x\| + \|y - T_\lambda y\|\right), \text{ for all } x, y \in X, \tag{15}$$

with $a_1, b \geq 0$ and $a_1 + 2b < 1$.

By (15), for $y = T_\lambda x$, we get (denote $d(x,y) = \|x - y\|$):

$$d(T_\lambda x, T_\lambda^2 x) \leq \alpha\, d(x, T_\lambda x),\ x \in X, \tag{16}$$

where $\alpha = \dfrac{a_1 + b}{1 - b} < 1$. Now denote

$$D = \{d(x, T_\lambda x) : x \in X\}, \quad \delta = \inf D$$

and

$$D_1 = \{d(T_\lambda x, T_\lambda^2 x) : x \in X\}.$$

Obviously, $D_1 \subseteq D$ and $\delta \geq 0$. Assume $\delta > 0$. Then, by using (16) we get

$$\delta = \inf D \leq \inf D_1 \leq \alpha \inf D = \alpha\delta < \delta,$$

a contradiction. This show that $\delta = 0$. Let $\{x_n\} \subset X$ be a sequence such that

$$d(x_n, T_\lambda x_n) \to 0, \text{ as } n \to \infty. \tag{17}$$

By (11) we get

$$d(x_n, x_m) \leq \frac{1 + b}{1 - a_1} \left(d(x_n, T_\lambda x_n) + d(x_m, T_\lambda x_m)\right),$$

which, by virtue of (17), shows that $\{x_n\}$ is a Cauchy sequence. Let

$$\lim_{n \to \infty} x_n = p. \tag{18}$$

By (11), we obtain

$$d(p, T_\lambda p) \leq \frac{1 + a_1}{1 - b} d(x_n, p) + \frac{1 + b}{1 - b} d(x_n, T_\lambda x_n),$$

which, by (17) and (18), proves that $T_\lambda p = p$, i.e., $p \in Fix\,(T_\lambda)$.

Assume that $q \neq p$ is another fixed point of $T_\lambda$. Then, by (11)

$$0 < d(p, q) = d(T_\lambda p, T_\lambda q) \leq a_1 d(p, q) + b(d(p, T_\lambda p) + d(q, T_\lambda q)) < d(p, q),$$

a contradiction. So $Fix\,(T_\lambda) = \{p\}$, which proves (i).

Consider the sequence $\{y_n\}_{n=0}^{\infty}$ defined by the iterative process (13), that is,

$$y_{n+1} = T_\lambda y_n,\ n \geq 0 \tag{19}$$

Then, by (15),

$$d(y_{n+1}, p) \leq \alpha d(y_n, p), n \geq 0 \tag{20}$$

and by (16),

$$d(y_n, y_{n+1}) \leq \alpha d(y_{n-1}, y_n), n \geq 1. \tag{21}$$

By induction, from (20) one obtains the a priori error estimate

$$d(y_n, p) \leq \alpha^n d(y_0, p), n \geq 0. \tag{22}$$

This also proves that $\{y_n\}$ converges to $p$ as $n \to \infty$.

From (21) one obtains

$$d(y_{n+m}, y_n) \leq (\alpha + \alpha^2 + \cdots + \alpha^{n+m}) d(y_{n-1}, y_n), n \geq 1,\ m \geq 0, \tag{23}$$

which, by letting $m \to \infty$, leads to the a posteriori error estimate

$$d(y_n, p) \leq \frac{\alpha}{1-\alpha} d(y_{n-1}, y_n), n \geq 1, \tag{24}$$

which together with (22) proves (iii).

□

**Remark 2.1.** *1) From Theorem 2 we obtain Theorem 1, if* $b = 0$, *and Theorem 2.1 in [11], if* $a = 0$.

*2) Since* $a + 2b < 1$ *implies* $\dfrac{a}{k+1} + 2b < 1$, *we also note that Theorem 2.3 in [12] is a particular case of Theorem 2.*

*3) The proof of Theorem 2 is different of those commonly used in metric fixed point theory and is based on the ideas in [31].*

It is possible to establish a more general fixed point theorem by allowing the constant coefficients $a$ and $b$ in (11) to depend on $x$ and $y$, like in Theorem 2.5 of Ćirić [22].

**Definition 3.** *Let* $(X, \|\cdot\|)$ *be a linear normed space and* $T : X \to X$ *a self mapping of* $X$. $T$ *is a generalized* $(k, a, b)$*-enriched Ćirić-Reich-Rus contraction if, there exist* $k \in [0, \infty)$ *and, for every* $x, y \in X$, *the non-negative functions* $a, b : X^2 \to [0, \infty)$ *satisfying*

$$\sup_{x,y \in X} \left( \frac{a(x,y)}{k+1} + 2b(x,y) \right) = \theta < 1$$

*such that, for all* $x, y \in X$,

$$\|k(x-y) + Tx - Ty\| \leq a(x,y)\|x-y\| + b(x,y)\left(\|x - Tx\| + \|y - Ty\|\right). \tag{25}$$

The next result is a generalization of Theorem 2.4 in [12].

**Theorem 3.** *Let* $(X, \|\cdot\|)$ *be a Banach space and* $T : X \to X$ *a generalized* $(k, a, b)$*-enriched Ćirić-Reich-Rus contraction. Then*

*(i)* $Fix\,(T) = \{p\}$, *where* $p \in X$;

*(ii) There exists* $\lambda \in (0,1]$ *such that the iterative method* $\{x_n\}_{n=0}^{\infty}$*, given by*

$$x_{n+1} = (1-\lambda)x_n + \lambda T x_n,\ n \geq 0,$$

*converges to* $p$*, for any* $x_0 \in X$*;*

*(iii) The following estimate holds*

$$\|x_{n+i-1} - p\| \leq \frac{\theta^i}{1-\theta} \cdot \|x_n - x_{n-1}\|, \quad n = 0, 1, 2, \dots;\ i = 1, 2, \dots$$

*where where* $\theta = \sup_{x,y \in X} \left( \frac{a(x,y)}{k+1} + 2b(x,y) \right)$.

*Proof.* We follow the pattern in the proof of Theorem 2. □

There is another interesting question about the class of enriched Ćirić-Reich-Rus contractions, in terms of the discussion in [17]: is it saturated or not? The following example plays a double role. It shows on the one hand the relationship between the classes of Ćirić-Reich-Rus contractions and enriched Ćirić-Reich-Rus contractions (in both versions mentioned in this paper), and on the other hand it will help us in answering the question we have just posed.

**Example 1.** *Let* $X = [0,1]$ *with the usual distance and* $T : X \to X$ *be given by* $Tx = 1 - x$*, for all* $x \in [0,1]$*. Then* $T$ *does not satisfy the Ćirić-Reich-Rus contraction condition* (8) *but it satisfies both enriched Ćirić-Reich-Rus contraction conditions* (9) *and* (11).

*Indeed, to see that* $T$ *is not a Ćirić-Reich-Rus contraction, let us assume the contrary, i.e., that there exist* $a, b \geq 0$*, satisfying* $a + 2b < 1$ *such that* (8) *holds. This means that*

$$|Tx - Ty| \leq a|x-y| + b(|x - Tx| + |y - Ty|) \Leftrightarrow$$

$$|x-y| \leq a|x-y| + b(|1-2x| + |1-2y|) \Leftrightarrow (1-a)|x-y| \leq b(|1-2x| + |1-2y|)$$

*Now take* $x = \frac{1}{4}$, $y = \frac{3}{4}$ *in the previous inequality to get*

$$(1-a) \cdot \frac{1}{2} \leq b \Leftrightarrow a + 2b \geq 1,$$

*a contradiction.*

*Now, since* $T$ *is an enriched contraction (and also an enriched Kannan mapping), see Example in [10], and also Example 2.1 in [11], it follows that* $T$ *is an enriched Ćirić-Reich-Rus contraction, too (in the sense of both Definitions 1 and 2).*

We close this section by discussing the new results obtained in the current paper by means of the concept of *unsaturated class* of mappings, introduced in [17].

**Definition 4** (Definition 2, [17]). *Let $(X, \|\cdot\|)$ be a linear normed space and let $\mathcal{M}$ be a subset of the family of all self-mappings of $X$. A mapping $T : X \to X$ is said to be* $\mathcal{M}$-enriched *or* enriched with respect to $\mathcal{M}$ *if there exists $\lambda \in (0, 1]$ such that $T_\lambda \in \mathcal{M}$.*

*By $\mathcal{M}^e$ we denote the set of all enriched mappings with respect to $\mathcal{M}$.*

**Remark 2.2.** *From Definition 4 and the fact that $T_\lambda = (1 - \lambda)I + \lambda T$, for $\lambda \in (0, 1]$, it follows immediately that $\mathcal{M} \subseteq \mathcal{M}^e$.*

**Definition 5** (Definition 3, [17]). *Let $X$ be a linear vector space and let $\mathcal{M}$ be a subset of the family of all self-mappings of $X$. If $\mathcal{M} = \mathcal{M}^e$, we say that $\mathcal{M}$ is a* saturated class of mappings*; otherwise, $\mathcal{M}$ is said to be unsaturated.*

Note that $\mathcal{M}$ is unsaturated if and only if the inclusion $\mathcal{M} \subset \mathcal{M}^e$ is strict.

The following classes of mappings: Banach contractions [10], Kannan mappings [11], Chatterjea mappings [13], almost contractions [15] and nonexpansive mappings [6], [7], are all **unsaturated** (see [17]), while strictly pseudocontractive mappings [5] and demicontractive mappings [17] are examples of **saturated** classes of mappings.

The interpretation of this fact is that the classes of Banach contractions, Kannan mappings, Chatterjea mappings, almost contractions and nonexpansive mappings can be enlarged by means of the technique of enriching contractive type mappings, while the classes of strictly pseudocontractive mappings and demicontractive mappings cannot.

The problem of whether the class of Ćirić-Reich-Rus contractions is saturated or unsaturated remained unanswered in [11]. So, in the spirit of [17], we may conclude this section with an answer to that question.

**Theorem 4.** *Let $X$ be a real Banach space and let $\mathcal{M}_{CRR}(X)$ denote the class of Ćirić-Reich-Rus contractions $T : X \to X$. Then, $\mathcal{M}_{CRR}(X)$ is an unsaturated class of contractive mappings.*

*Proof.* For $T$ in Example 1, we have that $T \in \mathcal{M}^e_{CRR}(X)$ but $T \notin \mathcal{M}_{CRR}(X)$. □

## 3 Conclusions and further study

1. We introduced the class of enriched Ćirić-Reich-Rus contractions (Definition 2), which includes the enriched Ćirić-Reich-Rus contractions previously considered in [11] (Definition 1);

2. It has been proven that any enriched Ćirić-Reich-Rus contraction in the sense of Definition 2 has a unique fixed point and this fixed point can be approximated by means of a suitable Kransnoselskij iteration.
3. By means of Example 1 we show that the class of enriched Ćirić-Reich-Rus contractions (both Definition 1 and Definition 2) strictly includes the class of Ćirić-Reich-Rus contractions.
4. It is worth mentioning that enriched Ćirić-Reich-Rus contractions preserve a fundamental property of Banach contractions, Kannan mappings and Ćirić-Reich-Rus contractions, namely any enriched Ćirić-Reich-Rus contraction has a unique fixed point, but they do not preserve another important property, i.e., the fact that the unique fixed point can be approximated by using the Picard iteration.
5. The fixed point theorems established in this paper could be applied for solving split feasibility problems and variational inequality problems, thus generalizing the existence and approximation results for enriched Kannan mappings given in [11]. Also, it is well known that Kannan fixed point theorem characterizes the completeness of the metric space. So, it is an open question wether or not this property is valid for the class of enriched Ćirić-Reich-Rus contractions (which includes Kannan mappings).
6. For other possible developments using the technique of enriching nonexpansive mappings, we refer to [4], [8]-[9], [18]-[16], [43], [44], [45],...

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