DOI: 10.2478/auom-2022-0017
**An. Şt. Univ. Ovidius Constanţa** **Vol. 30(2),2022, 25–44**

# Central and local limit theorems for the weighted Delannoy numbers

**Igoris Belovas**

**Abstract**

In this research we generalize our result for numbers satisfying the Delannoy triangle. We obtain a central limit theorem and a local limit theorem for weighted numbers of the triangle and establish the rate of convergence to the limiting (normal) distribution.

## 1 Introduction

The Delannoy numbers $D_{n,k}$ (named after French army officer and amateur mathematician Henri Delannoy) count the number of lattice paths from the southwest corner $(0,0)$ to the northeast corner $(n,k)$, using only single steps north, northeast, or east [16]. The Delannoy numbers constitute the Delannoy array (see Table 1a) and (for nonnegative integers $n$ and $k$) satisfy the recurrence relation [10, 15],

$$D_{n,k} = \begin{cases} 1, & \text{if } n = 0 \text{ or } k = 0, \\ D_{n-1,k} + D_{n-1,k-1} + D_{n,k-1}, & \text{otherwise .} \end{cases} \quad (1)$$

Key Words: Delannoy triangle, central limit theorem, local limit theorem, double generating function, triangular array
2010 Mathematics Subject Classification: Primary 05A15; 05A16; 60F05; Secondary 39A06; 39A14
Received: 01.04.2021
Accepted: 15.09.2021

Table 1: The Delannoy array and the tribonacci triangle

a) $D_{n,k}$ numbers

| | 0 | 1 | 2 | 3 | 4 | ... |
|---|---|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 | 1 | ... |
| 1 | 1 | 3 | 5 | 7 | 9 | ... |
| 2 | 1 | 5 | 13 | 25 | 41 | ... |
| 3 | 1 | 7 | 25 | 63 | 129 | ... |
| 4 | 1 | 9 | 41 | 129 | 321 | ... |
| ... | ... | ... | ... | ... | ... | ... |

b) $\hat{D}_{n,k}$ numbers

| | 0 | 1 | 2 | 3 | 4 | ... |
|---|---|---|---|---|---|---|
| 0 | 1 | 0 | 0 | 0 | 0 | ... |
| 1 | 1 | 1 | 0 | 0 | 0 | ... |
| 2 | 1 | 3 | 1 | 0 | 0 | ... |
| 3 | 1 | 5 | 5 | 1 | 0 | ... |
| 4 | 1 | 7 | 13 | 7 | 1 | ... |
| ... | ... | ... | ... | ... | ... | ... |

It is convenient to arrange the Delannoy numbers in a triangular array resembling Pascal's triangle [3, 11], in which each modified number $\hat{D}_{n,k}$ (see Table 1b) is the sum of the three numbers above it: $\hat{D}_{n,k} = D_{n-k,k}$ [16], or

$$\hat{D}_{n,k} = \begin{cases} 1, & \text{if } n = 0 \text{ and } k = 0, \\ 0, & \text{if } \min(n-k, n, k) < 0, \\ \hat{D}_{n-1,k-1} + \hat{D}_{n-1,k} + \hat{D}_{n-2,k-1}, & \text{otherwise} . \end{cases} \tag{2}$$

Thus, the weighted Delannoy numbers $d_{n,k}$ are defined by the following recurrent expression [2, 14]:

$$d_{n,k} = \begin{cases} 1, & \text{if } n = 0 \text{ and } k = 0, \\ 0, & \text{if } \min(n-k, n, k) < 0, \\ \alpha d_{n-1,k-1} + \beta d_{n-1,k} + \gamma d_{n-2,k-1}, & \text{otherwise} . \end{cases} \tag{3}$$

Here the weights $\alpha, \beta, \gamma$ are positive. The definition (3) is equivalent to

$$d_{n,k} = \begin{cases} 0, & \text{if } \min(n-k, n, k) < 0, \\ \alpha^n, & \text{if } n = k, \\ \beta^n, & \text{if } k = 0, \\ \alpha d_{n-1,k-1} + \beta d_{n-1,k} + \gamma d_{n-2,k-1}, & \text{otherwise} . \end{cases} \tag{4}$$

The closed form for the weighted Delannoy numbers (cf. Table 2) is [2]

$$d_{n,k} = \sum_{j=0}^{\min(k,n-k)} C_k^j C_{n-j}^k \alpha^{k-j} \beta^{n-k-j} \gamma^j. \tag{5}$$

Table 2: The generalized Delannoy triangle ($d_{n,k}$ numbers)

| | 0 | 1 | 2 | 3 | 4 | ... |
|---|---|---|---|---|---|---|
| 0 | $1$ | $0$ | $0$ | $0$ | $0$ | ... |
| 1 | $\beta$ | $\alpha$ | $0$ | $0$ | $0$ | ... |
| 2 | $\beta^2$ | $2\alpha\beta+\gamma$ | $\alpha^2$ | $0$ | $0$ | ... |
| 3 | $\beta^3$ | $3\alpha\beta^2+2\beta\gamma$ | $3\alpha^2\beta+2\alpha\gamma$ | $\alpha^3$ | $0$ | ... |
| 4 | $\beta^4$ | $4\alpha\beta^3+3\beta^2\gamma$ | $6\alpha^2\beta^2+6\alpha\beta\gamma+\gamma^2$ | $4\alpha^3\beta+3\alpha^2\gamma$ | $\alpha^4$ | ... |
| ... | ... | ... | ... | ... | ... | ... |

The Delannoy numbers and their generalizations are intensively investigated now. Amrouche, Belbachir and Ramrez have studied the unimodality of sequences located in the Delannoy triangle's infinite transversals and derived the explicit formulation of the linear recurrence sequence satisfied by the sum of the elements lying over any finite ray of the generalized Delannoy matrix [1, 2]. The total positivity of Delannoy-like triangles has been analyzed by Mu and Zheng [13]. Yang, Zheng and Yuan have examined the inverses of the generalized Delannoy matrices [17]. In [14] Noble has obtained asymptotic expansions for the central weighted Delannoy numbers $u_{r,r}$ and the numbers along the diagonal with slope 2 ($u_{r,2r}$). In this paper we extend the investigations of asymptotic expansions for the Delannoy numbers (cf. limit theorems for combinatorial numbers in [4, 5, 7, 8]) and generalize our result for numbers satisfying the ordinary Delannoy triangle [6].

The paper is organized as follows. The first part is the introduction. In the second part, we specify the moment-generating function of the weighted Delannoy numbers and calculate exact expressions for the first and second moments. The third and fourth sections are devoted to the central and local limit theorems. We establish the asymptotic normality of the numbers along with the rate of convergence to the limiting distribution.

Throughout this paper, we denote by $\Phi_{\mu,\sigma}(x)$ the cumulative distribution function of the normal distribution with the mean $\mu$ and the standard deviation $\sigma$; by $\varphi_{\mu,\sigma}(x)$ we denote the corresponding density function. All limits in the paper, unless specified, are taken as $n \to \infty$.

## 2 Moment-generating function

Let us identify the closed form of the generating function of the numbers (3),

$$F(x,y)=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}d_{n,k}x^ny^k=\sum_{n=0}^{\infty}\sum_{k=0}^{n}d_{n,k}x^ny^k. \tag{6}$$

**Lemma 2.1.** *The double ordinary generating function of the weighted Delannoy numbers is*

$$F(x,y)=\frac{1}{1-\beta x-\alpha xy-\gamma x^2y}. \tag{7}$$

*Proof.* Substituting the expression (4) into the generating function (6), we obtain, that

$$\begin{aligned}F(x,y)&=\sum_{n=0}^{\infty}d_{n,0}x^n+\sum_{n=1}^{\infty}d_{n,1}x^ny\\&+\sum_{n=2}^{\infty}\sum_{k=2}^{\infty}\alpha d_{n-1,k-1}+\beta d_{n-1,k}+\gamma d_{n-2,k-1}x^ny^k\\&=\sum_{n=0}^{\infty}d_{n,0}x^n+\sum_{n=1}^{\infty}d_{n,1}x^ny+\alpha xy\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}d_{n,k}x^ny^k\\&+\beta x\sum_{n=2}^{\infty}\sum_{k=2}^{\infty}d_{n,k}x^ny^k+\gamma x^2y\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}d_{n,k}x^ny^k\\&=(1-\beta x-\alpha xy-\gamma x^2y)\underbrace{\sum_{n=0}^{\infty}d_{n,0}x^n}_{=\frac{1}{1-\beta x}}+(y-\beta xy)\underbrace{\sum_{n=1}^{\infty}d_{n,1}x^n}_{=\frac{\alpha x+\gamma x^2}{(1-\beta x)^2}}\\&+\alpha xyF(x,y)+\beta xF(x,y)+\gamma x^2yF(x,y),\end{aligned} \tag{8}$$

yielding us the statement of the lemma. □

Let $A_n$ be an integral random variable with the probability mass function

$$P(A_n=k):=\frac{d_{n,k}}{\sum_{k=0}^{n}d_{n,k}}. \tag{9}$$

The moment-generating function of the random variable $A_n$ is

$$M_n(s)=E(e^{A_ns})=\sum_{k=0}^{n}P(A_n=k)e^{ks}=S_n^{-1}\sum_{k=0}^{n}d_{n,k}e^{ks}, \tag{10}$$

where $S_n$ stands for the sum of the $n$-th row of the triangle. Combining the definition of the generating function (6) and (10), we receive

$$F(x,e^s)=\sum_{n=0}^{\infty}x^n\sum_{k=0}^{n}d_{n,k}e^{sk}=\sum_{n=0}^{\infty}x^nS_nM_n(s).$$

Hence, the partial differentiation of the generating function $F(x,y)$ at $x=0$, yields us the moment-generating function

$$M_n(s)=\frac{1}{S_n n!}\,\frac{\partial^n}{\partial x^n}F(x,e^s)\bigg|_{x=0}. \tag{11}$$

Because of $M_n(0)=1$, we have formula for the sum of $n$-th row,

$$S_n=\frac{1}{n!}\,\frac{\partial^n}{\partial x^n}F(x,e^s)\bigg|_{(0,0)}. \tag{12}$$

Next, we prove the following result for the moment-generating function.

**Lemma 2.2.** *The moment generating function of the random variable $A_n$ is*

$$M_n(s)=\frac{\theta_0}{\theta(s)}\,\frac{(\alpha e^s+\beta+\theta(s))^{n+1}-(\alpha e^s+\beta-\theta(s))^{n+1}}{(\alpha+\beta+\theta_0)^{n+1}-(\alpha+\beta-\theta_0)^{n+1}}, \tag{13}$$

*here*

$$\begin{aligned}\theta(s)&=\sqrt{\alpha^2e^{2s}+(2\alpha\beta+4\gamma)e^s+\beta^2},\\ \theta_0=\theta(0)&=\sqrt{\alpha^2+2\alpha\beta+4\gamma+\beta^2}.\end{aligned} \tag{14}$$

*Proof.* Let us consider the denominator in the formula for the generating function $F(x,e^s)$ (7),

$$\begin{aligned}1-(\beta+\alpha e^s)x-\gamma e^sx^2&=-\gamma e^s\left(x^2+\frac{\beta+\alpha e^s}{\gamma e^s}x-\frac{1}{\gamma e^s}\right)\\&=-\gamma e^s(x-x_1)(x-x_2),\end{aligned}$$

where

$$x_1=-\frac{\alpha e^s+\beta+\theta(s)}{2\gamma e^s},\qquad x_2=-\frac{\alpha e^s+\beta-\theta(s)}{2\gamma e^s}.$$

Using the formula for the $n$-th derivative of the rational function,

$$\left(\frac{1}{x^2+px+q}\right)^{(n)}_{x=0}=\frac{n!}{x_2-x_1}\,\frac{x_2^{n+1}-x_1^{n+1}}{(x_1x_2)^{n+1}},$$

(here $x_1$ and $x_2$ are roots of the quadratic trinomial), we get

$$\frac{\partial^n}{\partial x^n} F(x, e^s)\bigg|_{x=0} = n! \frac{(\alpha e^s + \beta + \theta(s))^{n+1} - (\alpha e^s + \beta - \theta(s))^{n+1}}{2^{n+1}\theta(s)}.$$

Thus, by (12), the sum of the $n$-th row of the Delannoy triangle is

$$S_n = \frac{(\alpha + \beta + \theta_0)^{n+1} - (\alpha + \beta - \theta_0)^{n+1}}{2^{n+1}\theta_0},$$

yielding us the statement of the lemma.

□

**Lemma 2.3.** *Let*

$$\begin{aligned} h &= \frac{\alpha + \beta - \theta_0}{\alpha + \beta + \theta_0}, \\ \theta_0 &= \sqrt{\alpha^2 + 2\alpha\beta + 4\gamma + \beta^2}, \\ \theta_0' &= \frac{\alpha^2 + \alpha\beta + 2\gamma}{\theta_0}, \\ \theta_0'' &= \frac{\alpha^4 + 3\alpha^3\beta + 6\alpha^2\gamma + 3\alpha^2\beta^2 + 4\alpha\beta\gamma + 4\gamma^2 + \alpha\beta^3 + 2\beta^2\gamma}{\theta_0^3}, \end{aligned}$$

*then the expectation and the variance of the random variable $A_n$ are*

$$\begin{aligned} \mu_n &= n\frac{(\alpha + \beta + \theta_0)\alpha + 2\gamma}{(\alpha + \beta + \theta_0)\theta_0} + \frac{2\gamma(\alpha - \beta)}{(\alpha + \beta + \theta_0)\theta_0^2} + O\left(nh^n\right), \\ \sigma_n^2 &= (n+1)\left(\frac{\alpha + \theta_0''}{\alpha + \beta + \theta_0} - \frac{(\alpha + \theta_0')^2}{(\alpha + \beta + \theta_0)^2}\right) + \frac{{\theta_0'}^2 - \theta_0\theta_0''}{\theta_0^2} + O(n^2h^n), \end{aligned} \tag{15}$$

*respectively.*

*Proof.* Calculating the first and the second derivatives of the moment-generating function, we receive

$$\begin{aligned} M'(s) = &\frac{(\alpha e^s + \beta + \theta(s))^n(\alpha e^s + \theta'(s)) - (\alpha e^s + \beta - \theta(s))^n(\alpha e^s - \theta'(s))}{(n+1)^{-1}((\alpha + \beta + \theta_0)^{n+1} - (\alpha + \beta - \theta_0)^{n+1})\theta_0^{-1}\theta(s)} \\ &- \frac{((\alpha e^s + \beta + \theta(s))^{n+1} - (\alpha e^s + \beta - \theta(s))^{n+1})\theta'(s)}{((\alpha + \beta + \theta_0)^{n+1} - (\alpha + \beta - \theta_0)^{n+1})\theta_0^{-1}\theta^2(s)} \end{aligned} \tag{16}$$

and

$$
\begin{aligned}
&M''(s) = \\
&= \frac{n(\alpha e^s+\beta+\theta(s))^{n-1}(\alpha e^s+\theta'(s))^2+(\alpha e^s+\beta+\theta(s))^n(\alpha e^s+\theta''(s))}{(n+1)^{-1}((\alpha+\beta+\theta_0)^{n+1}-(\alpha+\beta-\theta_0)^{n+1})\theta_0^{-1}\theta(s)} \\
&- \frac{(\alpha e^s+\beta+\theta(s))^n(\alpha e^s+\theta'(s))\theta'(s)}{(n+1)^{-1}((\alpha+\beta+\theta_0)^{n+1}-(\alpha+\beta-\theta_0)^{n+1})\theta_0^{-1}\theta^2(s)} \\
&- \frac{n(\alpha e^s+\beta-\theta(s))^{n-1}(\alpha e^s-\theta'(s))^2+(\alpha e^s+\beta-\theta(s))^n(\alpha e^s-\theta''(s))}{(n+1)^{-1}((\alpha+\beta+\theta_0)^{n+1}-(\alpha+\beta-\theta_0)^{n+1})\theta_0^{-1}\theta(s)} \\
&+ \frac{(\alpha e^s+\beta-\theta(s))^n(\alpha e^s-\theta'(s))\theta'(s)}{(n+1)^{-1}((\alpha+\beta+\theta_0)^{n+1}-(\alpha+\beta-\theta_0)^{n+1})\theta_0^{-1}\theta^2(s)} \\
&- \frac{(n+1)(\alpha e^s+\beta+\theta(s))^n(\alpha e^s+\theta'(s))\theta'(s)+(\alpha e^s+\beta+\theta(s))^{n+1}\theta''(s)}{((\alpha+\beta+\theta_0)^{n+1}-(\alpha+\beta-\theta_0)^{n+1})\theta_0^{-1}\theta^2(s)} \\
&+ \frac{2(\alpha e^s+\beta+\theta(s))^{n+1}(\theta'(s))^2}{((\alpha+\beta+\theta_0)^{n+1}-(\alpha+\beta-\theta_0)^{n+1})\theta_0^{-1}\theta^3(s)} \\
&+ \frac{(n+1)(\alpha e^s+\beta-\theta(s))^n(\alpha e^s-\theta'(s))\theta'(s)+(\alpha e^s+\beta-\theta(s))^{n+1}\theta''(s)}{((\alpha+\beta+\theta_0)^{n+1}-(\alpha+\beta-\theta_0)^{n+1})\theta_0^{-1}\theta^2(s)} \\
&- \frac{2(\alpha e^s+\beta-\theta(s))^{n+1}(\theta'(s))^2}{((\alpha+\beta+\theta_0)^{n+1}-(\alpha+\beta-\theta_0)^{n+1})\theta_0^{-1}\theta^3(s)}.
\end{aligned}
\tag{17}
$$

Next, we have

$$
\begin{aligned}
&\theta'(s) = \frac{\alpha^2e^{2s}+(\alpha\beta+2\gamma)e^s}{\sqrt{\alpha^2e^{2s}+(2\alpha\beta+4\gamma)e^s+\beta^2}} = \frac{\alpha^2e^{2s}+(\alpha\beta+2\gamma)e^s}{\theta(s)}, \\
&\theta''(s) = \\
&= \frac{\alpha^4e^{4s}+3\alpha^2(\alpha\beta+2\gamma)e^{3s}+(3\alpha^2\beta^2+4\alpha\beta\gamma+4\gamma^2)e^{2s}+\beta^2(\alpha\beta+2\gamma)e^s}{\theta^3(s)},
\end{aligned}
\tag{18}
$$

and

$$\theta'(0) = \theta_0', \qquad \theta''(0) = \theta_0''. \tag{19}$$

Hence, the expectation (cf. (16) and (19)) is

$$\begin{aligned}
\mu_n = M_n'(0) &= \frac{(\alpha+\beta+\theta_0)^n(\alpha+\theta_0') - (\alpha+\beta-\theta_0)^n(\alpha-\theta_0')}{(n+1)^{-1}((\alpha+\beta+\theta_0)^{n+1} - (\alpha+\beta-\theta_0)^{n+1})} - \frac{\theta_0'}{\theta_0} \\
&= (n+1)\frac{\frac{\alpha(\alpha+\beta+\theta_0)+2\gamma}{(\alpha+\beta+\theta_0)\theta_0} - \frac{\alpha\theta_0-(\alpha^2+\alpha\beta+2\gamma)}{(\alpha+\beta+\theta_0)\theta_0}h^n}{1-h^{n+1}} - \frac{\alpha^2+\alpha\beta+2\gamma}{\theta_0^2} \\
&= (n+1)\frac{(\alpha+\beta+\theta_0)\alpha+2\gamma}{(\alpha+\beta+\theta_0)\theta_0}(1+O(h^n)) - \frac{\alpha^2+\alpha\beta+2\gamma}{\theta_0^2},
\end{aligned}$$

yielding us the first statement of the lemma.

Calculating the variance (cf. (17) and (19)), we obtain

$$\begin{aligned}
M_n''(0) &= (n+1)\frac{n(\alpha+\beta+\theta_0)^{n-1}(\alpha+\theta_0')^2 + (\alpha+\beta+\theta_0)^n(\alpha+\theta_0'')}{(\alpha+\beta+\theta_0)^{n+1} - (\alpha+\beta-\theta_0)^{n+1}} \\
&\quad - \frac{(n+1)(\alpha+\beta+\theta_0)^n(\alpha+\theta_0')\theta_0' - (n+1)(\alpha+\beta-\theta_0)^n(\alpha-\theta_0')\theta_0'}{((\alpha+\beta+\theta_0)^{n+1} - (\alpha+\beta-\theta_0)^{n+1})\theta_0} \\
&\quad - (n+1)\frac{n(\alpha+\beta-\theta_0)^{n-1}(\alpha-\theta_0')^2 + (\alpha+\beta-\theta_0)^n(\alpha-\theta_0'')}{(\alpha+\beta+\theta_0)^{n+1} - (\alpha+\beta-\theta_0)^{n+1}} \\
&\quad - \frac{(n+1)(\alpha+\beta+\theta_0)^n(\alpha+\theta_0')\theta_0' + (\alpha+\beta+\theta_0)^{n+1}\theta_0''}{((\alpha+\beta+\theta_0)^{n+1} - (\alpha+\beta-\theta_0)^{n+1})\theta_0} \\
&\quad + \frac{2(\alpha+\beta+\theta_0)^{n+1}{\theta_0'}^2 - 2(\alpha+\beta-\theta_0)^{n+1}{\theta_0'}^2}{((\alpha+\beta+\theta_0)^{n+1} - (\alpha+\beta-\theta_0)^{n+1})\theta_0^2} \\
&\quad + \frac{(n+1)(\alpha+\beta-\theta_0)^n(\alpha-\theta_0')\theta_0' + (\alpha+\beta-\theta_0)^{n+1}\theta_0''}{((\alpha+\beta+\theta_0)^{n+1} - (\alpha+\beta-\theta_0)^{n+1})\theta_0},
\end{aligned}$$

and

$$\begin{aligned}
M_n''(0) &= (n+1)\frac{\frac{n(\alpha+\theta_0')^2}{(\alpha+\beta+\theta_0)^2} + \frac{\alpha+\theta_0''}{\alpha+\beta+\theta_0} - \frac{n(\alpha-\theta_0')^2}{(\alpha+\beta+\theta_0)^2}h^{n-1} - \frac{\alpha-\theta_0''}{\alpha+\beta+\theta_0}h^n}{1-h^{n+1}} \\
&\quad - (n+1)\frac{\frac{(\alpha+\theta_0')\theta_0'}{(\alpha+\beta+\theta_0)\theta_0} - \frac{(\alpha-\theta_0')\theta_0'}{(\alpha+\beta+\theta_0)\theta_0}h^n}{1-h^{n+1}} - \frac{\frac{(n+1)(\alpha+\theta_0')\theta_0'}{(\alpha+\beta+\theta_0)\theta_0} + \frac{\theta_0''}{\theta_0}}{1-h^{n+1}} \\
&\quad + \frac{\frac{2{\theta_0'}^2}{\theta_0^2} - \frac{2{\theta_0'}^2}{\theta_0^2}h^{n+1}}{1-h^{n+1}} + \frac{\frac{(n+1)(\alpha-\theta_0')\theta_0'}{(\alpha+\beta+\theta_0)\theta_0}h^n + \frac{\theta_0''}{\theta_0}h^{n+1}}{1-h^{n+1}} \\
&= \frac{(n+1)n(\alpha+\theta_0')^2}{(\alpha+\beta+\theta_0)^2} + \frac{(n+1)(\alpha+\theta_0'')}{\alpha+\beta+\theta_0} \\
&\quad - \frac{2(n+1)(\alpha+\theta_0')\theta_0'}{(\alpha+\beta+\theta_0)\theta_0} + \frac{2{\theta_0'}^2 - \theta_0\theta_0''}{\theta_0^2} + O(n^2h^n).
\end{aligned}$$

Thus,

$$
\begin{aligned}
\sigma_n^2 = M_n''(0) - \mu_n^2 &= \frac{(n+1)n(\alpha+\theta_0')^2}{(\alpha+\beta+\theta_0)^2} + \frac{(n+1)(\alpha+\theta_0'')}{\alpha+\beta+\theta_0} \\
&- \frac{2(n+1)(\alpha+\theta_0')\theta_0'}{(\alpha+\beta+\theta_0)\theta_0} + \frac{2{\theta_0'}^2 - \theta_0\theta_0''}{\theta_0^2} - (n+1)^2\frac{(\alpha+\theta_0')^2}{(\alpha+\beta+\theta_0)^2} \\
&+ 2(n+1)\frac{(\alpha+\theta_0')\theta_0'}{(\alpha+\beta+\theta_0)\theta_0} - \frac{{\theta_0'}^2}{\theta_0^2} + O(n^2h^n) \\
&= (n+1)\left(\frac{\alpha+\theta_0''}{\alpha+\beta+\theta_0} - \frac{(\alpha+\theta_0')^2}{(\alpha+\beta+\theta_0)^2}\right) + \frac{{\theta_0'}^2 - \theta_0\theta_0''}{\theta_0^2} + O(n^2h^n),
\end{aligned}
$$

concluding the proof of the lemma. □

**Remark 2.1.** *The expectation and the variance of the random variable $A_n$ can be represented as*

$$
\begin{aligned}
\mu_n &= n\tilde{\mu}\left(1 + O\left(\frac{1}{n}\right)\right), \\
\sigma_n^2 &= n\tilde{\sigma}^2\left(1 + O\left(\frac{1}{n}\right)\right),
\end{aligned}
\tag{20}
$$

,

*respectively. Here*

$$
\begin{aligned}
\tilde{\mu} &= \frac{(\alpha+\beta+\theta_0)\alpha + 2\gamma}{(\alpha+\beta+\theta_0)\theta_0}, \\
\tilde{\sigma}^2 &= \frac{\alpha+\theta_0''}{\alpha+\beta+\theta_0} - \frac{(\alpha+\theta_0')^2}{(\alpha+\beta+\theta_0)^2} > 0.
\end{aligned}
\tag{21}
$$

Indeed, calculating $\tilde{\sigma}^2$, we obtain

$$
\begin{aligned}
\tilde{\sigma}^2 &= \frac{2\alpha\beta\theta_0^2 + 2\gamma(\alpha-\beta)^2}{\theta_0^2(\alpha+\beta+\theta_0)^2} + \frac{2(\alpha\beta+\gamma)(\alpha+\beta)((\alpha+\beta)^2+2\gamma)}{\theta_0^3(\alpha+\beta+\theta_0)^2} \\
&= \frac{(\alpha+\beta)(\alpha\beta+\gamma)}{\theta_0^3} > 0.
\end{aligned}
\tag{22}
$$

## 3 Central limit theorem

Let $\{\Omega_n\}$ be a sequence of integral random variables. Suppose that the moment generating function satisfies the asymptotic expression:

$$
M_n(s) = e^{H_n(s)}\left(1 + O\left(\kappa_n^{-1}\right)\right),
$$

the $O$-term being uniform for $|s| \leqslant \tau,\ s \in \mathbb{C},\ \tau > 0$, where

(i) $H_n(s) = u(s)\phi(n) + v(s)$, with $u(s)$ and $v(s)$ analytic for $|s| \leqslant \tau$ and independent of $n$, $u''(0) \neq 0$;

(ii) $\phi(n) \to \infty$;

(iii) $\kappa_n \to \infty$.

We will use Hwang's general limit theorem [12] to prove the asymptotic normality of the weighted Delannoy numbers and establish the rate of convergence to the limiting distribution.

**Theorem 3.1.** (Hwang). *Under assumptions* (i)-(iii),

$$\left| P\left(\frac{\Omega_n - u'(0)\phi(n)}{\sqrt{u''(0)\phi(n)}} < x\right) - \Phi(x)\right| = O\left(\frac{1}{\kappa_n} + \frac{1}{\sqrt{\phi(n)}}\right), \tag{23}$$

*uniformly with respect to $x$, $x \in \mathbb{R}$.*

Let us formulate an auxiliary lemma.

**Lemma 3.1.** *For $x \in \mathbb{R}$,*

$$\frac{\sqrt{\alpha\beta} - \sqrt{\alpha\beta+\gamma}}{\sqrt{\alpha\beta} + \sqrt{\alpha\beta+\gamma}} \leqslant \frac{\alpha e^x + \beta - \theta(x)}{\alpha e^x + \beta + \theta(x)} < 0.$$

*Proof.* Let

$$q(x) = \frac{\alpha e^x + \beta - \theta(x)}{\alpha e^x + \beta + \theta(x)}.$$

Calculating the first derivative, we obtain

$$q'(x) = \frac{2\alpha e^x \theta(x) - 2\alpha e^x \theta'(x) - 2\beta\theta'(x)}{(\alpha e^x + \beta + \theta(x))^2}.$$

Next, solving the equation

$$\theta(x) - \theta'(x) - \frac{\beta}{\alpha} e^{-x}\theta'(x) = 0,$$

we get (cf.(14) and (18))

$$\theta^2(x) - \alpha^2 e^{2x} - (\alpha\beta + 2\gamma)e^x - \alpha\beta e^x - \frac{\beta}{\alpha}(\alpha\beta+2\gamma) = 0 \quad \Rightarrow \quad e^x = \frac{\beta}{\alpha},$$

yielding us the stationary point $x_0 = \log(\beta/\alpha)$. Note that

$$\begin{aligned}\theta(x_0) &= 2\sqrt{\beta^2 + \frac{\beta\gamma}{\alpha}},\\ \theta'(x_0) &= \sqrt{\beta^2 + \frac{\beta\gamma}{\alpha}},\\ \theta''(x_0) &= \frac{4\beta^2(2\alpha^2\beta^2 + 3\alpha\beta\gamma + \gamma^2)}{\alpha^2\theta^3(x_0)}.\end{aligned}$$

Since

$$q''(x_0) = \frac{16\beta^3\gamma(\alpha\beta + \gamma)}{\alpha^2\theta^3(x_0)(2\beta + \theta(x_0))^2} > 0,$$

we have

$$\min_{x\in\mathbb{R}} q(x) = q(x_0) = \frac{\sqrt{\alpha\beta} - \sqrt{\alpha\beta + \gamma}}{\sqrt{\alpha\beta} + \sqrt{\alpha\beta + \gamma}},$$

thus concluding the proof. □

**Theorem 3.2.** *Suppose that $F_n(x)$ is the cumulative distribution function of the random variable $A_n$, then*

$$\left| F_n\left(\frac{x - \tilde{\mu}n}{\tilde{\sigma}\sqrt{n}}\right) - \Phi(x) \right| = O\left(\frac{1}{\sqrt{n}}\right),$$

*uniformly with respect to $x$, $x \in \mathbb{R}$.*

*Proof.* The logarithm of the moment-generating function equals

$$\begin{aligned}\log M_n(s) = \underbrace{n}_{:=\phi(n)} \underbrace{\log \frac{\alpha e^s + \beta + \theta(s)}{\alpha + \beta + \theta_0}}_{:=u(s)} + \underbrace{\log \frac{\alpha e^s + \beta + \theta(s)}{\alpha + \beta + \theta_0} + \log \frac{\theta_0}{\theta(s)}}_{:=v(s)} \\ + \underbrace{\log\left(1 - \left(\frac{\alpha e^s + \beta - \theta(s)}{\alpha e^s + \beta + \theta(s)}\right)^{n+1}\right)}_{:=O\left(\left(\frac{\alpha e^s + \beta - \theta(s)}{\alpha e^s + \beta + \theta(s)}\right)^{n+1}\right)} \underbrace{- \log(1 - h^{n+1})}_{:=O(h^{n+1})}.\end{aligned}$$

By Lemma 3.1,

$$\left|\frac{\alpha e^s + \beta - \theta(s)}{\alpha e^s + \beta + \theta(s)}\right| \leqslant \underbrace{\left|\frac{\sqrt{\alpha\beta} - \sqrt{\alpha\beta + \gamma}}{\sqrt{\alpha\beta} + \sqrt{\alpha\beta + \gamma}}\right|}_{:=r} < 1.$$

Thus,

$$M_n(s) = \exp\left(u(s)\phi(n) + v(s)\right)\left(1 + O\left(\kappa_n^{-1}\right)\right).$$

Here $\kappa_n = (\max(r,h))^{n+1}$. Note that the functions $u(s)$, $v(s)$, $\phi(n)$ and $\kappa_n$ satisfy the conditions (i)-(iii). Indeed,

$$u'(s) = \frac{\alpha e^s + \theta'(s)}{\alpha e^s + \beta + \theta(s)},$$
$$u''(s) = \frac{(\alpha e^s + \theta''(s))(\alpha e^s + \beta + \theta(s)) - (\alpha e^s + \theta'(s))^2}{(1 + e^s + \theta(s)^2}.$$

Hence,

$$\begin{aligned} u'(0) &= \frac{\alpha^2 + \alpha\beta + 2\gamma + \alpha\theta_0}{(\alpha + \beta + \theta_0)\theta_0} = \tilde{\mu}, \\ u''(0) &= \frac{\alpha + \theta_0''}{\alpha + \beta + \theta_0} - \frac{(\alpha + \theta_0')^2}{(\alpha + \beta + \theta_0)^2} = \tilde{\sigma}^2 > 0, \end{aligned} \tag{24}$$

yielding us, by (21) and (23), the statement of the theorem. □

## 4 Local limit theorem

We will use Bender's general local limit theorem [9], based on the nature of the generating function (6)-(7).

**Theorem 4.1.** (Bender) *Let $f(z,w)$ have a power series expansion*

$$f(z,w) = \sum_{n,k \geqslant 0} u_{n,k} z^n w^k \tag{25}$$

*with non-negative coefficients and let $a < b$ be real numbers. Define*

$$R(\varepsilon) = \{z : a \leqslant \Re z \leqslant b, \quad |\Im z| \leqslant \varepsilon\}. \tag{26}$$

*Suppose there exists $\varepsilon > 0, \delta > 0$, a non-negative integer $m$, and functions $A(s)$, $r(s)$ such that*

(i) *an $A(s)$ is continuous and non-zero for $s \in R(\varepsilon)$,*

(ii) *an $r(s)$ is non-zero and has a bounded third derivative for $s \in R(\varepsilon)$,*

(iii) *for $s \in R(\varepsilon)$ and $|z| \leqslant |r(s)|(1+\delta)$ function*

$$\left(1 - \frac{z}{r(s)}\right)^m f(z, e^s) - \frac{A(s)}{1 - z/r(s)} \tag{27}$$

*is analytic and bounded,*

(iv) $(r'(t)/r(t))^2 - r''(t)/r(t) \neq 0$ *for* $a \leqslant t \leqslant b$,

(v) $f(z, e^s)$ *is analytic and bounded for*

$$|z| \leqslant |r(\Re s)|(1+\delta), \qquad \varepsilon \leqslant |\Im s| \leqslant \pi.$$

*Then we have*

$$u_{n,k} \sim \frac{n^m e^{-tk} A(t)}{m! r^n(t) \vartheta_t \sqrt{2\pi n}} \tag{28}$$

*uniformly for* $a \leqslant t \leqslant b$, *where*

$$\frac{k}{n} = -\frac{r'(t)}{r(t)}, \qquad \vartheta_t^2 = \left(\frac{k}{n}\right)^2 - \frac{r''(t)}{r(t)}. \tag{29}$$

The following local limit theorem specifies an asymptotic for the weighted Delannoy numbers.

**Theorem 4.2.** *Let* $\tilde{\mu}_n = \tilde{\mu} n$ *and* $\tilde{\sigma}_n^2 = n\tilde{\sigma}^2$, *then for all* $k$, *such that*

$$|k - \tilde{\mu}_n| = o(\tilde{\sigma}_n^{4/3}), \tag{30}$$

*we have*

$$d_{n,k} \sim \frac{(2\gamma)^{n+1}}{\theta_0(\theta_0 - \alpha - \beta)^{n+1}} \varphi_{\tilde{\mu}_n, \tilde{\sigma}_n}(k). \tag{31}$$

*Proof.* By Lemma 2.1, the generating function of the weighted Delannoy numbers equals

$$f(z, e^s) = \frac{1}{1 - \beta z - \alpha z e^s - \gamma z^2 e^s}. \tag{32}$$

Let $r(s)$ be a root of the denominator in (32),

$$\begin{aligned} 1 - (\beta + \alpha e^s) z - \gamma e^s z^2 &= -\gamma e^s \left( z^2 + \frac{\beta + \alpha e^s}{\gamma e^s} z - \frac{1}{\gamma e^s} \right) \\ &= -\gamma e^s (z - z_1)(z - z_2). \end{aligned}$$

The function has two roots,

$$z_1 = r_1(s) = -\frac{\alpha e^s + \beta + \theta(s)}{2\gamma e^s}, \qquad z_2 = r_2(s) = -\frac{\alpha e^s + \beta - \theta(s)}{2\gamma e^s}. \tag{33}$$

Using (18), we calculate derivatives,

$$\begin{aligned} r_1'(s) &= \frac{\beta - \theta'(s) + \theta(s)}{2\gamma e^s} = \frac{\beta\theta(s) + (\alpha\beta + 2\gamma)e^s + \beta^2}{2\gamma e^s \theta(s)}, \\ r_2'(s) &= \frac{\beta + \theta'(s) - \theta(s)}{2\gamma e^s} = \frac{\beta\theta(s) - (\alpha\beta + 2\gamma)e^s - \beta^2}{2\gamma e^s \theta(s)}. \end{aligned}$$

By Theorem 1 (Bender, in [9]), the mean of the random variable $A_n$ is

$$\mu_n = n\mu, \qquad \mu = -r'(0)/r(0).$$

Let $r(s) = r_2(s)$. Now we have

$$\begin{aligned}\frac{r'(s)}{r(s)} &= \frac{\beta^2 + (\alpha\beta + 2\gamma)e^s - \beta\theta(s)}{(\alpha e^s + \beta - \theta(s))\theta(s)},\\ \frac{r'(0)}{r(0)} &= \frac{\beta^2 + \alpha\beta + 2\gamma - \beta\theta_0}{(\alpha + \beta - \theta_0)\theta_0} = -\underbrace{\frac{(\alpha + \beta + \theta_0)\alpha + 2\gamma}{(\alpha + \beta + \theta_0)\theta_0}}_{=\tilde{\mu}},\end{aligned} \tag{34}$$

and

$$\begin{aligned}\frac{r''(s)}{r(s)} &= -\frac{\beta^2\theta(s) + \beta^2\theta'(s) - \beta\theta^2(s) + (\alpha\beta + 2\gamma)e^s\theta'(s)}{(\alpha e^s + \beta - \theta(s))\theta^2(s)}\\ &= \frac{\alpha^2(\alpha\beta + 2\gamma)e^{3s} + (2\alpha^2\beta^2 + (\alpha\beta + 2\gamma)^2)e^{2s}}{(\theta(s) - \alpha e^s - \beta)\theta^3(s)}\\ &\quad + \frac{3\beta^2(\alpha\beta + 2\gamma)e^s + \beta^4 - \beta\theta^3(s)}{(\theta(s) - \alpha e^s - \beta)\theta^3(s)}.\end{aligned} \tag{35}$$

Thus,

$$\frac{r''(0)}{r(0)} = \frac{\alpha^2(\alpha\beta + 2\gamma) + 2\alpha^2\beta^2 + (\alpha\beta + 2\gamma)^2 + 3\beta^2(\alpha\beta + 2\gamma) + \beta^4 - \beta\theta_0^3}{(\theta_0 - \alpha - \beta)\theta_0^3}.$$

Next, consider the function $A(s)$ (cf. (27) in Theorem 4.1) as the limit

$$A(s) = \lim_{z \to r(s)} f(z, e^s)\left(1 - \frac{z}{r(s)}\right)^{m+1}. \tag{36}$$

Here $m + 1$ is the order of the pole. Note that, if the pole is simple, then $m = 0$. Calculating $A(s)$ we receive

$$\begin{aligned}A(s) &= \lim_{z \to r(s)} \frac{1}{1 - \beta z - \alpha z e^s - \gamma z^2 e^s}\left(1 - \frac{z}{r(s)}\right)\\ &= -\frac{1}{\gamma e^s}\lim_{z \to r_2(s)} \frac{r_2(s) - z}{(z - r_1(s))(z - r_2(s))r_2(s)} = \frac{1}{r(s)\theta(s)}.\end{aligned} \tag{37}$$

The function (27)

$$\begin{aligned}\left(1 - \frac{z}{r(s)}\right)^m f(z, e^s) - \frac{A(s)}{1 - z/r(s)} &= \frac{-\theta(s) + \gamma e^s(z - r_1(s))}{e^s(z - r_1(s))(z - r_2(s))\theta(s)}\\ &= \frac{1}{(z - r_1(s))\theta(s)}\end{aligned}$$

is analytic and bounded for

$$|s| < \varepsilon, \qquad |z| < |r(0)| + \delta = \frac{|\alpha + \beta - \theta_0|}{2\gamma} + \delta. \tag{38}$$

Thus, conditions (i)-(iii) and (v) of Theorem 4.1 are satisfied. To verify the condition (iv), we calculate the expression $(r'(t)/r(t))^2 - r''(t)/r(t)$. By (34) and (35), we receive

$$\left(\frac{r'(t)}{r(t)}\right)^2 - \frac{r''(t)}{r(t)} =$$
$$= \underbrace{\frac{2(\alpha\beta+\gamma)e^t(\alpha e^t+\beta)}{(\alpha e^t+\beta-\theta(t))^2\theta^3(t)}}_{>0}((\alpha e^t+\beta)^2 + 2\gamma e^t - \theta(t)(\alpha e^t+\beta)) > 0.$$

Indeed, we obtain

$$\alpha^2 e^{2t} + 2(\alpha\beta+\gamma)e^t + \beta^2 > \theta(t)(\alpha e^t + \beta),$$

since

$$(C+\beta^2)^2 > \underbrace{(C + 2\gamma e^t + \beta^2)}_{=\theta^2(\alpha)}(C - 2\gamma e^t + \beta^2).$$

Here $C = \alpha^2 e^{2t} + 2(\alpha\beta+\gamma)e^t$. We obtain the parameter $t$ by solving the equation

$$\frac{r'(t)}{r(t)} = -\frac{k}{n}. \tag{39}$$

Let $\rho = k/n$. By (34), we get

$$\frac{\beta^2 + (\alpha\beta+2\gamma)e^t - \beta\theta(t)}{(\alpha e^t+\beta-\theta(t))\theta(t)} = -\rho,$$

$$(\beta^2 + (\alpha\beta+2\gamma)e^t - \rho\theta^2(t))^2 = (\beta - \rho(\alpha e^t+\beta))^2\theta^2(t),$$

and

$$-4\gamma e^t(\rho(1-\rho)\alpha^2e^{2t} - ((1-2\rho+2\rho^2)\alpha\beta + (1-4\rho+4\rho^2)\gamma)e^t + \rho(1-\rho)\beta^2) = 0.$$

Thus,

$$\tau = e^t = \frac{(1-2H)\alpha\beta + (1-4H)\gamma + (2\rho-1)\sqrt{\alpha\beta+\gamma}\sqrt{\alpha\beta+\gamma(1-4H)}}{2\alpha^2 H}. \tag{40}$$

Here $H = \rho - \rho^2$. Note that $H \leqslant 1/4$.

Next, combining (29), (35) and (37), we get

$$
\begin{aligned}
&d_{n,k} \sim \\
&\sim \frac{(2\pi n)^{-1/2} e^{-kt}}{r^{n+1}(t)\theta(t)\sqrt{\frac{k^2}{n^2} - \frac{\alpha^2(\alpha\beta+2\gamma)e^{3t}+(2\alpha^2\beta^2+(\alpha\beta+2\gamma)^2)e^{2t}+3\beta^2(\alpha\beta+2\gamma)e^t+\beta^4-\beta\theta^3(t)}{(\theta(t)-\alpha e^t-\beta)\theta^3(t)}}} \\
&= \frac{(2\pi n)^{-1/2}(2\gamma)^{n+1}\tau^{n-k+1}(\theta(\log\tau)-\alpha\tau-\beta)^{-n-1}}{\sqrt{\frac{k^2}{n^2}\theta^2(\log\tau) - \frac{\alpha^2(\alpha\beta+2\gamma)\tau^3+(2\alpha^2\beta^2+(\alpha\beta+2\gamma)^2)\tau^2+3\beta^2(\alpha\beta+2\gamma)\tau+\beta^4-\beta\theta^3(\log\tau)}{(\theta(\log\tau)-\alpha\tau-\beta)\theta(\log\tau)}}} \\
&= \frac{\gamma\sqrt{2\theta_0}(2\gamma)^n}{\sqrt{\pi n}(\theta_0-\alpha-\beta)^{n+1}\sqrt{(\alpha+\beta)(\alpha\beta+\gamma)}} \underbrace{\frac{(\theta_0-\alpha-\beta)^n\tau^{n-k}}{(\theta(\log\tau)-\alpha\tau-\beta))^n}}_{:=\delta_{n,k}} \Theta_{n,k}.
\end{aligned}
\tag{41}
$$

Here

$$
\begin{aligned}
&\Theta_{n,k} = \\
&= \frac{(\theta_0-\alpha-\beta)(\theta(\log\tau)-\alpha\tau-\beta)^{-1}\sqrt{(\alpha+\beta)(\alpha\beta+\gamma)\theta_0^{-1}\tau}}{\sqrt{\frac{k^2}{n^2}\theta^2(\log\tau) - \frac{\alpha^2(\alpha\beta+2\gamma)\tau^3+(2\alpha^2\beta^2+(\alpha\beta+2\gamma)^2)\tau^2+3\beta^2(\alpha\beta+2\gamma)\tau+\beta^4-\beta\theta^3(\log\tau)}{(\theta(\log\tau)-\alpha\tau-\beta)\theta(\log\tau)}}}.
\end{aligned}
\tag{42}
$$

Note that by (21) and (30), we have

$$
\left|\frac{k}{n} - \tilde{\mu}\right| = o\left(\frac{1}{\sqrt[3]{n}}\right), \tag{43}
$$

hence $k/n \to \tilde{\mu}$ and $\tau \to 1$, while $n \to \infty$. Indeed,

$$
\lim_{\rho\to\tilde{\mu}} H = \frac{2((\alpha+\beta)(\alpha+\beta+\theta_0)+2\gamma)(\alpha\beta+\gamma)}{(\alpha+\beta+\theta_0)^2\theta_0^2} = \frac{\alpha\beta+\gamma}{\theta_0^2}. \tag{44}
$$

By (40), we get

$$
\begin{aligned}
\lim_{\rho\to\tilde{\mu}} \tau &= \frac{(\alpha\beta+\gamma) - 2(\alpha\beta+2\gamma)H + (2\tilde{\mu}-1)\sqrt{(\alpha\beta+\gamma)^2 - 4\gamma(\alpha\beta+\gamma)H}}{2\alpha^2 H} \\
&= \frac{\theta_0^2 - 2(\alpha\beta+2\gamma) + \theta_0(2\tilde{\mu}-1)\sqrt{\theta_0^2-4\gamma}}{2\alpha^2}.
\end{aligned}
\tag{45}
$$

Noticing that $\sqrt{\theta_0^2-4\gamma} = \alpha+\beta$, we receive, that $\tau \to 1$, while $n \to \infty$. Thus (cf. (42)), $\Theta_{n,k} \to 1$.

Next, let us denote

$$x = \frac{k - \tilde{\mu}_n}{\tilde{\sigma}_n}. \tag{46}$$

By (21), we have

$$\rho = \frac{k}{n} = \tilde{\mu} + \frac{x\tilde{\sigma}}{\sqrt{n}}, \tag{47}$$

and, by (43), we get

$$|x| = o(\sqrt[6]{n}). \tag{48}$$

Calculating the logarithm of $\delta_{n,k}$ (cf. (41)), we receive

$$\log \delta_{n,k} = n \log(\theta_0 - \alpha - \beta) + n\,(1 - \rho) \log \tau - n \log(\theta(\log \tau) - \alpha\tau - \beta). \tag{49}$$

Using Taylor series expansions and noticing that

$$\tilde{\mu} - \tilde{\mu}^2 = \frac{\alpha\beta + \gamma}{\theta_0^2}, \qquad 1 - 2\tilde{\mu} = -\frac{\alpha - \beta}{\theta_0},$$

we obtain for $n$ large enough,

$$\begin{aligned} \tau &= 1 + \frac{x}{\tilde{\sigma}\sqrt{n}} + \frac{c_1 x^2}{n} + O\left(\frac{x^3}{n\sqrt{n}}\right), \\ \log \tau &= \frac{x}{\tilde{\sigma}\sqrt{n}} + \frac{c_2 x^2}{n} + O\left(\frac{x^3}{n\sqrt{n}}\right), \\ \theta(\log \tau) &= \theta_0 \left(1 + \frac{b_2 x}{\sqrt{n}} + \frac{c_3 x^2}{n} + O\left(\frac{x^3}{n\sqrt{n}}\right)\right). \end{aligned} \tag{50}$$

Here

$$\begin{aligned} c_1 &= \frac{(\alpha+\beta)(\alpha^2\theta_0^2 + (\alpha\beta+\gamma)\theta_0^2 + 2\beta\gamma(\alpha+\beta)) + 8\beta\gamma^2}{(\alpha+\beta)^2(\alpha\beta+\gamma)\theta_0}, \\ c_2 &= c_1 - \frac{1}{2\tilde{\sigma}^2} = \frac{((\alpha+\beta)^2 - 2\gamma)(\alpha-\beta)\theta_0}{2(\alpha\beta+\gamma)(\alpha+\beta)^2} \\ c_3 &= \frac{(\alpha^2 + 2\alpha^2 c_1\tilde{\sigma}^2 + (2\alpha\beta + 4\gamma)c_1\tilde{\sigma}^2)\theta_0^2 - (\alpha^2 + \alpha\beta + 2\gamma)^2}{2\theta_0^4\tilde{\sigma}^2}, \\ b_2 &= \frac{\alpha^2 + \alpha\beta + 2\gamma}{\theta_0^2\tilde{\sigma}}. \end{aligned}$$

Substituting (50) into (49), we get

$$\begin{aligned}
\log \delta_{n,k} &= n\log(\theta_0-\alpha-\beta)\\
&+n(1-\tilde{\mu})\left(1-\frac{x\tilde{\sigma}}{(1-\tilde{\mu})\sqrt{n}}\right)\left(\frac{x}{\tilde{\sigma}\sqrt{n}}+\frac{c_2x^2}{n}+O\left(\frac{x^3}{n\sqrt{n}}\right)\right)\\
&-n\log\left((\theta_0-\alpha-\beta)+\frac{(\theta_0b_2\tilde{\sigma}-\alpha)x}{\tilde{\sigma}\sqrt{n}}+\frac{(\theta_0c_3-\alpha c_1)x^2}{n}+O\left(\frac{x^3}{n\sqrt{n}}\right)\right)\\
&=n\left(\frac{(1-\tilde{\mu})x}{\tilde{\sigma}\sqrt{n}}+\frac{((1-\tilde{\mu})c_2-1)x^2}{n}+O\left(\frac{x^3}{n\sqrt{n}}\right)\right)\\
&-n\log\left(1+\frac{(\theta_0b_2\tilde{\sigma}-\alpha)x}{(\theta_0-\alpha-\beta)\tilde{\sigma}\sqrt{n}}+\frac{(\theta_0c_3-\alpha c_1)x^2}{(\theta_0-\alpha-\beta)n}+O\left(\frac{x^3}{n\sqrt{n}}\right)\right)\\
&=n\Bigg[\underbrace{\left(1-\tilde{\mu}-\frac{\theta_0b_2\tilde{\sigma}-\alpha}{\theta_0-\alpha-\beta}\right)}_{=0}\frac{x}{\tilde{\sigma}\sqrt{n}}\\
&+\underbrace{\left((1-\tilde{\mu})c_2-1-\frac{\theta_0c_3-\alpha c_1}{\theta_0-\alpha-\beta}+\frac{(\theta_0b_2\tilde{\sigma}-\alpha)^2}{2(\theta_0-\alpha-\beta)\tilde{\sigma}^2}\right)}_{=-1/2}\frac{x^2}{n}+O\left(\frac{x^3}{n\sqrt{n}}\right)\Bigg]\\
&=-\frac{x^2}{2}+o(1),
\end{aligned} \tag{51}$$

which, combined with (41) and (48), yields us the statement of the theorem. □

**Acknowledgements.** The author would like to thank the anonymous reviewer for careful reading of the manuscript and providing constructive comments and suggestions, which have helped improve the paper's quality.

## References

[1] S. Amrouche, H. Belbachir, *Unimodality and linear recurrences associated with rays in the Delannoy triangle*, Turk. J. Math., 44, (2020), 118 130. DOI: 10.3906/mat-1811-109

[2] S. Amrouche, H. Belbachir, J. L. Ramrez, *Unimodality, linear recurrences and combinatorial properties associated to rays in the generalized Delannoy matrix*, J. Differ. Equ., 25 (8), (2019), 1200-1215, DOI: 10.1080/10236198.2019.1662413

[3] P. Barry, *Generalized Catalan Numbers Associated with a Family of Pascal-like Triangles*, J. Integ. Seq., Vol. 22 (5), Article 19.5.8 (2019), [1–50].

[4] I. Belovas, *A central limit theorem for coefficients of the modified Borwein method for the calculation of the Riemann zeta-function*, Lith. Math. J., vol. 59(1) (2019), 17–23. DOI: 10.1007/s10986-019-09421-4

[5] I. Belovas, *A local limit theorem for coefficients of modified Borwein's method*, Glas. Mat. Ser. III, Vol. 54 (1) (2019), [1–9]. DOI: 10.3336/gm.54.1.01

[6] I. Belovas, *Central and local limit theorems for numbers of the Tribonacci triangle*, Mathematics, vol. 9, no. 8, art. no. 880 (2021), [1-11]. DOI: 10.3390/math9080880

[7] I. Belovas, M. Sabaliauskas, *Series with binomial-like coefficients for the evaluation and 3D visualization of zeta functions*, Informatica, vol. 31(4), (2020), 659–680. DOI: 10.15388/20-INFOR434

[8] I. Belovas, L. Sakalauskas, *Limit theorems for the coefficients of the modified Borwein method for the calculation of the Riemann zeta-function values*, Colloq. Math., vol. 151, no. 2 (2018), 217–227. DOI: 10.4064/cm7086-2-2017

[9] E. A. Bender, *Central and local limit theorems applied to asymptotic enumeration*, Journal of combinatorial theory (A), **15** (1973), 91-111.

[10] K.-W. Chen, *Delannoy Numbers and Preferential Arrangements*, Mathematics, vol. 7, no. 3, art. no. 238 (2019), 1–9. DOI: 10.3390/math7030238

[11] C. Cobeli, A. Zaharescu, *Promenade around Pascal Triangle Number Motives*, Bull. Math. Soc. Sci. Math. Roumanie, Vol. 56 (1), (2013), 73–98.

[12] H.-K. Hwang, *On Convergence Rates in the Central Limit Theorems for Combinatorial Structures*, European Journal of Combinatorics, **19**(3) (1998), 329–343.

[13] L. Mu, S. Zheng, *On the total positivity of Delannoy-like triangles*, Journal of Integer Sequences, Vol. 20, (2017), Article 17.1.6

[14] R. Noble, *Asymptotics of the weighted Delannoy numbers*, Int. J. Number Theory, 8 (1), (2012), 175–188. DOI: 10.1142/S1793042112500108

[15] Sloane, N. J. A. (ed.), *Square array of Delannoy numbers* $D(i,j)$ $(i \geqslant 0,\ j \geqslant 0)$ *read by antidiagonals. OEIS Number sequence A008288*, The On-Line Encyclopedia of Integer Sequences. https://oeis.org/A008288

[16] Y. Wang, S. Zheng and X. Chen, *Analytic aspects of Delannoy numbers*, Discrete Math., Vol. 342 (8), (2019), 2270–2277. DOI: 10.1016/j.disc.2019.04.003

[17] S. Yang, S. Zheng, S. Yuan, T. He, *Schröder matrix as inverse of Delannoy matrix*, Linear Algebra Appl., 439 (11), (2013), 3605–3614. DOI: 10.1016/j.laa.2013.09.044

Igoris BELOVAS,
Institute of Data Science and Digital Technologies,
Vilnius University,
Akademijos str. 4, LT-04812 Vilnius, Lithuania.
Email: Igoris.Belovas@mif.vu.lt