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Periodic and Solitary Wave Solutions for the One-Dimensional Cubic Nonlinear Schrödinger Model

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Abstract

Using a similar approach as Korteweg and de Vries, [19], we obtain periodic solutions expressed in terms of the Jacobi elliptic function cn, [3], for the self-focusing and defocusing one-dimensional cubic nonlinear Schrödinger equations. We will show that solitary wave solutions are recovered through a limiting process after the elliptic modulus of the Jacobi elliptic function cn that describes the periodic solutions for the self-focusing nonlinear Schrödinger model.

1 Introduction

The very well-known Korteweg-de Vries equation, [19], possesses periodic solutions expressed in terms of the Jacobi elliptic function cn, [3]. Korteweg and de Vries called these profiles *cnoidal waves*. They showed that in the limiting case when the elliptic modulus approaches 1, Russell's solitary wave (known as soliton since 1972, [31]), [24], is recovered. Drazin, [10], gives an excellent account for the Korteweg and de Vries' cnoidal waves and how Russell's solitary wave is recovered from them.

Key Words: NLS, self-focusing, defocusing, dispersive, nonlinearity, carrier waves, solution profile, envelope, cnoidal waves, solitary waves, surface gravity waves, sound waves, water-air interface, sonic layer depth.
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This paper will follow a similar type of thinking as Korteweg and de Vries in 1895. We will obtain periodic solutions expressed in terms of the Jacobi elliptic function cn for the nonlinear Schrödinger model (NLS) below,

$$iu_t \pm \frac{\alpha}{2} u_{xx} - \alpha u|u|^2 = 0,\ \alpha > 0 \text{ constant.} \tag{1}$$

The NLS model (1) we work within the present paper is suggested by the works in [21], [23], and [32]. For the self-focusing NLS, we will derive solitary wave solutions from the periodic solutions by considering a limiting process onto the elliptic modulus of the Jacobi elliptic function *cn*.

It is important to note that the function u in the model (1) is complex-valued. As explained in [11], the nonlinear model (1) is far from describing the quantum state of a particle, as its linear counterpart does, i.e., Schrödinger's wave-equation for describing dispersive wave phenomena suitable for micro-mechanical problems, [25]. Its typical physical applications are primarily in nonlinear optics, [4], [5], [6], but the model can be used widely for other phenomena like surface gravity waves in deepwater, as an example. An example that refers to the dispersive hydrodynamics concerning this model is [8].

The results obtained in this paper align with the continuous endeavour of studying the periodic wave nature and solitary wave nature described by nonlinear Schrödinger models, two relevant examples being [27] and [30].

2 Periodic Solutions for Self-Focusing NLS

While the one-dimensional Schrödinger equation, [25], is universally known to describe the wave function for a free particle, the NLS models are far from representing the quantum state of a particle. It is already common knowledge that the NLS models describe waves in nonlinear optics and deepwater. Thus, the NLS models are subject to Galilean invariance, i.e., the laws of motion of an object represent the same motion of the object in all inertial reference frames. In this section, we are interested in obtaining periodic solutions for the self-focusing NLS model (2) as follows.

We consider the self-focusing NLS

$$iu_t - \frac{\alpha}{2} u_{xx} - \alpha u|u|^2 = 0,\ \alpha > 0 \text{ constant.} \tag{2}$$

Using separation of variables in (2)

$$u(x,t) = r(x)T(t), \tag{3}$$

such that $r(x)$ is a real-valued function and $T(t)$ is a complex-valued function such that $|T| = 1$, we obtain

$$\frac{iT'}{T} = \frac{\frac{\alpha}{2}r'' + \alpha r^3}{r} = \mu,\ \mu \text{ constant}, \tag{4}$$

and we will be interested to analyze the case when $\mu > 0$.

From (4) we obtain

$$T(t) = e^{-i\mu t}, \tag{5}$$

and the following second order nonlinear ordinary differential equation for $r = r(x)$

$$r'' + \frac{2}{\alpha}r(\alpha r^2 - \mu) = 0. \tag{6}$$

Multiplying (6) by r' and simplifying, we obtain

$$\frac{d}{dx}\left(\frac{(r')^2}{2} - \frac{\mu}{\alpha}r^2 + \frac{1}{2}r^4\right) = 0. \tag{7}$$

Integrating the equation (7), we readily obtain

$$(r')^2 - \frac{2\mu}{\alpha}r^2 + r^4 = K,\ K \text{ integration constant}. \tag{8}$$

Solving the equation (8) for r' , we obtain

$$r' = \pm\sqrt{P(r)},\ P(r) = -r^4 + \frac{2\mu}{\alpha}r^2 + K. \tag{9}$$

We are interested in the case when the polynomial P factors as follows

$$P(r) = (r^2 + r_1)(r_2 - r^2),\ 0 \le r_1 < r_2,\ r_2 - r_1 = \frac{2\mu}{\alpha},\ K = r_1 r_2. \tag{10}$$

From (9) and (10), we obtain

$$\frac{dr}{\sqrt{(r^2 + r_1)(r_2 - r^2)}} = \pm dx. \tag{11}$$

Integrating (11), we obtain

$$\int_{\sqrt{r_2}}^{r} \frac{dw}{\sqrt{(w^2 + r_1)(r_2 - w^2)}} = \pm(x + C),\ C \text{ integration constant}. \tag{12}$$

Making the substitution

$$w = \sqrt{r_2}\cos\theta, \tag{13}$$

and performing all the calculations, the equation (12) becomes

$$\mp\sqrt{r_1+r_2}(x+C) = \int_0^{\phi} \frac{1}{\sqrt{1-m\sin^2\theta}}d\theta,\ m = \frac{r_2}{r_1+r_2}$$

$$\Downarrow$$

$$\operatorname{cn}\left(\sqrt{r_1+r_2}(x+C)|m\right) = \cos\phi, \tag{14}$$

where cn is the Jacobi elliptic function defined as follows, [3], [10],

$$\operatorname{cn}(\tau|m) = \cos\phi,\ \tau = \int_0^{\phi} \frac{d\theta}{\sqrt{1-m\sin^2\theta}},\ m \in [0,\ 1]. \tag{15}$$

As well, we used the fact that the cn function is an even function. Then, referring back to the substitution (13), we finally obtain

$$r = \sqrt{r_2}\cos\phi = \sqrt{r_2}\operatorname{cn}\left(\sqrt{r_1+r_2}(x+C)|m\right). \tag{16}$$

Thus, the solution we were looking for the equation (6) is

$$\begin{aligned} r(x) = \sqrt{r_2}\operatorname{cn}\left(\sqrt{r_1+r_2}(x+C)|m\right),\ m = \frac{r_2}{r_1+r_2}, \\ 0 \le r_1 < r_2,\ r_2 - r_1 = \frac{2\mu}{\alpha},\ C \in \mathbb{R}. \end{aligned} \tag{17}$$

From (3) and (17), the NLS (2) has the following solution

$$\begin{aligned} u(x,t) = r(x)e^{-i\mu t}, \\ r(x) = \sqrt{r_2}\operatorname{cn}\left(\sqrt{r_1+r_2}(x+C)|m\right). \end{aligned} \tag{18}$$

Applying scale symmetry, [22], onto the solution (18), we obtain the following scaled solution for the NLS (2)

$$u(x,t) \longmapsto u(x,t|\delta) = \delta u(\delta x, \delta^2 t), \delta \neq 0. \tag{19}$$

The NLS (2) is Galilean invariant as follows: If $u(x,t)$ is a solution of the NLS (2) then we can obtain a new solution by changing the inertial reference frame, and adding a phase factor as follows

$$u(x,t) \longmapsto u(x,t|v) = u(x-vt,t)e^{-\frac{i}{2\mu}\lambda^2 v\left(x+C-\frac{vt}{2}\right)}, \lambda = \sqrt{\frac{2\mu}{\alpha}},\ v \in \mathbb{R}. \tag{20}$$

Applying the Galilean invariance (20) onto the scaled solution (19), we obtain the following complex-valued solution of the NLS (2)

$$\begin{aligned} u(x,t) &= \delta r(\delta(x-vt))e^{-\frac{i}{4\mu}\left(2\lambda^2 v(x+C)-(\lambda^2 v^2-4\delta^2\mu^2)t\right)}, \\ &\alpha>0,\ \mu>0,\ \lambda=\sqrt{\frac{2\mu}{\alpha}},\ \delta\neq 0,\ v\in\mathbb{R}, \\ r(x) &= \sqrt{r_2}\,\mathrm{cn}\left(\sqrt{r_1+r_2}(x+C)|m\right),\ m=\frac{r_2}{r_1+r_2}, \\ &0\le r_1<r_2,\ r_2-r_1=\frac{2\mu}{\alpha},\ C\in\mathbb{R}. \end{aligned} \tag{21}$$

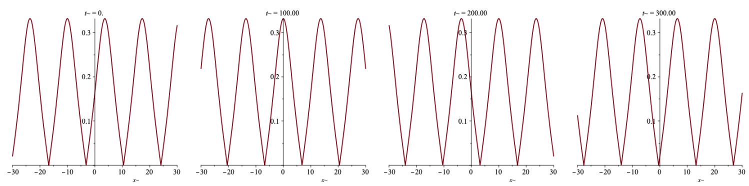

Figure 1: Cnoidal wave of the NLS (2) for $\alpha=0.2$, $\mu=1$, $\delta=0.1$, $v=0.1$, $r_1=1$, and $C=1$.

Figure 1 illustrates the time evolution of a periodic solution for the NLS (2) traveling from left to right with the velocity $v=0.1$. Physically speaking, the cnoidal wave depicted in Figure 1 is nothing else but the envelope of the modulated carrier waves (i.e., the modulated oscillatory components) of the solution (21). The envelope of the modulated carrier waves propagates with the group velocity $v_g = v$, and it is given by the graphs of the functions $\pm A=\pm|u|=\pm\sqrt{u\bar{u}}$. We will call the upper part of the envelope *the profile of a cnoidal wave*, given by the formula below,

$$A(x,t)=\sqrt{\frac{2\mu}{\alpha}+r_1}\left|\delta\,\mathrm{cn}\left(\sqrt{\frac{2\mu}{\alpha}+2r_1}\,(\delta(x-vt)+C)|\,\frac{2\mu+\alpha r_1}{2\mu+2\alpha r_1}\right)\right|. \tag{22}$$

Figure 2 illustrates the time evolution of the modulated carrier waves and their envelope for the NLS (2), traveling from left to right with the group velocity $v_g=0.1$.

When a dispersive harmonic wave of the NLS model (2) is subject to the cubic nonlinearity $u|u|^2$, the wave will be subject to a "force" that will act against the dispersion process. In other words, the nonlinearity will cancel out the dispersive effect so that the wave will steepen its wavefront. When the wave reaches a "perfect" balance between dispersion and nonlinearity, its oscillatory

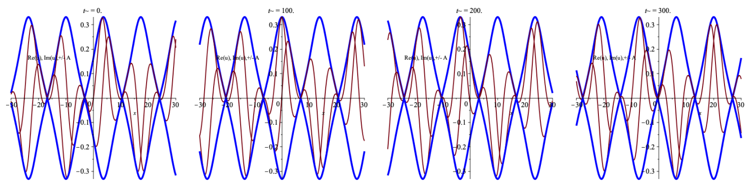

Figure 2: Modulated carrier waves and their envelope for the NLS (2) with $\alpha = 0.2$, $\mu = 1$, $\delta = 0.1$, $v = 0.1$, $r_1 = 1$, and $C = 1$.

components will become modulated waves with a localized shaped envelope that decays at infinity. In other words, they will become wave packets. The envelope of these modulated carrier waves is known as the profile of a solitary wave, or a soliton. In the next section, we will obtain solitary wave solutions for the self-focusing NLS model (2) through a limiting process for the elliptic modulus, m, in (21).

3 Solitary Wave Solutions for Self-Focusing NLS

Taking the limiting process $r_1 \longrightarrow 0$ in (21), the elliptic modulus m will approach 1, and the solution (21) will have the profile of a solitary wave given by the formula below,

$$\begin{aligned} u(x,t) &= \delta r(\delta(x-vt)) e^{-\frac{i}{4\mu}\left(2\lambda^2 v(x+C) - (\lambda^2 v^2 - 4\delta^2\mu^2)t\right)}, \\ r(x) &= \lambda \operatorname{sech}\left(\lambda(x+C)\right), \ \lambda = \sqrt{\frac{2\mu}{\alpha}}, \ \delta \neq 0, \ v \in \mathbb{R}, \ C \in \mathbb{R}. \end{aligned} \tag{23}$$

The solitary wave described by formula (23), i.e., the profile of the complex-valued function u, $A = |u| = \sqrt{u\bar{u}}$, satisfies the expected boundary conditions mentioned below,

$$\begin{aligned} A_x(-C/\delta + vt, t) &= 0, \\ A(x,t) \to 0 \text{ as } |x| &\to \infty. \end{aligned} \tag{24}$$

The dispersion and the nonlinearity of the solitary wave described by the formula (23) are in "perfect" balance, and its oscillatory components are modulated carrier waves given by

$$\begin{aligned}
\operatorname{Re}(u(x,t)) &= \lambda\delta \frac{\cos\left(\frac{1}{2\mu}\lambda^2 v(x+C) - \left(\frac{1}{4\mu}\lambda^2 v^2 - \delta^2\mu\right)t\right)}{\cosh\left(\lambda(\delta(x-vt)+C)\right)}, \\
\operatorname{Im}(u(x,t)) &= -\lambda\delta \frac{\sin\left(\frac{1}{2\mu}\lambda^2 v(x+C) - \left(\frac{1}{4\mu}\lambda^2 v^2 - \delta^2\mu\right)t\right)}{\cosh\left(\lambda(\delta(x-vt)+C)\right)}, \qquad (25) \\
\lambda &= \sqrt{\frac{2\mu}{\alpha}},\ \delta \neq 0,\ v \in \mathbb{R},\ C \in \mathbb{R}.
\end{aligned}$$

The envelope of the modulated carrier waves (25) propagates with the group velocity $v_g = v$, and it is given by the graphs of the functions $\pm A = \pm|u| = \pm\sqrt{u\overline{u}}$. We will call the upper part of the envelope *the profile of a solitary wave*. These types of solitary waves are known in the nonlinear dispersive waves literature as "bright" solitons, [31], [1], [2].

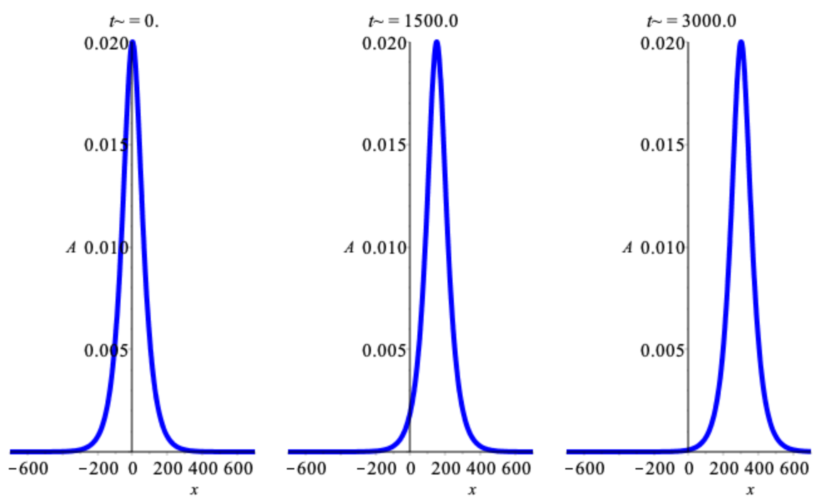

Figure 3: Solitary wave of the NLS (2) for $\alpha = 2$, $\mu = 1$, $\delta = 0.02$, $v_g = 0.1$, and $C = 0$.

Figure 3 illustrates the time evolution of a solitary wave of the NLS (2) traveling from left to right with the group velocity $v_g = 0.1$.

Figure 4 illustrates the time evolution of the modulated carrier waves (25) and their envelope for the NLS (2) traveling from left to right with the group velocity $v_g = 0.1$.

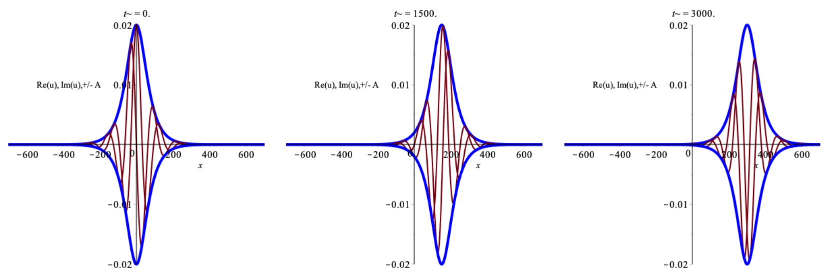

Figure 4: Modulated carrier waves and solitary wave for the NLS (2) for $\alpha = 2$, $\mu = 1$, $\delta = 0.02$, $v_g = 0.1$, and $C = 0$.

4 Singular Periodic Solutions for Defocusing NLS

We consider the defocusing NLS

$$iu_t + \frac{\alpha}{2}u_{xx} - \alpha u|u|^2 = 0,\ \alpha > 0 \text{ constant.} \tag{26}$$

In this section, we are interested in obtaining singular periodic solutions for the model (26) as follows.

Using separation of variables in (26)

$$u(x,t) = r(x)T(t), \tag{27}$$

such that $r(x)$ is a real-valued function and $T(t)$ is a complex-valued function such that $|T| = 1$, we obtain

$$\frac{i\,T'}{T} = \frac{-\frac{\alpha}{2}r'' + \alpha r^3}{r} = \mu,\ \mu \text{ constant,} \tag{28}$$

and we will be interested to analyze the case when $\mu > 0$.

From (28) we obtain

$$T(t) = e^{-i\mu t}, \tag{29}$$

and the following second order nonlinear ordinary differential equation for $r = r(x)$

$$r'' + \frac{2}{\alpha}r(\mu - \alpha r^2) = 0. \tag{30}$$

Multiplying (30) by r' and simplifying, we obtain

$$\frac{d}{dx}\left(\frac{(r')^2}{2} + \frac{\mu}{\alpha}r^2 - \frac{1}{2}r^4\right) = 0. \tag{31}$$

Integrating the equation (31), we readily obtain

$$(r')^2 + \frac{2\mu}{\alpha} r^2 - r^4 = K,\ K \text{ integration constant.} \tag{32}$$

Solving for r' the equation (32), we obtain

$$r' = \pm\sqrt{P(r)},\ P(r) = r^4 - \frac{2\mu}{\alpha} r^2 + K. \tag{33}$$

We are interested in the case when the polynomial P factors as follows

$$P(r) = (r^2 + r_1)(r^2 - r_2),\ 0 \leq r_1 < r_2,\ r_2 - r_1 = \frac{2\mu}{\alpha},\ K = -r_1 r_2. \tag{34}$$

From (33) and (34), we obtain

$$\frac{dr}{\sqrt{(r^2 + r_1)(r^2 - r_2)}} = \pm dx. \tag{35}$$

Integrating (35), we obtain

$$\int_{\sqrt{r_2}}^{r} \frac{dw}{\sqrt{(w^2 + r_1)(w^2 - r_2)}} = \pm(x + C),\ C \text{ integration constant.} \tag{36}$$

Making the substitution

$$w = \sqrt{r_2} \sec\theta, \tag{37}$$

and performing all the calculations, the equation (36) becomes

$$\pm\sqrt{r_1 + r_2}(x + C) = \int_0^{\phi} \frac{1}{\sqrt{1 - m\sin^2\theta}} d\theta,\ m = \frac{r_1}{r_1 + r_2}$$

$$\Downarrow$$

$$\operatorname{cn}\left(\sqrt{r_1 + r_2}(x + C)|m\right) = \cos\phi, \tag{38}$$

where we used the fact that the cn function is an even function. Then, referring back to the substitution (37), we finally obtain

$$r = \frac{\sqrt{r_2}}{\cos\phi} = \frac{\sqrt{r_2}}{\operatorname{cn}\left(\sqrt{r_1 + r_2}(x + C)|m\right)}. \tag{39}$$

Thus, the solution we were looking for the equation (30) is

$$r(x) = \frac{\sqrt{r_2}}{\operatorname{cn}\left(\sqrt{r_1 + r_2}(x + C)|m\right)},\ m = \frac{r_1}{r_1 + r_2}, \qquad 0 \leq r_1 < r_2,\ r_2 - r_1 = \frac{2\mu}{\alpha},\ C \in \mathbb{R}. \tag{40}$$

From (27) and (40), the NLS (26) has the following solution

$$\begin{aligned} u(x,t) &= r(x)e^{-i\mu t}, \\ r(x) &= \frac{\sqrt{r_2}}{\operatorname{cn}\left(\sqrt{r_1+r_2}(x+C)|m\right)}, \end{aligned} \tag{41}$$

which is a complex-valued singular periodic solution for the NLS (26).

Applying scale symmetry, [22], onto the solution (41), we obtain the following scaled solution for the NLS (26)

$$u(x,t) \longmapsto u(x,t|\delta) = \delta u(\delta x, \delta^2 t), \delta \neq 0. \tag{42}$$

The NLS (26) is Galilean invariant as follows: If $u(x,t)$ is a solution of the NLS (26) then we can obtain a new solution by changing the inertial reference frame, and adding a phase factor as follows

$$u(x,t) \longmapsto u(x,t|v) = u(x-vt,t)e^{\frac{i}{2\mu}\lambda^2 v\left(x+C-\frac{vt}{2}\right)}, \lambda = \sqrt{\frac{2\mu}{\alpha}}, v \in \mathbb{R}. \tag{43}$$

Applying the Galilean invariance (43) onto the scaled solution (42), we obtain the complex-valued singular periodic solution for the NLS (26)

$$\begin{aligned} u(x,t) &= \delta r(\delta(x-vt))e^{\frac{i}{4\mu}\left(2\lambda^2 v(x+C)-(\lambda^2v^2+4\delta^2\mu^2)t\right)}, \\ &\alpha > 0,\ \mu > 0,\ \lambda = \sqrt{\frac{2\mu}{\alpha}},\ \delta \neq 0,\ v \in \mathbb{R}, \\ r(x) &= \frac{\sqrt{r_2}}{\operatorname{cn}\left(\sqrt{r_1+r_2}(x+C)|m\right)},\ m = \frac{r_1}{r_1+r_2}, \\ &0 \leq r_1 < r_2,\ r_2 - r_1 = \frac{2\mu}{\alpha},\ C \in \mathbb{R}. \end{aligned} \tag{44}$$

Regarding the singularities in the solution (44), we use the fact that the cn function is periodic with period $4K$, and it has simple zeros at $\pm K$, [29], where K is the constant given by the following integral

$$K = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-m^2t^2)}}, m \in [0,\, 1]. \tag{45}$$

Thus, the singularities in the solution (44) within a period of the cn function will be at the following x-values, where $\operatorname{cn}(\pm K|m) = 0$,

$$\left\{ x \,\middle|\, x = vt + \frac{1}{\delta}\left(\pm\frac{K}{2\sqrt{r_1+r_2}} - C\right),\ t \in \mathbb{R},\ v \in \mathbb{R} \right\}. \tag{46}$$

The simplest case for determining the singularities in the solution (44) is when $m = 0$, $\operatorname{cn}(\sqrt{r_1 + r_2}(\delta(x - vt) + C)|0) = \cos(\sqrt{r_1 + r_2}(\delta(x - vt) + C))$, and the set of singularities will be at the following x-values

$$\left\{ x \,\middle|\, x = vt + \frac{1}{\delta}\left(\frac{(2n+1)\pi}{2\sqrt{r_1 + r_2}} - C\right),\ t \in \mathbb{R},\ v \in \mathbb{R},\ n \in \mathbb{Z} \right\}. \tag{47}$$

The dispersion and the nonlinearity are not in balance in the formula (44). The steepening of the wavefronts, due to nonlinearity, of the oscillatory components of the nonlinear dispersive waves given by (44) is so strong that the dispersive effect almost diminishes, and the waves will develop singularities. The oscillatory components given by (44) represent modulated singular carrier waves. The envelope of these modulated singular carrier waves propagates with the group velocity $v_g = v$, and it is represented by the graphs of the functions $\pm A = \pm|u| = \pm\sqrt{u\bar{u}}$. We will call the upper part of the envelope *a singular periodic profile* of the NLS (26) given by the formula below,

$$A(x,t) = \sqrt{\frac{2\mu}{\alpha} + r_1}\left|\frac{\delta}{\operatorname{cn}\left(\sqrt{\frac{2\mu}{\alpha} + 2r_1}\ (\delta(x - vt) + C)|\ \frac{\alpha r_1}{2\mu + 2\alpha r_1}\right)}\right|. \tag{48}$$

It is important to notice that the singularities in the solution (44), the modulated singular carrier waves given by (44), and the envelope (48) travel with the group velocity $v_g = v$. If the singularities in the solution (44) did not display a regular pattern, the solutions would not be of much interest to study because their behaviour would indicate a level of randomness that could potentially conceal the behaviour of the nonlinear process.

Figure 5 illustrates the time evolution of a singular periodic profile of the NLS (26) traveling from left to right with the group velocity $v_g = 3.5$.

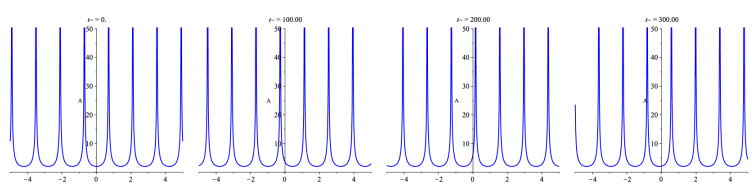

Figure 5: Singular periodic profile of the NLS (26) for $\alpha = 0.5$, $\mu = 1$, $\delta = 0.92$, $C = 0$, and $v_g = 3.5$.

Figure 6 illustrates the time evolution of modulated singular carrier waves and their envelope for the NLS (26) traveling from left to right with the group velocity $v_g = 3.5$.

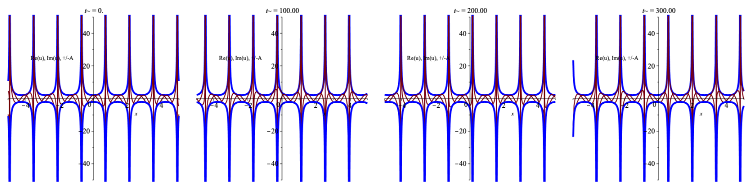

Figure 6: Modulated singular carrier waves and their envelope for the NLS (26) for $\alpha = 0.5$, $\mu = 1$, $\delta = 0.92$, $C = 0$, and $v_g = 3.5$.

The strong effect of nonlinearity on the dispersive effect shows in the time evolution of the modulated singular carrier waves given by (44), which propagate as the process described by (26) would almost be "dispersive-free."

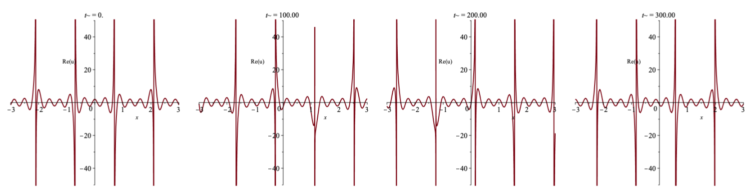

Figure 7: Modulated singular carrier waves, Re(u), for the NLS (26) for $\alpha = 0.5$, $\mu = 1$, $\delta = 0.92$, $C = 0$, and $v_g = 3.5$.

Figures 7 and 8 show the strong effect of nonlinearity onto the oscillatory components given by (44).

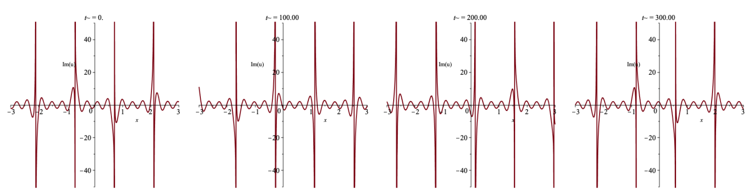

Figure 8: Modulated singular carrier waves, Im(u), for the NLS (26) for $\alpha = 0.5$, $\mu = 1$, $\delta = 0.92$, $C = 0$, and $v_g = 3.5$.

Haines, [14], describes the water-air interface (the free surface) as a "virtually" complete barrier for sound waves, like a reflective hard surface. Sound waves originating from underneath the free surface are "trapped" in the water medium and, if properly started, they will experience a process of reflection (at the free surface) and refraction (due to the gradient) "bouncing" up and down with very steep gradients at the water-air interface. Suppose the sound wave originates from above the sonic layer depth (SLD), and the gradient is steep. In that case, the wave will refract upward until it hits the free surface, after which it "bounces" back (i.e., reflects) to refract upward again by its gradient. The type of propagation of a sound wave in between the water-air interface and the sonic layer depth, in underwater acoustics, is called surface duct sound propagation, [28]. The surface duct sound propagation belongs to the general class of sound propagation in ducts, [26]. The duct surface between the water-air interface and the sonic layer depth varies, and the authors in [16] provide examples for SLD estimations. Figure (9) shows similarities

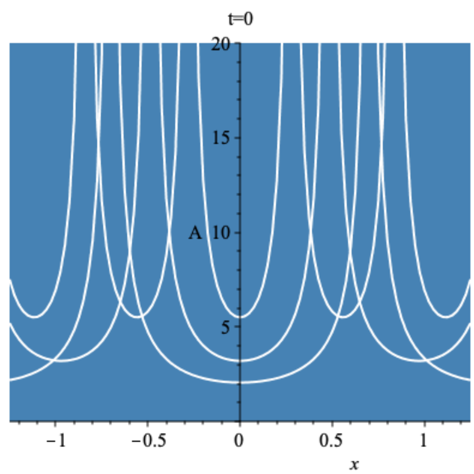

Figure 9: Scenario of singular profiles of the NLS (26) as sound waves propagating in the duct channel between water-air interface and SLD for $\alpha = 0.05,\ 0.1,\ 0.5$, $\delta = 0.5,\ 0.92,\ 1.2$, $\mu = 1$, $C = 0$, and $v_g = 1$.

of the singular periodic profiles (48) with the duct sound propagation found in oceanographic literature, [7]. We interpret the wave refracted/reflected at the water-air interface with a very steep gradient when the profile (48) becomes singular. The refraction/reflection points along the water-air interface are given by the set of singular points (47).

5 Summary and Discussions

The NLS models (2) and (26) are of great interest in studying nonlinear waves emerging from areas of Physics such as nonlinear optics and deepwater wave propagation phenomena. Zakharov and Shabat described the solitary wave profiles for the NLS models in 1972, [31]. They used the scattering method to obtain them, [13], [20]. Peregrine, [23], gives an account of solutions for the nonlinear Schrödinger equations up to the time when the article was published. Eighteen years later, in [17], the author mentioned the surging importance of studying these models, with their widespread applications in Physics mainly.

In this paper, we succeeded in using elliptic functions and obtaining periodic solutions for both self-focusing and defocusing nonlinear Schrödinger models (2) and (26). An essential aspect of these periodic solutions is that they correlate rather interestingly; the periodic solutions for the defocusing NLS model are "reciprocals" of the periodic solutions for the self-focusing NLS and vice-versa. The relevance in looking for periodic profiles for these models resonates with the facts described in [17]. Two examples of works on periodic solutions and solitary wave solutions for different NLS models are [27] and [30]. As well, the authors of [27] used similar techniques for finding exact solutions for the nonlinear Schrödinger model in an optical fiber. In the end, we used a classic technique, i.e., the Fourier method, in approaching the NLS model.

One significant contribution of this article is that we obtained the solitary wave solution for the self-focusing nonlinear Schrödinger model (2) through a limiting process using a sequence of periodic waves, i.e., the cnoidal waves. So far, consulting available literature, we did not find studies on obtaining the solitary wave for the self-focusing nonlinear Schrödinger model through a limiting process using a sequence of cnoidal waves. Our question was: "Could the solitary wave of the self-focusing nonlinear Schrödinger model be obtained similarly as Korteweg and de Vies obtained the solitary wave solution for their famous KdV model, [19]?" As described by Korteweg and de Vries, the solitary wave formation in shallow water is a consequence of a limiting process using a sequence of cnoidal waves. It was a "curiosity" question to see whether the deepwater scenario modeled by the self-focusing nonlinear Schrödinger equation would reveal a similar process.

Note: We read extensive literature on this subject, but we cannot say we read everything, as it is simply impossible.

Another significant contribution of this article is that we succeeded in obtaining "well-behaved" singular solutions for the defocusing nonlinear Schrödinger model (26). The applicability of the singular solutions (44) in underwater acoustic, as explained in Section 4, may be very realistic because the water-air interface behaves as a virtually impenetrable surface for the sound waves hitting it with a very steep gradient. The periodicity of the singular profiles (48) agrees as well with the fact that the sound trapped in the duct surface will propagate for as long as its physical properties will allow it. The physical properties of sound propagation trapped in the duct surface depend on the depth of the duct surface and the frequency of the sound, [14].

Our interest in discovering and studying singular solutions for the NLS model (1) aligns with the systematic interest by scientists to study them, [9], [12], [15], [21].

Because of the high nonlinearity displayed by the solutions (44), the singular periodic profile (48) may explain the periodic train of mechanical waves that create the so-called ocean swell. As well, they may explain wavebreaking is hydrodynamics, [18]. Still, these matters require further investigation, and they are brought here only as observations.

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