DOI: 10.2478/auom-2022-0020
An. Şt. Univ. Ovidius Constanţa **Vol. 30(2),2022, 91–101**

Motion of Inextensible Quaternionic Curves and Modified Korteweg-de Vries Equation

Kemal Eren

Abstract

Many curve evolutions have been determined which are integrable in recent times. The motion of curves can be defined by certain integrable equations including the modified Korteweg-de Vries. In this study, the quaternionic curves in 3 and 4-dimensional Euclidean spaces have been considered and the motions of inextensible quaternionic curves have been characterized by the modified Korteweg-de Vries (mKdV) equations. For this purpose, the basic concepts of the quaternions and quaternionic curves have been summarized. Then the evolutions of inextensible quaternionic curves with reference to the Frenet formulae have been obtained. Finally, the mKdV equations have been generated with the help of their evolutions

1 Introduction

The obscurities of many physical events encountered in nature have been disambiguated by mathematical models. Problems describing the physical and natural phenomena are usually expressed in nonlinear partial differential equations. The Korteweg-de Vries (KdV) equation is a fundamental differential equation for modeling and describing waves in nature. Debuting of the KdV equation was seen in the water waves observed by Scottish engineer J. Scott Russell in 1834. After Russell, many researchers such as Stokes (1847) and

Key Words: Modified Korteweg-de Vries equation, quaternionic curve, evolution curve, inextensible curve.
2010 Mathematics Subject Classification: Primary 53A04; Secondary 35Q53.
Received: 10.08.2021
Accepted: 10.11.2021

Boussinesq (1872) mentioned these waves. One of the most important properties of these waves is that they collide and keep their shape after a collision. Because of these properties, Kruskal and Zabusky (1965) called these waves solitons. Soliton concept of waves expresses that they move like particles. After Russell's observations, Dutch mathematician Diederik Johannes Korteweg and his student Gustav de Vries (1895) proposed a model to explain the observed event. According to this model, a wave with a higher height than the waves moving in shallow water moves faster, collides with the slight wave in front, and passes it. However, the exciting thing about these two waves is that each of them returns to the state before the collision [1, 2]. In addition, the KdV equation is a critical nonlinear partial differential equation that corresponds to physical systems such as shallow-water waves, large inner waves in densely layered oceans, ionic sound waves in plasma, sound waves in crystal lattices [3]. The standard KdV equation is a third-order nonlinear equation. Furthermore, it was extended to higher odd orders. The family of third-order KdV equation of the function $u\left(x,t\right)$ is given by

$$u_t + \varphi\left(u\right) u_x + u_{xxx} = 0,$$

where x is space and t is time variable, The coefficients of $\varphi\left(u\right) u_x$ and u_{xxx} can be constant numbers. By taking $\varphi\left(u\right) = \rho u^2$ in this last equation, the equation of modified Korteweg-de Vries (mKdV) is defined as

$$u_t + \rho u^2 u_x + u_{xxx} = 0, \tag{1}$$

where ρ is nonzero constant. The mKdV equation has applications in some areas such as electrodynamics, traffic flow, electro-magnetic waves in size-quantized films, elastic media, electric circuits, and multi-component plasmas [4, 5]. The mKdV equation has been considered from different geometric and algebraic perspectives in [6, 7, 8, 9, 10].

On the other hand, the quaternions were found in 1843 during the work of Irish mathematician William Rowan Hamilton to generalize complex numbers to 3-dimensional space. While finding quaternions, Hamilton first studied complex numbers and then concluded that complex numbers consist of two real numbers as algebra. Based on this result, Hamilton concentrated his studies on the triple numbers as $(q = q_0 + q_1\mathbf{e_1} + q_2\mathbf{e_2})$, including two complex numbers and one real number component. Although he defined addition and multiplication on this system, he could not develop a method for division. In the meantime, he realized that in the number system of these numbers, the commutative property of multiplication does not occur. By giving up this property of multiplication, he defined three imaginary units that are satisfying $\mathbf{e_1}^2 = \mathbf{e_2}^2 = \mathbf{e_3}^2 = -1$. Thus, Hamilton discovered 4-dimensional division algebra, and its elements are called quaternions [11]. The quaternion theory has renewed itself over time and diversified. The base elements of the quaternions discovered by Hamilton are imaginary, and their components are real numbers, so they are called real quaternions. The real quaternions are the simplest in terms of structure and lead to defining new types of quaternions [12, 13]. The Serret-Frenet formulas of a curve in 3-dimensional Euclidean space R^3 have been re-derived by Bharathi and Nagaraj for quaternionic curves [14]. In recent years, the application areas of quaternions have spread to almost all branches of science. It is seen that the quaternions are being used in a wide range, including the investigation of molecular structures, DNA and protein structures, the definition of eye movements, dynamics, astronomy, and optics, etc. [15, 16, 17]. The carried-out studies are the source of our inspiration to obtain the mKdv equation for the moving quaternionic curves.

2 Preliminaries

Below, we give the basic notions of the theory of quaternions. More comprehensive information on quaternions and quaternionic curves are available in [11] and [14].
A quaternion q is defined as the sum of a scalar q_0 and a vector $\mathbf{q} = (q_1,\ q_2,\ q_3)$ such as

$$q = q_0 + \mathbf{q} = q_0 + q_1\mathbf{e_1} + q_2\mathbf{e_2} + q_3\mathbf{e_3}$$

where $\{\mathbf{e_i} |\, 1 \leq i \leq 3\}$ is the standard orthonormal basis for R^3, and it is used to represent the orientation of a rigid body or coordinate frame in R^3. For a real quaternion q, the components q_0, q_1, q_2 and q_3 are real numbers and $\mathbf{e_i}$, $(1 \leq i \leq 3)$ are quaternionic units that satisfy the non-commutative multiplication rules $\mathbf{e_i} \times \mathbf{e_j} = \mathbf{e_k} = -\mathbf{e_j} \times \mathbf{e_i}$ and $\mathbf{e_i} \times \mathbf{e_i} = -1$ for all $1 \leq i, j, k \leq 3$ where $\times$ denotes cross product of vectors in R^3. The complex conjugate of a quaternion q, denoted $\bar{q}$, is defined by

$$\bar{q} = q_0 - \mathbf{q} \text{ or } \bar{q} = q_0 - q_1\mathbf{e_1} - q_2\mathbf{e_2} - q_3\mathbf{e_3}.$$

If we get two quaternions $p = p_0 + \mathbf{p}$ and $q = q_0 + \mathbf{q}$, then their quaternionic product is defined as follows;

$$p \times q = p_0 q_0 - \langle \mathbf{p}, \mathbf{q} \rangle + p_0\mathbf{q} + q_0\mathbf{p} + \mathbf{p} \times \mathbf{q},$$

where $\langle , \rangle$ denotes the inner product in R^3. Let Q denotes the quaternion set. Now, we can define the quaternion inner product by the following form

$$\begin{array}{l} f : Q \times Q \to R \\ (p, q) \to f(p, q) = \frac{1}{2}(p \times \bar{q} + q \times \bar{p}) \end{array}$$

which is a real-valued, symmetric, and bilinear form. In addition, the norm of a quaternion q is given by

$$\|q\|^2 = f(q,q) = q \times \bar{q} = \bar{q} \times q = {q_0}^2 + {q_1}^2 + {q_2}^2 + {q_3}^2.$$

Also, if $\|q\| = 1$, then the quaternion q is called unit quaternion. The inverse of a quaternion q with a nonzero norm is expressed by

$$q^{-1} = \tfrac{\bar{q}}{\|q\|^2}.$$

The three-dimensional Euclidean space R^3 is identified by the space of the spatial quaternions $\{\, q \in Q |\, q + \overline{q} = 0\}$ where Q denotes quaternion set [14].

Definition 1. A spatial quaternionic curve α is defined by

$$\begin{aligned} &\alpha : I \to Q, \\ &\quad s \;\; \to \alpha(s) = \sum_{i=1}^{3} \alpha_i(s)\, \mathbf{e_i} \end{aligned}$$

where $I = [0,1]$ is an interval in real line R and $s \in [0,1]$ is the arc-length parameter [14].

Let α be a spatial quaternionic curve with the arc-length parameter $s \in I = [0,1]$. Then the FrenetSerret vectors (also commonly referred to as the Frenet vectors) of the curve α at a point $\alpha(s)$ are

$$\mathbf{t}(s) = \alpha'(s), \mathbf{n}(s) = \tfrac{\alpha''(s)}{\|\alpha''(s)\|}, \mathbf{b}(s) = \mathbf{t}(s) \times \mathbf{n}(s).$$

Here the vectors $\mathbf{t}$, $\mathbf{n}$, and $\mathbf{b}$ are called unit tangent, unit principal normal, and unit binormal vectors of the quaternionic curve α, respectively [14]. This identification produces the Frenet formulas given in the following theorem:

Theorem 1. *Let $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ be Frenet frame and $\{k, r\}$ be curvatures of a quaternionic curve α in R^3 with arc-length parameter $s \in I = [0,1]$. Then, the relationship between the Frenet vectors and the curvatures are given as*

$$\begin{aligned} \mathbf{t}_s &= k\mathbf{n}, \\ \mathbf{n}_s &= -k\mathbf{t} + r\mathbf{b}, \\ \mathbf{b}_s &= -r\mathbf{n}, \end{aligned} \tag{2}$$

where k is the principal curvature and r is the torsion of α [14].

On the other hand, four-dimensional Euclidean space R^4 is identified by the space of the quaternions.

Definition 2. A quaternionic curve β is defined by

$$\beta : I \to Q,$$
$$s \to \beta(s) = \sum_{i=0}^{3} \beta_i(s)\, \mathbf{e_i}$$

where $I = [0,1]$ is an interval in the real line R. Let $s \in [0,1]$ be the arc-length parameter such that the tangent $\mathbf{T} = \beta'(s)$ has unit magnitude [14].

The Frenet formulas of the quaternionic curve β are given in the following theorem:

Theorem 2. *Let β be a quaternionic curve with the arc-length parameter $s \in I = [0,1]$ in R^4. If $\{\mathbf{T},\mathbf{N},\mathbf{B},\mathbf{E}\}$ denotes the Frenet frame of β, then the relationship between the Frenet vectors and the curvatures of β is given by*

$$\begin{aligned} \mathbf{T}_s &= \kappa \mathbf{N}, \\ \mathbf{N}_s &= -\kappa \mathbf{T} + k\mathbf{B}, \\ \mathbf{B}_s &= -k\mathbf{N} + (r-\kappa)\,\mathbf{E}, \\ \mathbf{E}_s &= -(r-\kappa)\,\mathbf{B}, \end{aligned} \tag{3}$$

where $\kappa = \|\mathbf{T}'\|$, $\mathbf{N} = \mathbf{t}\times\mathbf{T}, \mathbf{B} = \mathbf{n}\times\mathbf{T}, \mathbf{E} = \mathbf{b}\times\mathbf{T}$. Here κ, k, and $(r-\kappa)$ are called the principal curvature, torsion, and bitorsion of β, respectively [14].

3 The mKdV equation by the motion of inextensible quaternionic curves

In this section, the mKdV equation is obtained by the motions of inextensible quaternionic curves in both 3 and 4-dimensional Euclidean spaces.

3.1 Spatial inextensible quaternionic curves in R^3

Let $\alpha = \alpha(s)$ be a spatial quaternionic curve in R^3 and $\alpha(s,t)$ denotes the position vector of the curve $\alpha(s)$ at time t. The time evolution of the Frenet frame $\{\mathbf{t},\mathbf{n},\mathbf{b}\}$ of α is written in the following form

$$\begin{aligned} \mathbf{t}_t &= \delta_1 \mathbf{n} + \delta_2 \mathbf{b}, \\ \mathbf{n}_t &= -\delta_1 \mathbf{t} + \gamma_1 \mathbf{b}, \\ \mathbf{b}_t &= -\delta_2 \mathbf{t} - \gamma_1 \mathbf{n}, \end{aligned} \tag{4}$$

where δ_1, δ_2 and γ_1 are smooth functions of s and t.

The symmetry of the second-order derivatives with respect to the arc-length parameter s and time t implies that the quaternionic curve α has to be inextensible [9]. Accordingly, using the conditions $\mathbf{t}_{ts} = \mathbf{t}_{st}$, $\mathbf{n}_{ts} = \mathbf{n}_{st}$, and

$\mathbf{b}_{ts} = \mathbf{b}_{st}$ for the spatial inextensible quaternionic curve, one easily obtains from the equations (2) and (4) the following relations;

$$\delta_{1s} = k_t + r\delta_2, \tag{5}$$

$$\delta_{2s} = k\gamma_1 - r\delta_1, \tag{6}$$

$$\gamma_{1s} = r_t - k\delta_2. \tag{7}$$

Thus, we can give the following theorem.

Theorem 3. *Let $\alpha(s,t)$ be a position vector of a moving spatial inextensible quaternionic curve α in R^3, then the time evolution of the Frenet frame $\{\mathbf{t},\mathbf{n},\mathbf{b}\}$ is*

$$\begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix}_t = \begin{bmatrix} 0 & \delta_1 & \delta_2 \\ -\delta_1 & 0 & \gamma_1 \\ -\delta_2 & -\gamma_1 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix},$$

such that

$$\delta_1 = f_1 k + f_{2s} - f_3 r,\ \delta_2 = f_2 r + f_{3s},$$
$$\gamma_1 = \frac{1}{k}\left((f_2 r + f_{3s}) + r\left(f_1 k + f_{2s} - f_3 r\right)\right)$$

where f_1, f_2 and f_3 are the components of velocity vector of α and k, r are the principal curvature and torsion of α, respectively.

Proof. Let α be a moving spatial inextensible quaternionic curve and f_1, f_2 and f_3 be the components of velocity vector α. If we denote the velocity vector as $\mathbf{v} = \alpha_t$, then we can write

$$\mathbf{v} = f_1\mathbf{t} + f_2\mathbf{n} + f_3\mathbf{b}, \tag{8}$$

If we take the derivative of $\mathbf{v} = \alpha_t$ and $\alpha_s = \mathbf{t}$ with respect to s and t, we have

$$\mathbf{v}_s = \alpha_{ts} = \left(f_{1s} - f_2 k\right)\mathbf{t} + \left(f_1 k + f_{2s} - f_3 r\right)\mathbf{n} + \left(f_2 r + f_{3s}\right)\mathbf{b}$$

and

$$\alpha_{st} = \mathbf{t}_t = \delta_1\mathbf{n} + \delta_2\mathbf{b}.$$

Then by using the compatibility conditions $\alpha_{ts} = \alpha_{st}$ of inextensibility, we find

$$\left(f_{1s} - f_2 k\right)\mathbf{t} + \left(f_1 k + f_{2s} - f_3 r\right)\mathbf{n} + \left(f_2 r + f_{3s}\right)\mathbf{b} = \delta_1\mathbf{n} + \delta_2\mathbf{b}.$$

From here, the following statements are satisfied;

$$f_{1s} - f_2 k = 0, \tag{9}$$

$$f_1 k + f_{2_s} - f_3 r = \delta_1, \tag{10}$$

$$f_2 r + f_{3_s} = \delta_2. \tag{11}$$

By substituting the equations (10) and (11) into (6), we obtain the smooth function γ_1 as follows

$$\gamma_1 = \frac{1}{k}\left(\left(f_2 r + f_{3_s}\right) + r\left(f_1 k + f_{2_s} - f_3 r\right)\right). \tag{12}$$

This completes the proof. □

Also, we can determine the time evolution of the curvature k and the torsion r of the moving spatial inextensible quaternionic curve α with respect to the components of velocity f_1, f_2 and f_3. By substituting the formulas (10) and (11) into (5) and (7), the following corollary is obvious.

Corollary 1. *The time evolutions of the curvature k and the torsion r of the moving spatial inextensible quaternionic curve α are given by*

$$k_t = \left(f_1 k - f_{2_s} - f_3 r\right)_s - r\left(f_2 r + f_{3_s}\right) \tag{13}$$

and

$$r_t = {\gamma_1}_s + k\left(f_2 r + f_{3_s}\right). \tag{14}$$

Theorem 4. *Let $\alpha(s,t)$ be a position vector of the spatial inextensible quaternionic curve α. If the time evolution of α is given by*

$$\begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix}_t = \begin{bmatrix} 0 & -\frac{k^3}{2} - k_{ss} & f_{3_s} \\ \frac{k^3}{2} + k_{ss} & 0 & \frac{f_{3_{ss}}}{k} \\ -f_{3_s} & -\frac{f_{3_{ss}}}{k} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix},$$

then α satisfies the mKdV equation $k_t + k_{sss} + \frac{3}{2}k^2 k_s = 0$.

Proof. If we consider $\delta_1 = -\frac{k^3}{2} - k_{ss}$, $\delta_2 = f_{3_s}$, $\gamma_1 = \frac{f_{3_{ss}}}{k}$, then $f_2 = -k_s$ and the torsion of the moving inextensible spatial inextensible quaternionic curve α vanishes, i.e., $r = 0$. Then from the equation (9), we find that $f_1 = -\frac{k^2}{2} + c_1(t)$ where $c_1(t)$ is arbitrary. Since $r = 0$, the quaternionic inextensible curve α is a planar quaternionic curve for all time t. Here, if $c_1(t) = 0$, then from the last equation, it is $f_1 = -\frac{k^2}{2}$.

By considering (13) and (14) with $f_1 = -\frac{k^2}{2}$ and $f_2 = -k_s$, the mKdV equation related to arc-lengthed spatial inextensible quaternionic curve $\alpha(s) = \sum_{i=1}^{3} \alpha_i(s)\,\mathbf{e_i}$ in R^3 is obtained as $k_t + k_{sss} + \frac{3}{2}k^2 k_s = 0$. Also, $\left(\frac{f_{3_{ss}}}{k}\right)_s = -k f_{3_s}$ is satisfied for the binormal component of the velocity of the moving inextensible quaternionic curve. □

3.2 Inextensible quaternionic curves in R^4

Let $\beta = \beta(s)$ be a quaternionic curve in R^4 and $\beta(s,t)$ be the position vector of the curve $\beta(s)$ at time t.

The time evolution of the Frenet frame $\{\mathbf{T}, \mathbf{N}, \mathbf{B}, \mathbf{E}\}$ can be written in the following form

$$\begin{aligned} \mathbf{T}_t &= \lambda_1 \mathbf{N} + \lambda_2 \mathbf{B} + \lambda_3 \mathbf{E}, \\ \mathbf{N}_t &= -\lambda_1 \mathbf{T} + \mu_1 \mathbf{B} + \mu_2 \mathbf{E}, \\ \mathbf{B}_t &= -\lambda_2 \mathbf{T} - \mu_1 \mathbf{N} + \eta_1 \mathbf{E}, \\ \mathbf{E}_t &= -\lambda_3 \mathbf{T} - \mu_2 \mathbf{N} - \eta_1 \mathbf{B}, \end{aligned} \tag{15}$$

where λ_i, μ_i and η_i $(1 \leq i \leq 3)$ are smooth functions of s and t.

If β is an inextensible quaternionic curve in R^4, then there are the conditions $\mathbf{T}_{ts} = \mathbf{T}_{st}$, $\mathbf{N}_{ts} = \mathbf{N}_{st}$, $\mathbf{B}_{ts} = \mathbf{B}_{st}$, and $\mathbf{E}_{ts} = \mathbf{E}_{st}$. From the equations (3) and (15), we get

$$\begin{aligned} \lambda_{1s} &= \kappa_t + k\lambda_2, \\ \lambda_{2s} &= \kappa\mu_1 - k\lambda_1 + (r-\kappa)\lambda_3, \\ \lambda_{3s} &= \kappa\mu_2 - (r-\kappa)\lambda_2, \\ \mu_{1s} &= k_t - \kappa\lambda_2 + (r-\kappa)\mu_2, \\ \mu_{2s} &= k\eta_1 - \kappa\lambda_3 - (r-\kappa)\mu_1, \\ \eta_{1s} &= (r-\kappa)_t - k\mu_2. \end{aligned} \tag{16}$$

Theorem 5. *Let $\beta(s,t)$ be the position vector of an inextensible quaternionic curve β in R^4. If the time evolution of the Frenet frame $\{\mathbf{T}, \mathbf{N}, \mathbf{B}, \mathbf{E}\}$ of β satisfies*

$$\begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \\ \mathbf{E} \end{bmatrix}_t = \begin{bmatrix} 0 & \lambda_1 & \lambda_2 & \lambda_3 \\ -\lambda_1 & 0 & \mu_1 & \mu_2 \\ -\lambda_2 & -\mu_1 & 0 & \eta_1 \\ -\lambda_3 & -\mu_2 & -\eta_1 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \\ \mathbf{E} \end{bmatrix},$$

where

$$\begin{aligned} \lambda_1 &= -\tfrac{\kappa^3}{2} - \kappa_{ss}, \\ \lambda_2 &= g_{3s} - (r-\kappa)\, g_4, \\ \lambda_3 &= g_{4s} + (r-\kappa)\, g_3, \\ \mu_1 &= \tfrac{1}{\kappa}\left((r-\kappa)\left((r-\kappa)\, g_3 + g_{4s}\right) - \left(g_{3s} - (r-\kappa)\, g_4\right)_s\right), \\ \mu_2 &= \tfrac{1}{\kappa}\left(\left((r-\kappa)\, g_3 + g_{4s}\right)_s + (r-\kappa)\left(g_{3s} - (r-\kappa)\, g_4\right)\right), \\ \eta_1 &= \int (r-\kappa)_t ds, \end{aligned}$$

and κ, $(r-\kappa)$ are the curvatures, and g_i $(3 \leq i \leq 4)$ are the components of the velocity vector of β, respectively, then the curvature of β provides the mKdV equation

$$\kappa_t + \kappa_{sss} + \frac{3}{2}\kappa^2 \kappa_s = 0.$$

Proof. Let β be a moving inextensible quaternionic curve in R^4. If we denote the velocity vector of β by $\mathbf{u} = \beta_t$, then it can be written as

$$\mathbf{u} = g_1\mathbf{T} + g_2\mathbf{N} + g_3\mathbf{B} + g_4\mathbf{E}, \tag{17}$$

where g_i $(1 \leq i \leq 4)$ are the components of the velocity vectors of the moving quaternionic curve β.

If we take the derivatives of $\mathbf{u} = \beta_t$ and $\beta_s = \mathbf{T}$ with respect to s and t, respectively, we have

$$\mathbf{u}_s = \beta_{ts} = (g_{1s} - \kappa g_2)\,\mathbf{T} + (\kappa g_1 + g_{2s} - kg_3)\,\mathbf{N} + (kg_2 + g_{3s} - (r-\kappa)\,g_4)\,\mathbf{B} + ((r-\kappa)\,g_3 + g_{4s})\,\mathbf{E}$$

and

$$\beta_{st} = \lambda_1\mathbf{N} + \lambda_2\mathbf{B} + \lambda_3\mathbf{E}.$$

By using the compatibility conditions $\beta_{ts} = \beta_{st}$, we find that the following statements are satisfied;

$$\begin{aligned} &g_{1s} - \kappa g_2 = 0, \\ &\kappa g_1 + g_{2s} - kg_3 = \lambda_1, \\ &kg_2 + g_{3s} - (r-\kappa)\,g_4 = \lambda_2, \\ &(r-\kappa)\,g_3 + g_{4s} = \lambda_3. \end{aligned} \tag{18}$$

Substituting the formulas (18) into the equations (16), the time evolution of the curvatures κ, k and $(r-\kappa)$ of the moving quaternionic curve β are found as

$$\begin{aligned} &\kappa_t = (\kappa g_1 + g_{2s} - kg_3)_s - k\,(kg_2 + g_{3s} - (r-\kappa)\,g_4)\,, \\ &k_t = {\mu_1}_s + \kappa\,(kg_2 + g_{3s} - (r-\kappa)\,g_4) - (r-\kappa)\,\mu_2, \\ &(r-\kappa)_t = {\eta_1}_s + k\mu_2. \end{aligned} \tag{19}$$

On the other hand, considering the equations (16) and (18) together, we can see that the smooth functions μ_1, μ_2 and η_1 of β satisfy the equalities

$$\begin{aligned} &\mu_1 = \tfrac{1}{\kappa}\begin{pmatrix} -k\,(\kappa g_1 + g_{2s} - kg_3) + (r-\kappa)\,((r-\kappa)\,g_3 + g_{4s}) \\ -(kg_2 + g_{3s} - (r-\kappa)\,g_4)_s \end{pmatrix}, \\ &\mu_2 = \tfrac{1}{\kappa}\,(((r-\kappa)\,g_3 + g_{4s})_s + (r-\kappa)\,kg_2 + g_{3s} - (r-\kappa)\,g_4)\,, \\ &\eta_1 = \int (r-\kappa)_t - k\left(\tfrac{1}{\kappa}\begin{pmatrix} ((r-\kappa)\,g_3 + g_{4s})_s + (r-\kappa)\,kg_2 \\ +g_{3s} - (r-\kappa)\,g_4 \end{pmatrix}\right)ds. \end{aligned} \tag{20}$$

So by the hypothesis $g_2 = -\kappa_s$ and the curvature k of the moving inextensible quaternionic curve β vanishes, i.e., $k = 0$. Then from equation (18), we find

$$g_1 = -\tfrac{\kappa^2}{2} + c_2\,(t)\,,$$

where $c_2(t)$ is arbitrary. Here, if $c_2(t) = 0$, then from the last equation, we get $g_1 = -\frac{\kappa^2}{2}$. For $g_1 = -\frac{\kappa^2}{2}$, $g_2 = -\kappa_s$ and $k = 0$. Also, substituting the equations $g_1 = -\frac{\kappa^2}{2}$ and $g_2 = -\kappa_s$, $k = 0$ into equation (19), we obtain the mKdV equation related to arc-lengthed quaternionic curve $\beta(s) = \sum_{i=0}^{3} \beta_i(s)\,\mathbf{e_i}$ in R^4 with curvatures $\{\kappa, k, (r-\kappa)\}$ is found as $\kappa_t + \kappa_{sss} + \frac{3}{2}\kappa^2\kappa_s = 0$. □

4 Conclusions

A curve evolution is called integrable if the motion is defined by an integrable partial differential equation. The integrable evolutions of curves have been studied widely in recent times. The motion of curves can be defined by integrable equations including mKdV equations. The mKdV equations have been studied in detail by algebraic and geometric approaches, but the motions of inextensible quaternionic curves have not been studied in terms of mKdV equations. In this regard, we obtained the mKdV equation under certain conditions by using the evolution of the curvatures of the inextensible quaternionic curves in 3 and 4-dimensional Euclidean spaces. For this purpose, we expressed the evolution equations related to the Frenet frames and curvatures of the inextensible quaternionic curves. The obtained relations may be used in the study of physical phenomena including the motions of particles under certain conditions.

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Kemal EREN,
Department of Mathematics,
Sakarya University,
54187 Sakarya, Turkey.
Email: kemal.eren1@ogr.sakarya.edu.tr