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Ramanujan-type congruences modulo 4 for partitions into distinct parts

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Abstract

In this paper, we consider the partition function $Q(n)$ counting the partitions of n into distinct parts and investigate congruence identities of the form

$$Q\left(p\cdot n+\frac{p^2-1}{24}\right)\equiv 0 \pmod 4,$$

where $p\geqslant 5$ is a prime.

1 Introduction

Recall that a composition of a positive integer n is a sequence of positive integers $(\lambda_1,\lambda_2,\ldots,\lambda_k)$ whose sum is n, i.e.,

$$n=\lambda_1+\lambda_2+\cdots+\lambda_k. \tag{1}$$

When the order of integers λ_i does not matter, the representation (1) is known as an integer partition [1] and can be rewritten as

$$n=t_1+2t_2+\cdots+nt_n,$$

where each positive integer i appears t_i times in the partition. For consistency, we consider a partition of n a non-increasing sequence of natural numbers whose sum is n. For example, the partitions of 4 are given as:

$$(4),\ (3,1),\ (2,2),\ (2,1,1),\ (1,1,1,1). \tag{2}$$

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The fastest algorithms for enumerating all the partitions of an integer have recently been presented by Merca [8, 9]. As usual, we denote by $p(n)$ the number of integer partitions of n and we have the generating function

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty}.$$

Here and throughout this paper, we use the following customary q-series notation:

$$(a;q)_n = \begin{cases} 1, & \text{for } n = 0, \\ (1-a)(1-aq)\cdots(1-aq^{n-1}), & \text{for } n > 0; \end{cases}$$
$$(a;q)_\infty = \lim_{n\to\infty} (a;q)_n.$$

Because the infinite product $(a;q)_\infty$ diverges when $a \neq 0$ and $|q| \geqslant 1$, whenever $(a;q)_\infty$ appears in a formula, we shall assume $|q| < 1$.

The famous Ramanujan congruences for the partition function $p(n)$, which were proved by Atkin, Ramanujan and Watson [2, 3, 18], assert that

$$p\big(5^j n + \beta_5(j)\big) \equiv 0 \pmod{5^j},$$
$$p\big(7^j n + \beta_7(j)\big) \equiv 0 \pmod{7^{[j/2]+1}},$$
$$p\big(11^j n + \beta_{11}(j)\big) \equiv 0 \pmod{11^j}$$

for every non-negative integer n where $\beta_m(j) := 1/24 \pmod{m^j}$. Congruences modulo power of 5 and 7 for the partition function $Q(n)$ counting the partitions of n into distinct parts can be seen in a paper by B. Gordon and K. Hughes [4].

From Euler's pentagonal number theorem

$$(q;q)_\infty = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2}$$

we known that almost all values of $Q(n)$ are even, i.e,.

$$\sum_{n=0}^{\infty} Q(n)\, q^n = (-q;q)_\infty \equiv (q;q)_\infty \pmod 2.$$

Thus $Q(n)$ is odd if and only if n is a generalized pentagonal number. This fact was generalized by B. Gordon and K. Ono [5, Theorem 1], who demonstrated that, for any positive integer k, almost all values of $Q(n)$ are divisible by 2^k. More precisely, if k is a positive integer, then

$$Q(n) \equiv 0 \pmod{2^k}$$

for a subset of non-negative integers n with arithmetic density one. In [13], K. Ono and D. Penniston provided an exact formula for $Q(n)$ modulo 8.

In this paper, we remark some congruences modulo 4 for the partition function $Q(n)$. Surprisingly, these congruences have not been noticed so far.

Theorem 1. *For all* $n \not\equiv 0 \pmod 5$,

$$Q(5n+1) \equiv 0 \pmod 4.$$

Having $Q(6) = 4$, $Q(11) = 12$, $Q(16+25) = 1260$ and $Q(21) = 76$, for $\alpha \in \{6, 11, 16, 21\}$ we notice that

$$\sum_{n=0}^{\infty} Q(25n+\alpha)\, q^n \not\equiv 0 \pmod 8.$$

Theorem 1 follows directly from the following two identities where for any positive integer k, f_k is defined by

$$f_k := (q^k; q^k)_\infty.$$

Theorem 2. *For* $|q| < 1$,

$$\begin{aligned}
&\left(\sum_{n=0}^{\infty} Q(25n+6)\, q^n\right)\left(\sum_{n=0}^{\infty} Q(25n+21)\, q^n\right)\\
&\quad = 16\left(19\,\frac{f_2^{18}\, f_5^{38}}{f_1^{40}\, f_{10}^{16}} + 1431\, q\,\frac{f_2^{17}\, f_5^{33}}{f_1^{39}\, f_{10}^{11}} + 19164\, q^2\,\frac{f_2^{16}\, f_5^{28}}{f_1^{38}\, f_{10}^{6}} + 95176\, q^3\,\frac{f_2^{15}\, f_5^{23}}{f_1^{37}\, f_{10}}\right.\\
&\qquad + 261104\, q^4\,\frac{f_2^{14}\, f_5^{18}\, f_{10}^{4}}{f_1^{36}} + 553344\, q^5\,\frac{f_2^{13}\, f_5^{13}\, f_{10}^{9}}{f_1^{35}} + 838656\, q^6\,\frac{f_2^{12}\, f_5^{8}\, f_{10}^{14}}{f_1^{34}}\\
&\qquad \left. + 804864\, q^7\,\frac{f_2^{11}\, f_5^{3}\, f_{10}^{19}}{f_1^{33}} + 434176\, q^8\,\frac{f_2^{10}\, f_{10}^{24}}{f_1^{32}\, f_5^{2}} + 98304\, q^9\,\frac{f_2^{9}\, f_{10}^{29}}{f_1^{31}\, f_5^{7}}\right)
\end{aligned}$$

and

$$\begin{aligned}
&\left(\sum_{n=0}^{\infty} Q(25n+11)\, q^n\right)\left(\sum_{n=0}^{\infty} Q(25n+16)\, q^n\right)\\
&\quad = 16\left(24\,\frac{f_2^{18}\, f_5^{38}}{f_1^{40}\, f_{10}^{16}} + 1321\, q\,\frac{f_2^{17}\, f_5^{33}}{f_1^{39}\, f_{10}^{11}} + 20129\, q^2\,\frac{f_2^{16}\, f_5^{28}}{f_1^{38}\, f_{10}^{6}} + 91056\, q^3\,\frac{f_2^{15}\, f_5^{23}}{f_1^{37}\, f_{10}}\right.\\
&\qquad + 268704\, q^4\,\frac{f_2^{14}\, f_5^{18}\, f_{10}^{4}}{f_1^{36}} + 554624\, q^5\,\frac{f_2^{13}\, f_5^{13}\, f_{10}^{9}}{f_1^{35}} + 816896\, q^6\,\frac{f_2^{12}\, f_5^{8}\, f_{10}^{14}}{f_1^{34}}\\
&\qquad \left. + 815104\, q^7\,\frac{f_2^{11}\, f_5^{3}\, f_{10}^{19}}{f_1^{33}} + 454656\, q^8\,\frac{f_2^{10}\, f_{10}^{24}}{f_1^{32}\, f_5^{2}} + 98304\, q^9\,\frac{f_2^{9}\, f_{10}^{29}}{f_1^{31}\, f_5^{7}}\right).
\end{aligned}$$

Upon reflection, one expects that there might be an infinite family of congruence identities where the congruence identity given by Theorem 1 is the first entry.

Theorem 3. *Let $p \geqslant 5$ be a prime number such that $p \not\equiv 1 \pmod{24}$. For all $n \not\equiv 0 \pmod{p}$, we have*

$$Q\left(p \cdot n + \frac{p^2-1}{24}\right) \equiv 0 \pmod{4}.$$

Theorem 4. *Let $p \equiv 1 \pmod{24}$ be a prime. For all $n \not\equiv 0 \pmod{p}$, we have*

$$Q\left(p \cdot n + \frac{p^2-1}{24}\right) \equiv \begin{cases} 2 \pmod{4}, & \text{if } n + \frac{p-1}{24} = \frac{k(3k+1)}{2},\ k \in \mathbb{Z}, \\ 0 \pmod{4}, & \text{otherwise.} \end{cases}$$

The case $p = 7$ of Theorem 3 reads as

$$Q(7n+2) \equiv 0 \pmod{4},$$

for all $n \not\equiv 0 \pmod{7}$. We remark that there is a stronger result.

Theorem 5. *For all $n \not\equiv 0 \pmod{7}$,*

$$Q(7n+2) \equiv 0 \pmod{8}.$$

The organization of this paper is as follows. We will first prove Theorems 2 and 5 in Sec. 2. In Sec. 3, we will prove Theorems 3 and 4 considering new connections between partitions and divisors. Some open problems are introduced in the last section.

2 Ramanujan-like congruences

Although the generating function for $p(n)$ was discovered by Euler in 1748, almost nothing was known of the arithmetic properties of $p(n)$ before the twentieth century. The first major discoveries in this area are due to Ramanujan [15, 16]:

$$\sum_{n=0}^{\infty} p(5n+4)\, q^n = 5\, \frac{f_5^5}{f_1^6},$$
$$\sum_{n=0}^{\infty} p(7n+5)\, q^n = 7\, \frac{f_7^3}{f_1^4} + 49q\, \frac{f_7^7}{f_1^8}.$$

These identities allowed the derivation of the famous congruences modulo 5 and 7 for the partition function $p(n)$.

In 1957, O. Kolberg [7] realized that these identities of Ramanujan could be extended to include a much larger variety of similar identities for $p(5n+j)$, $p(7n+j)$ and others. For example, Kolberg proved that

$$\left(\sum_{n=0}^{\infty} p(5n+1)\, q^n\right)\left(\sum_{n=0}^{\infty} p(5n+2)\, q^n\right) = 2\,\frac{f_5^4}{f_1^6} + 25q\,\frac{f_5^{10}}{f_1^{12}}.$$

In 2015, C.-S. Radu [14] constructed an algorithm to compute identities in the form of those discovered by Ramanujan and Kolberg above. He designed an algorithm which takes as input a generating function of the form

$$\sum_{n=0}^{\infty} a_r(n)\, q^n = \prod_{\delta|M}\prod_{n=1}^{\infty}(1-q^{\delta n})^{r_\delta}$$

and positive integers m and N, where M is a positive integer and $(r_\delta)_{\delta|M}$ is a sequence indexed by the positive divisors δ of M. With this data the algorithm attempts to produce a set $P_{m,r}(j) \subseteq \{0,1,\ldots,m-1\}$ which contains j and is uniquely defined by m, $(r_\delta)_{\delta|M}$ and j. Next the algorithm decides if there exists a sequence $(s_\delta)_{\delta|N}$ such that

$$q^\alpha \prod_{\delta|N}\prod_{n=1}^{\infty}(1-q^{\delta n})^{s_\delta} \cdot \prod_{j'\in P_{m,r}(j)} \sum_{n=0}^{\infty} a(mn+j')\, q^n$$

is a modular function with certain restrictions on its behaviour on the boundary of $\mathbb{H}$. Very recently, N. A. Smoot [17] provided a successful Mathematica implementation of Radu's algorithm. The package is called `RaduRK` and requires `4ti2`, a software package for algebraic, geometric and combinatorial problems on linear spaces. Instructions for the proper installation for these packages can be found in [17]. In this section, we use the `RaduRK` program to prove Theorems 2 and 5.

The generating function for $Q(n)$ is given by

$$\sum_{n=0}^{\infty} Q(n)\, q^n = (-q,q)_\infty = \frac{(q^2;q^2)_\infty}{(q;q)_\infty}.$$

This can be described by setting $M=2$ and $r=\{-1,1\}$.

Proof of Theorem 2. If we now take $m=25$, guess $N=10$ and take $j=6$, then we obtain

```
In[1] := RK[10,2,{-1,1},25,6]
```

$$\prod_{\delta|\mathtt{M}}(\mathtt{q}^{\delta};\mathtt{q}^{\delta})_{\infty}^{\mathtt{r}_{\delta}} = \sum_{\mathtt{n}=0}^{\infty}\mathtt{a}(\mathtt{n})\,\mathtt{q}^{\mathtt{n}}$$

$$\boxed{\mathtt{f}_1(\mathtt{q})\cdot\prod_{\mathtt{j}'\in\mathtt{P}_{\mathtt{m},\mathtt{r}}(\mathtt{j})}\sum_{\mathtt{n}=0}^{\infty}\mathtt{a}(\mathtt{mn}+\mathtt{j}')\,\mathtt{q}^{\mathtt{n}} = \sum_{\mathtt{g}\in\mathtt{AB}}\mathtt{g}\cdot\mathtt{p}_{\mathtt{g}}(\mathtt{t})}$$

Modular Curve: $\mathtt{X}_0(\mathtt{N})$

```
Out[2] =
```

N:	10
$\{\mathtt{M},(\mathtt{r}_\delta)_{\delta\vert\mathtt{M}}\}$:	$\{2,\{-1,1\}\}$
m:	25
$\mathtt{P}_{\mathtt{m},\mathtt{r}}(\mathtt{j})$:	$\{6,21\}$
$\mathtt{f}_1(\mathtt{q})$:	$\frac{((q;q)_\infty)^{30}\left((q^5;q^5)_\infty\right)^{12}}{q^{10}\left((q^2;q^2)_\infty\right)^{8}\left((q^{10};q^{10})_\infty\right)^{34}}$
t:	$\frac{\left((q^2;q^2)_\infty\right)\left((q^5;q^5)_\infty\right)^{5}}{q\left((q;q)_\infty\right)\left((q^{10};q^{10})_\infty\right)^{5}}$
AB:	$\{1\}$
$\{\mathtt{p}_{\mathtt{g}}(\mathtt{t}):\mathtt{g}\in\mathtt{AB}\}$:	$\{1\,572\,864\,t+6\,946\,816\,t^2+12\,877\,824\,t^3+13\,418\,496\,t^4+8\,853\,504\,t^5+4\,177\,664\,t^6+1\,522\,816\,t^7+306\,624\,t^8+22\,896\,t^9+304\,t^{10}\}$
Common Factor:	16

This gives us

$$\mathtt{f}_1(q)\cdot\left(\sum_{n=0}^{\infty}Q(25n+6)\,q^n\right)\left(\sum_{n=0}^{\infty}Q(25n+21)\,q^n\right)$$
$$= 1572864\,t + 6946816\,t^2 + 12877824\,t^3 + 13418496\,t^4 + 8853504\,t^5 + 4177664\,t^6 + 1522816\,t^7 + 306624\,t^8 + 22896\,t^9 + 304\,t^{10},$$

which yields the first identity on rearrangement.

If we now take $m = 25$, guess $N = 10$ and take $j = 11$, then we obtain

```
In[1] := RK[10,2,{-1,1},25,11]
```

$$\prod_{\delta|\mathtt{M}}(\mathtt{q}^{\delta};\mathtt{q}^{\delta})_{\infty}^{\mathtt{r}_\delta} = \sum_{\mathtt{n}=0}^{\infty}\mathtt{a}(\mathtt{n})\,\mathtt{q}^{\mathtt{n}}$$

$$\boxed{\mathtt{f}_1(\mathtt{q})\cdot\prod_{\mathtt{j}'\in\mathtt{P}_{\mathtt{m},\mathtt{r}}(\mathtt{j})}\sum_{\mathtt{n}=0}^{\infty}\mathtt{a}(\mathtt{mn}+\mathtt{j}')\,\mathtt{q}^{\mathtt{n}} = \sum_{\mathtt{g}\in\mathtt{AB}}\mathtt{g}\cdot\mathtt{p}_{\mathtt{g}}(\mathtt{t})}$$

Modular Curve: $\mathtt{X}_0(\mathtt{N})$

```
Out[2] =
```

N:	10
$\{\mathtt{M},(\mathtt{r}_\delta)_{\delta\mid\mathtt{M}}\}$:	$\{2,\{-1,1\}\}$
m:	25
$\mathtt{P}_{\mathtt{m},\mathtt{r}}(\mathtt{j})$:	$\{6,21\}$
$\mathtt{f}_1(\mathtt{q})$:	$\dfrac{((q;q)_\infty)^{30}\left(\left(q^5;q^5\right)_\infty\right)^{12}}{q^{10}\left(\left(q^2;q^2\right)_\infty\right)^{8}\left(\left(q^{10};q^{10}\right)_\infty\right)^{34}}$
t:	$\dfrac{\left(\left(q^2;q^2\right)_\infty\right)\left(\left(q^5;q^5\right)_\infty\right)^{5}}{q\left((q;q)_\infty\right)\left(\left(q^{10};q^{10}\right)_\infty\right)^{5}}$
AB:	$\{1\}$
$\{\mathtt{p}_{\mathtt{g}}(\mathtt{t}):\mathtt{g}\in\mathtt{AB}\}$:	$\{1\,572\,864\,t+7\,274\,496\,t^2+13\,041\,664\,t^3+13\,070\,336\,t^4+8\,873\,984\,t^5+4\,299\,264\,t^6+1\,456\,896\,t^7+322\,064\,t^8+21\,136\,t^9+384\,t^{10}\}$
Common Factor:	16

This gives us

$$\mathtt{f}_1(q)\cdot\left(\sum_{n=0}^{\infty}Q(25n+11)\,q^n\right)\left(\sum_{n=0}^{\infty}Q(25n+16)\,q^n\right)$$

$$\begin{aligned} &= 1572864\,t + 7274496\,t^2 + 13041664\,t^3 + 13070336\,t^4 + 8873984\,t^5 \\ &\quad + 4299264\,t^6 + 1456896\,t^7 + 322064\,t^8 + 21136\,t^9 + 384\,t^{10}, \end{aligned}$$

which yields the second identity on rearrangement.

Proof of Theorem 5. Having $Q(9) = 8$, $Q(16+49) = 18200$, $Q(23) = 104$, $Q(30) = 296$, $Q(37) = 760$ and $Q(44) = 1816$, for $\alpha \in \{9, 16, 23, 30, 37, 44\}$ we notice that

$$\sum_{n=0}^{\infty} Q(49n+\alpha)\, q^n \not\equiv 0 \pmod{16}.$$

Thus, Theorem 5 follows directly from the following lemma.

Lemma 6. *For* $|q| < 1$,

(i) $$\prod_{\alpha\in\{9,16,30\}} \sum_{n=0}^{\infty} Q(49n+\alpha)\, q^n \equiv 0 \pmod{2^9}$$

(ii) $$\prod_{\alpha\in\{23,37,44\}} \sum_{n=0}^{\infty} Q(49n+\alpha)\, q^n \equiv 0 \pmod{2^9}$$

Proof. The proof of this lemma is quite similar to the proof of the Theorem 2, so we omit the details. To obtain the first congruence identity, we use

```
RK[14,2,{-1,1},49,9].
```

This gives us

$$\begin{aligned} &\frac{(q;q)_\infty^{97}\,(q^7;q^7)_\infty^{38}}{q^{44}\,(q^2;q^2)_\infty^{37}\,(q^{14};q^{14})_\infty^{98}} \prod_{\alpha\in\{9,16,30\}} \sum_{n=0}^{\infty} Q(49n+\alpha)\, q^n \\ &\qquad = 512 \cdot \left(p_1(t) + p_2(t) \cdot \frac{(q^2;q^2)_\infty^{8}\,(q^7;q^7)_\infty^{4}}{q^2\,(q;q)_\infty (q^{14};q^{14})_\infty^{7}} \right), \end{aligned}$$

where

$$\begin{aligned} p_1(t) = &- 648518346341351424 - 1156641477899055005696\,t \\ &- 5353985521969251588505 6\,t^2 + 1054507420582477299712 00\,t^3 \\ &+ 4694768969587740888793088\,t^4 + 21390855376330998377611264\,t^5 \\ &+ 13991341992545467494301696\,t^6 + 11260505525461188675108864\,t^7 \\ &+ 44330252745473867191943168\,t^8 + 23643615579547387255848960\,t^9 \end{aligned}$$

$$
\begin{aligned}
&+ 837415585560856141129 3184\, t^{10} + 1805576016164080502964224\, t^{11}\\
&+ 148251452149552490217472\, t^{12} - 31553327537583703031808\, t^{13}\\
&- 988853602408867601 2032\, t^{14} - 720939207186607809024\, t^{15}\\
&+ 614690531552817283 20\, t^{16} + 104296024845671381 04\, t^{17}\\
&+ 350946686691788872\, t^{18} + 2453846372311302\, t^{19}\\
&+ 23458799564 01 t^{20} + 142473177 t^{21} + 148 t^{22}
\end{aligned}
$$

and

$$
\begin{aligned}
p_2(t) =& 648518346341351424 + 1298450822965697183744\, t\\
&+ 99220620593049041371136\, t^2 + 1206009969735756425461760\, t^3\\
&+ 5176826205924958811455488\, t^4 + 11097840813001246343430144\, t^5\\
&+ 13991341992545467494301696\, t^6 + 11260505525461188675108864\, t^7\\
&+ 5975272239407813722374144\, t^8 + 2057274390255339109875712\, t^9\\
&+ 410447470135012141039616\, t^{10} + 21990447479668472807424\, t^{11}\\
&- 9901472011422939742208\, t^{12} - 2239478040117778219008\, t^{13}\\
&- 87172452006829977600\, t^{14} + 19395135652819907072\, t^{15}\\
&+ 1787542742856116928\, t^{16} + 32903925181539592\, t^{17}\\
&+ 97052913403920\, t^{18} + 27860920174\, t^{19} + 264755\, t^{20},
\end{aligned}
$$

with

$$
t = \frac{(q^2;q^2)_\infty\,(q^7;q^7)_\infty^7}{q^2\,(q;q)_\infty\,(q^{12};q^{14})_\infty^7}.
$$

The second congruence identity can be obtain if we consider

```
RK[14,2,{-1,1},49,23].
```

This gives us

$$
\begin{aligned}
&\frac{(q;q)_\infty^{97}\,(q^7;q^7)_\infty^{38}}{q^{43}\,(q^2;q^2)_\infty^{37}\,(q^{14};q^{14})_\infty^{98}} \prod_{\alpha\in\{23,37,44\}} \sum_{n=0}^{\infty} Q(49n+\alpha)\,q^n\\
&\quad = 512\cdot\left(p_1(t) + p_2(t)\cdot\left(\frac{(q^2;q^2)_\infty^8\,(q^7;q^7)_\infty^4}{q^3\,(q;q)_\infty^4(q^{14};q^{14})_\infty^8} - \frac{4\,(q^2;q^2)_\infty\,(q^7;q^7)_\infty^7}{q^2\,(q;q)_\infty\,(q^{14};q^{14})_\infty^7}\right)\right),
\end{aligned}
$$

where

$$
p_1(t) = -122533037941570981068 8\, t - 5302921710921790259200 0\, t^2
$$

$$
\begin{aligned}
&+ 1059837741419475826114 56\, t^3 + 46934069072232376529059 84\, t^4 \\
&+ 2139170693624936870497484 8\, t^5 + 4508188831121107379132825 6\, t^6 \\
&+ 5566504147440694903753932 8\, t^7 + 4433026547932445799979417 6\, t^8 \\
&+ 2364363440732056494853324 8\, t^9 + 8374136065999589712330752\, t^{10} \\
&+ 1805578003899798851158016\, t^{11} + 1482496033399479426088 96\, t^{12} \\
&- 3155259621545466920960 0\, t^{13} - 988845147905594247577 6\, t^{14} \\
&- 7208857965397478993 92\, t^{15} + 6147075547815475648 0\, t^{16} \\
&+ 10429851285495028344\, t^{17} + 350957576681349999\, t^{18} \\
&+ 2453307890300554\, t^{19} + 2342867920924\, t^{20} + 152283143\, t^{21}
\end{aligned}
$$

and

$$
\begin{aligned}
p_2(t) =& 1349629729131135500288\, t + 995438915075764859699 20\, t^2 \\
&+ 1205372780933149538385920\, t^3 + 5177219767184086813114368\, t^4 \\
&+ 1109785082693313868542771 2\, t^5 + 1399123839777479006879744 0\, t^6 \\
&+ 1126054256264872838614220 8\, t^7 + 5975262039538114523824128\, t^8 \\
&+ 2057270115516818175033344\, t^9 + 410453459809019650637824\, t^{10} \\
&+ 21990450677627612037120\, t^{11} - 9901615893689232523264\, t^{12} \\
&- 2239649077347891544064\, t^{13} - 87188088020470964224\, t^{14} \\
&+ 19394405130087151616\, t^{15} + 1787438044885791360\, t^{16} \\
&+ 32905858181925128\, t^{17} + 97126492812313\, t^{18} \\
&+ 27745559755\, t^{19} + 280345\, t^{20}
\end{aligned}
$$

with

$$
t = \frac{(q^2;q^2)_\infty \, (q^7;q^7)_\infty^7}{q^2 \, (q;q)_\infty \, (q^{12};q^{14})_\infty^7}.
$$

$\square$

3 Connections between partitions and divisors

One of the well-known identities in the partition theory is given by the following theta series of Gauss

$$
1 + 2\sum_{n=1}^{\infty} (-q)^{n^2} = \frac{(q;q)_\infty}{(-q;q)_\infty}. \tag{3}
$$

Very recently [11], we consider this theta identity and obtained the following truncated form for it: For the positive integers k and r, we have:

$$(-q;q)_\infty \left(1+2\sum_{j=1}^{k}(-1)^j q^{r\cdot j^2}\right)$$
$$= \frac{(-q;q)_\infty (q^r;q^r)_\infty}{(-q^r;q^r)_\infty} + 2(-1)^k q^{r(k+1)^2} \frac{(q^r;q^{2r})_\infty}{(q;q^2)_\infty} \sum_{j=0}^{\infty} \frac{q^{(2k+2j+3)rj}}{(q^{2r};q^{2r})_j (q^r;q^{2r})_{k+j+1}}.$$

As a consequence of this theorem, we derived in [11] an infinite family of linear inequalities for the partition function $Q(n)$ counting the partitions of n into distinct parts: For $n \geqslant 0$, $m \geqslant 1$,

$$(-1)^m \left(Q(n) + 2\sum_{j=1}^{m}(-1)^j Q(n-3j^2) - \sum_{k=0}^{\infty} \delta_{n,G_k}\right) \geqslant 0,$$

where

$$G_k = \frac{1}{2}\left\lceil \frac{k}{2} \right\rceil \left\lceil \frac{3k+1}{2} \right\rceil$$

is the kth generalized pentagonal number.

The limiting case $m \to \infty$ of this inequality reads as: For $n \geqslant 0$,

$$Q(n) + 2\sum_{j=1}^{\infty}(-1)^j Q(n-3j^2) = \begin{cases} 1, & \text{if } n = G_k,\ k \in \mathbb{N}_0, \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

Another proof of this recurrence relation can be seen in [12]. We invoke the recurrence relation (4) in order to prove some unexpected congruences that combine the partition function $Q(n)$ and the divisor function $\tau(n)$ that counts the positive divisors of n.

Theorem 7. *Let $r \in \{0,2\}$. For any nonnegative integer n,*

$$Q(n) \equiv r \pmod 4 \quad \Longleftrightarrow \quad \tau(24n+1) \equiv r \pmod 4.$$

Proof. Let $R(n)$ be the number of representation of n as the sum of a generalized pentagonal number and thrice a square number. When n is not a generalized pentagonal number it is clear that $R(n)$ is even. In [6], Hirschhorn proved the following result:

$$R(n) = d_{1,8}(24n+1) + d_{3,8}(24n+1) - d_{5,8}(24n+1) - d_{7,8}(24n+1),$$

where $d_{\ell,m}(x)$ is the number of positive divisors d of x with $d \equiv \ell \pmod{m}$. Considering the recurrence relation (4), we deduce that

$$Q(n) \equiv 0 \pmod 4 \quad \Longleftrightarrow \quad \frac{R(n)}{2} \equiv 0 \pmod 2$$

and

$$Q(n) \equiv 2 \pmod 4 \quad \Longleftrightarrow \quad \frac{R(n)}{2} \equiv 1 \pmod 2.$$

If $\ell \in \{1,3\}$ and d is a divisor of $24n+1$ such that $d \equiv \ell$ (mod 8), then $(24n+1)/d \equiv \ell$ (mod 8). On the other hand, when n is not a generalized pentagonal number, the integer $24n+1$ is not a square. Thus we deduce that

$$\frac{R(n)}{2} + \frac{\tau(24n+1)}{2} = d_{1,8}(24n+1) + d_{3,8}(24n+1)$$

is an even number. This means that the integers $\frac{R(n)}{2}$ and $\frac{\tau(24n+1)}{2}$ have the same parity. □

Theorem 8. *Let n be a nonnegative integer.*

(i) If n is congruent to 0 or 3 mod 4, then

$$Q\left(\frac{3n^2+n}{2}\right) \equiv \pm 1 \pmod 4 \quad \Longleftrightarrow \quad \tau\big((6n+1)^2\big) \equiv \pm 1 \pmod 4.$$

(ii) If n is not congruent to 0 or 3 mod 4, then

$$Q\left(\frac{3n^2+n}{2}\right) \equiv \pm 1 \pmod 4 \quad \Longleftrightarrow \quad \tau\big((6n+1)^2\big) \equiv \mp 1 \pmod 4.$$

(iii) If n is congruent to 2 or 3 mod 4, then

$$Q\left(\frac{3n^2-n}{2}\right) \equiv \pm 1 \pmod 4 \quad \Longleftrightarrow \quad \tau\big((6n-1)^2\big) \equiv \pm 1 \pmod 4.$$

(iv) If n is not congruent to 2 or 3 mod 4, then

$$Q\left(\frac{3n^2-n}{2}\right) \equiv \pm 1 \pmod 4 \quad \Longleftrightarrow \quad \tau\big((6n-1)^2\big) \equiv \mp 1 \pmod 4.$$

Proof. The proof of this theorem is quite similar to the proof of Theorem 7. Considering our recurrence relations (4), we deduce that

$$Q\left(\frac{3n^2 \pm n}{2}\right) \equiv \begin{cases} 1 \pmod 4, & \iff \quad \dfrac{R\big((6n \pm 1)^2\big) - 1}{2} \text{ is even,} \\ \\ 3 \pmod 4, & \iff \quad \dfrac{R\big((6n \pm 1)^2\big) - 1}{2} \text{ is odd.} \end{cases}$$

Let $d_{\ell,m}(x)$ be the number of positive divisors d of x with $d \equiv \ell \pmod m$. On the one hand, it is not difficult to prove that

$$d_{1,8}\big((6n+1)^2\big) + d_{3,8}\big((6n+1)^2\big) - 1$$

is even if and only if n is congruent to 0 or 3 mod 4. On the other hand, it is not difficult to prove that

$$d_{1,8}\big((6n-1)^2\big) + d_{3,8}\big((6n-1)^2\big) - 1$$

is even if and only if n is congruent to 2 or 3 mod 4. This concludes the proof □

Proof of Theorems 3 and 4. According to Theorem 7, we have

$$Q\left(p \cdot n + \frac{p^2-1}{24}\right) \equiv 0 \pmod 4 \qquad \iff \qquad \tau(24 \cdot p \cdot n + p^2) \equiv 0 \pmod 4.$$

Taking into account that $n \not\equiv 0 \pmod p$, we obtain

$$\gcd(p, 24n + p) = 1.$$

Then we deduce that

$$\tau(24 \cdot p \cdot n + p^2) = \tau(p) \cdot \tau(24n + p) = 2 \cdot \tau(24n + p).$$

On the one hand, if p is a prime such that $p \not\equiv 1 \pmod{24}$ then it is not difficult to prove that $24n + p$ cannot be a square. On the other hand, it is well known that $24n+1$ is a square if and only if n is a generalized pentagonal number. When p is a prime such that $p \equiv 1 \pmod{24}$, we deduce that $24n+p$ is a square if and only if $n + \frac{p-1}{24}$ is a generalized pentagonal number. This concludes the proof.

4 Open problems and concluding remarks

Infinite families of Ramanujan-type congruences modulo 4 for the partition function $Q(n)$ have been obtained in this paper considering new connections between partitions and divisors.

Inspired by the case $p = 7$ of Theorem 2, we experimentally found the following congruence identities, where

$$\begin{aligned} S = \{&(11,5), (13,6), (17,8), (19,9), (23,11), (31,3), (37,6), \\ &(41,8), (43,9), (47,11), (59,6), (61,6), (67,10), (71,13), \\ &(79,3), (83,5), (89,9), (103,3), (107,6), (109,6), (113,9)\}. \end{aligned}$$

Conjecture. *Let $(p,k) \in S$. For all $n \not\equiv 0 \pmod{p}$, we have*

$$Q\left(p \cdot n + \frac{p^2-1}{24}\right) \equiv 0 \pmod{2^k}.$$

We were unable to prove these congruences due to the running time of the `RaduRK` program. Another approach to these congruences would be very interesting.

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