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A new existence results on fractional differential inclusions with state-dependent delay and Mittag-Leffler kernel in Banach space

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Abstract

In this manuscript the existence of the fractional-order functional differential inclusions [FFDI] with state-dependent delay [SDD] is investigated within the Mittag-Leffler kernel. We use both contractive and condensing maps to prove the existence of mild solutions through solution operator. Finally, an example is presented to illustrate the theoretical findings.

1 Introduction

In this manuscript, we establish the existence of mild solutions of FFDI with SDD and Mittag-Leffler kernel of the form

$$\mathscr{D}^{\vartheta}_{ABC}p(\varsigma) \in Ap(\varsigma) + \mathscr{F}(\varsigma, p_{\sigma(\varsigma,p_\varsigma)}), \quad \varsigma \in [0, \xi] \tag{1.1}$$

$$p(\varsigma) = \varphi(\varsigma) \in \mathscr{B}, \tag{1.2}$$

where $\xi > 0, \vartheta \in (0, 1)$, $A : D(A) \subset E \to E$ is the infinitesimal generator of an ϑ-resolvent family $\widehat{\mathscr{B}}_\vartheta(\varsigma)_{\varsigma\geq 0}$, the solution operator $\mathscr{B}_\vartheta(\varsigma)_{\varsigma\geq 0}$ is described on a complex Banach space E, $\mathscr{D}^{\vartheta}_{ABC}$ is the Atangana-Baleanu-Caputo derivative, $\mathscr{F}$ is a set-valued map and $\sigma : [0, \xi] \times \mathscr{B} \to (-\infty, \xi]$ are given functions that satisfy a later-specified assumption.

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We assume that $p_\varsigma : (-\infty, 0] \to E, p_\varsigma(x) = p(\varsigma + x),\ x \leq 0$, belongs to an abstract phase space $\mathscr{B}$.

Fractional differential equations, general enhancement of classical ordinary and partial differential equations which allows the real (or) even complex number to be its order of differentiation. The main benefits of employing the fractional derivatives is their ability to record their hereditary property of various processes which is mainly because of the non-locality nature of the operators. The hereditary property of fractional derivatives allows one to model the materials with intermediate properties such as viscoelasticity of the substances. The greatest breakthrough in the field of fractional differential equation found in its application in the abundant fields of diverse nature such as circuits in electrical engineering, control theory, statistical modeling, the vibration of earthquake motions and so on. Many excellent monographs were available which provides the essential conceptual tools to explore more hidden applications of fractional differential equations. Similarly, one can locate the distinguish properties of both classical and fractional differential representations, we refer to the monographs [32, 6] and the related research articles of fractional differential systems are given in [1, 4, 5, 33, 28, 7, 23, 25].

In recent times, by noticing the aforementioned peculiar properties and wide applicability of fractional derivatives in various scientific fields, it became handy and more applicable to model the real world problems. The essential thing to note is that the applications and outcomes of fractional derivatives and integrals differ depending on the definitions used, such as Riemann-Liouville, Hadamard, Grunwald Letnikov, Caputo, Riesz-Caputo, Chen, Weyl, Erd Iyi-Kober, and so on. In 2015, Caputo and Fabrizio [12] proposed the recent definition of non local derivatives with non-singular kernel in the non-necessarily Banach space $\mathcal{H}^1$. This concept posed a significant barrier to its implementation, but it rapidly made its way into a variety of fields, including thermal science and mechanical engineering, as well as groundwater research, see [3, 10, 11] for further information, as well as the references therein. A year later, the new definition of non-local derivatives with non singular kernel based on the Mittag-Leffler function was proposed by Atangana and Baleanu. The current concept confirmed the Caputo-Fabrizio's concept which is depends on exponential function. Atanganu-Baleanu definition achieved important applications in various fields established by the togetherness of strong relationship in among fractional calculus and Mittag-Leffler function. For more reference, the readers can see [7, 6, 2, 27].

In [15, 16], Gautam and Dabas proved the existence of mild solutions for fractional integro-differential equations with SDD by using the applicable fixed point theorem. In [31, 13], Das et al. investigated the class of second order partial neutral differential equations with SDD in Banach spaces and proved the existence of mild solution for the considered system along with the help of Hausdorff measures of non-compactness and Darbo fixed point theorem. In [34], the authors researched the abstract fractional integro-differential inclusions with infinite state-dependent delay in Banach spaces. In [36], the authors studied the fractional neutral differential systems with SDD in Banach spaces under non-compactness measure and showed its existence via contractive and condensing maps. In recent times, Aiemene et al. [2] studied the controllability results for fractional order semilinear differential equations with impulses having finite delay in Banach spaces. Recently, Mallika Arjunan et al. [29, 30] analyzed the existence results of various fractional differential systems through A-B derivative under suitable fixed point theorem. To the best of our knowledge, there is no results reported on the existence results for FFDI with SDD and Mittag-Leffler kernel in $\mathscr{B}$ phase space contexts. This encouraged us to investigate the existence results of the system (1.1) -(1.2) with SDD in Banach spaces. The existence findings for (1.1)-(1.2) are presented for the first time in this manuscript.

Now we will move on to a description of the work. We give some fundamental concepts on A-B fractional derivatives, phase space axioms ($\mathscr{B}$), sectorial operator in Section 2. The proof of our main findings and an example are given in Section 3.

2 Preliminaries

This part covers the fundamental definitions and results of the sectorial operator, set-valued mappings, measures of non-compactness [MNC], phase space axioms, and A-B fractional derivative, which will aid us in proving our key conclusions.

Let $(E, \|\cdot\|_E)$ be a complex Banach space. $L(E)$ is the Banach space of all bounded linear operators from X into X with $\|\cdot\|_{L(E)}$ as the corresponding norm.

$\mathscr{C}([0,\xi], E)$ is the Banach space of all continuous functions from $[0,\xi]$ into E with the norm

$$\|p\|_{\mathscr{C}([0,\xi],E)} = \sup\{\|p(\varsigma)\| : \varsigma \in [0,\xi]\}.$$

The functions $p : [0,\xi] \to E$ that are integrable in the Bochner notion with regard to the Lebesgue measure, equipped with

$$\|p\|_1 = \int_0^{\xi} \|p(x)||dx$$

is denoted by $L^1([0,\xi], E)$.

Here, we recall some fundamental definition of Atangnan-Baleanu fractional derivative.

Definition 2.1. *[7] The A-B fractional integral of order $\vartheta \in (0,1)$ of a function $r : (d,\xi) \to \mathbb{R}$ is described by*

$$^{AB}I^{\vartheta}_{d^+}r(\varsigma) = \frac{1-\vartheta}{B(\vartheta)}r(\varsigma) + \frac{\vartheta}{B(\vartheta)\Gamma(\vartheta)}\int_d^{\varsigma}(\varsigma - x)^{\vartheta-1}r(x)dx,$$

where $B(\vartheta) = (1-\vartheta) + \frac{\vartheta}{\Gamma(\vartheta)}$ is the normalising function that fulfills the requirement $B(0) = B(1) = 1$.

Definition 2.2. *[7] For $r \in H^1(d,\xi), d < \xi$, the A-B fractional derivative of order $\vartheta \in (0,1)$ of a function r in Caputo sense is characterized by*

$$^{ABC}\mathscr{D}^{\vartheta}_{d^+}r(\varsigma) = \frac{B(\vartheta)}{1-\vartheta}\int_d^{\varsigma} r'(s)E_{\vartheta}\left(-\frac{\vartheta}{1-\vartheta}(\varsigma - x)^{\vartheta}\right)dx$$

for each $\varsigma \in (d,\xi)$. Here E_{ϑ} is the Mittag-Leffler function.

We recommend readers to refer the following papers to prevent repeats of several definitions used in this paper: sectorial operator [18] and solution operator (see Definition 2.7 in [30]).

We urge that the reader to read [32, 2, 25, 7, 29, 35] for additional detail on this topic and its uses.

2.1 Set-valued maps and MNC

Assume that Θ is a metric space. All through this manuscript, $\mathcal{P}(\Theta)$ denotes a list of all nonempty subsets of Θ, whereas $\mathcal{P}_b(\Theta)$ denotes a list of all bounded nonempty subsets of Θ.

The idea of measure of non-compactness underpins several of our findings. With this purpose, we will remember a some characteristics of this idea next. As for basic information, the reader will refer [9, 14, 24, 34]. We just use Hausdorff measure of non-compactness [HMNC] concept throughout this manuscript.

Definition 2.3 ([9, 14] (HMNC))**.** *Let $\mathcal{U}$ be a family of bounded subset of Θ. Then HMNC is described by*

$$\beta(\mathcal{U}) := \inf\{\delta > 0 : \mathcal{U} = \cup_{i=1}^{k}\mathcal{U}_i \textit{ with } \operatorname{diam}(\mathcal{U}_i) \leq \delta \textit{ for } i = 1,2,\dots,k\}.$$

Lemma 2.1 ([9, 14])**.** *For any bounded sets $\mathcal{U}, \mathcal{U}_1$ and $\mathcal{U}_2$ of Θ, we obtain*

(i) $\beta(\mathcal{U}) = 0$ iff $\mathcal{U}$ is totally bounded;

(ii) $\beta(\mathcal{U}) = \beta(\overline{\mathcal{U}})$, *where* $\overline{\mathcal{U}}$ *means the closure of* $\mathcal{U}$*;*

(iii) *For each* $\mathcal{U}_1 \subset \mathcal{U}_2$ *implies* $\beta(\mathcal{U}_1) \leq \beta(\mathcal{U}_2)$*;*

(iv) $\beta(\mathcal{U}_1 + \mathcal{U}_2) \leq \beta(\mathcal{U}_1) + \beta(\mathcal{U}_2)$*;*

(v) $\beta(\mathcal{U}_1 \cup \mathcal{U}_2) = \max\{\beta(\mathcal{U}_1), \beta(\mathcal{U}_2)\}$*;*

(vi) $\beta(\lambda\mathcal{U}) = |\lambda|\beta(\mathcal{U})$ *for any* $\lambda \in \mathbb{R}$.

(vii) $\beta(\mathcal{U}) = \beta(\overline{co}(\mathcal{U}))$.

Let $\mathcal{Y}$ be a normed space. In order to signify the subsequent set listing, we now utilizing the terminology $\nu(\mathcal{Y})$ and $\mathcal{M}\nu(\mathcal{Y})$:

(i) $\nu(\mathcal{Y}) = \{\mathcal{D} \in \mathcal{P}(\mathcal{Y}) : \mathcal{D} \text{ is convex }\}$;

(ii) $\mathcal{M}\nu(\mathcal{Y}) = \{\mathcal{D} \in \nu(\mathcal{Y}) : \mathcal{D} \text{ is compact}\}$.

Definition 2.4. *A condensing map with regard to* η *(abbreviated,* η*-condensing) is a set-valued map* $\Upsilon : \Theta \to \mathcal{P}(\mathcal{Y})$ *if for any bounded set* $\mathcal{D} \subset \Theta, \eta(\mathcal{D}) > 0, \eta(\Upsilon(\mathcal{D})) < \eta(\mathcal{D})$.

Remark 2.1. *We can see that* Υ *is closed if* $\Upsilon : \Theta \to \mathcal{M}\nu(\mathcal{Y})$ *is u.s.c.*

Theorem 2.1 ([Corollary 3.3.1], [24])**.** *Suppose* $\Upsilon : \mathcal{N} \to \mathcal{M}\nu(\mathcal{N})$ *is a upper semi-continuous* β*-condensing multivalued map, then* $\mathrm{Fix}(\Upsilon) = \{z \in \Upsilon(z)\}$ *is a nonempty compact set, where* $\mathcal{N}$ *be a convex closed subset of* $\mathcal{Y}$.

Now, we collect some measure β properties from[34, Lemma 2.1-2.4] that will be used to verify our key findings.

We call a set $\Omega \subseteq L^1([0,\xi], E)$ is uniformly integrable if $\gamma > 0 \in L^1([0,\xi])$ in a way that $\|p(\varsigma)\| \leq \gamma(\varsigma)$ a.e. for $\varsigma \in [0,\xi]$ and all $p \in \Omega$.

2.2 Phase space axioms

To employ delay criteria, we must first establish the phase space axioms $\mathscr{B}$ introduced by Hale and Kato in [20] and utilize the terminology used in [22]. As a result, $(\mathscr{B}, \|\cdot\|_{\mathscr{B}})$ is a semi-normed linear space of functions mapping $(-\infty, 0]$ into E and satisfying the axioms below.

If $p :]-\infty, \xi] \to E, \xi > 0$, is such that $p_0 \in \mathscr{B}$, then for all $\varsigma \in [0,\xi]$, the subsequent assumptions hold:

(C1) $p_\varsigma \in \mathscr{B}$,

(C2) $\|p_\varsigma\|_{\mathscr{B}} \leq Q_1(\varsigma) \sup_{0\leq x\leq\varsigma} \|p(x)\| + Q_2(\varsigma)\|p_0\|_{\mathscr{B}}$,

(C3) $\|p(\varsigma)\| \leq \overline{W}\|p_\varsigma\|_{\mathscr{B}}$, where $\overline{W} > 0$ is a constant and $Q_1 : [0,\infty) \to [0,\infty)$ is continuous, $Q_2 : [0,\infty) \to [0,\infty)$ is locally bounded, and Q_1, Q_2 are independent of $p(\cdot)$. Furthermore, $\|\varphi(0)\| \leq \overline{W}\|\varphi\|_{\mathscr{B}}$ for every $\varphi \in \mathscr{B}$.

(C4) p_ς is a $\mathscr{B}$-valued continuous function on $[0,\xi]$ and $\mathscr{B}$ is complete. For more details, see [19].

Now, we define the space

$$Y_\varphi([0,\xi], E) = \{\kappa \in \mathscr{C}([0,\xi], E) : \kappa(0) = \varphi(0)\}.$$

For more details on phase space axioms and its examples, we suggest the reader to refer [22, 19, 26].

3 Main results

The existence findings for the inclusions (1.1)-(1.2) under contractive and condensing maps are presented and proved in this part.

We begin by imposing some appropriate constraints on the set-valued map $\mathscr{F}$.

$\mathscr{F}(i)$ For any $\omega \in \mathscr{B}$, the function $\mathscr{F}(\cdot,\omega) : [0,\xi] \to \mathcal{M}\nu(E)$ permits a strongly measurable selection.

$\mathscr{F}(ii)$ The function $\mathscr{F}(\varsigma,\cdot) : \mathscr{B} \to \mathcal{M}\nu(E)$ is upper semi-continuous for every $\varsigma \in [0,\xi]$.

$\mathscr{F}(iii)$ We can find a function $\gamma \in L^1([0,\xi])$ in a way that

$$\|\mathscr{F}(\varsigma,\omega)\| := \sup\{\|f\| : f \in \mathscr{F}(\varsigma,\omega)\} \leq \gamma(\varsigma)\Phi(\|\omega\|_{\mathscr{B}}), \quad \text{a.e. } \varsigma \in [0,\xi],$$

where $\Phi : [0,\infty) \to [0,\infty)$ is continuous non-decreasing function.

In addition, we suppose that $\sigma : [0,\xi]\times\mathscr{B} \to [0,\infty)$ is a continuous function in a way that $\sigma(\varsigma,\omega) \leq \varsigma$ for all $\varsigma \geq 0$ and $\omega \in \mathscr{B}$.

Remark 3.1. *In view of $\mathscr{F}(i)$ and $\mathscr{F}(ii)$, we notice that the set*

$$\Upsilon_{\mathscr{F},\sigma,\kappa} = \{p \in L^1([0,\xi], E) : p(\varsigma) \in \mathscr{F}(\varsigma, \kappa_{\sigma(\varsigma,\kappa_\varsigma)})\} \neq \emptyset,$$

and $\Upsilon_{\mathscr{F},\sigma,\kappa}$ is convex.

We can now describe the mild solution for the inclusions (1.1)-(1.2).

Definition 3.1. *A function $\kappa:(-\infty,\xi]\to E$ is called a mild solution of the inclusions* (1.1)-(1.2) *if the subsequent holds: $p_0=\varphi\in\mathscr{B}$ on $(-\infty,0]$, the restriction of $\kappa(\cdot)$ to the interval $[0,\xi]$ is continuous and satisfies the following integral equation:*

$$\kappa(\varsigma)=\begin{cases}\varphi(\varsigma), \quad \varsigma\in(-\infty,0]\\ \mathbb{S}\mathscr{B}_\vartheta(\varsigma)\varphi(0)+\dfrac{\mathbb{S}\mathbb{T}(1-\vartheta)}{B(\vartheta)\Gamma(\vartheta)}\displaystyle\int_0^\varsigma(\varsigma-s)^{\vartheta-1}p(s)ds\\ +\dfrac{\vartheta\mathbb{S}^2}{B(\vartheta)}\displaystyle\int_0^\varsigma\widehat{\mathscr{B}}_\vartheta(\varsigma-s)p(s)ds, \quad p\in\Upsilon_{\mathscr{F},\sigma,\kappa}, \text{ and all } \varsigma\in[0,\xi],\end{cases} \tag{3.1}$$

where $\mathbb{S}=\zeta(\zeta I-A)^{-1}$ and $\mathbb{T}=-\widetilde{\gamma}A(\zeta I-A)^{-1}$ with $\zeta=\frac{B(\vartheta)}{1-\vartheta}$, $\widetilde{\gamma}=\frac{\vartheta}{1-\vartheta}$ and

$$\mathscr{B}_\vartheta(\varsigma)=E_\vartheta(-\mathbb{T}\varsigma^\vartheta)=\frac{1}{2\pi i}\int_\Gamma e^{x\varsigma}x^{\vartheta-1}(x^\vartheta I-\mathbb{T})^{-1}dx, \tag{3.2}$$

$$\widehat{\mathscr{B}}_\vartheta(\varsigma)=\varsigma^{\vartheta-1}E_{\vartheta,\vartheta}(-\mathbb{T}\varsigma^\vartheta)=\frac{1}{2\pi i}\int_\Gamma e^{x\varsigma}(x^\vartheta I-\mathbb{T})^{-1}dx, \tag{3.3}$$

Γ *denotes the Bromwich path [8].*

Remark 3.2. *We must first establish the operator estimates described in* (3.2) *and* (3.3) *before we can present and prove the major conclusions of this section.*

If $\vartheta\in(0,1)$ and $A\in\mathscr{A}^\vartheta(\widetilde{\alpha}_0,\omega_0)$, then for any $p\in E$ and $\varsigma>0$, we have $\|\mathscr{B}_\vartheta(\varsigma)\|\le\widehat{\Lambda}e^{\omega\varsigma}$ and $\|\widehat{\mathscr{B}}_\vartheta(\varsigma)\|\le Ce^{\omega\varsigma}(1+\varsigma^{\vartheta-1})$, for every $\varsigma>0,\omega>\omega_0$. Hence, we get $\|\mathscr{B}_\vartheta(\varsigma)\|\le\widehat{M}_{\mathscr{B}}$ and $\|\widehat{\mathscr{B}}_\vartheta(\varsigma)\|\le\varsigma^{\vartheta-1}\widehat{M}_{\widehat{\mathscr{B}}}$. Since $\widehat{M}_{\mathscr{B}}=\sup_{0\le\varsigma\le\xi}\|\mathscr{B}_\vartheta(\varsigma)\|$ and $\widehat{M}_{\widehat{\mathscr{B}}}=\sup_{0\le\varsigma\le\xi}Ce^{\omega\varsigma}(1+\varsigma^{1-\vartheta})$. For additional details, see [35, 25, 18].

To establish our findings, we must examine an integral operator given on the set $\Upsilon_{\mathscr{F},\sigma,\kappa}$ for functions $\kappa\in Y_\varphi([0,\xi],E)$. We begin by discussing the characteristics of $\Upsilon_{\mathscr{F},\sigma,\kappa}$. The first outcome determines that $\Upsilon_{\mathscr{F},\sigma,\kappa}$ is closed. In specific, we recall the property from [34, Lemma 3.1].

Just from the other side, as a result of $\mathscr{F}(iii)$, the set $\Upsilon_{\mathscr{F},\sigma,\kappa}$ is uniformly integrable on $[0,\xi]$, i.e., we can find a function $\gamma_{\sigma,\kappa}>0\in L^1([0,\xi])$ in ways that $\|p(\varsigma)\|\le\gamma_{\sigma,\kappa}(\varsigma)$ a.e. for $\varsigma\in[0,\xi]$ and all $p\in\Upsilon_{\mathscr{F},\sigma,\kappa}$.

We introduce now the operator $\overline{\Upsilon}:L^1([0,\xi],E)\to\mathscr{C}([0,\xi],E)$ given by

$$\overline{\Upsilon}p(\varsigma)=\frac{\mathbb{S}\mathbb{T}(1-\vartheta)}{B(\vartheta)\Gamma(\vartheta)}\int_0^\varsigma(\varsigma-s)^{\vartheta-1}p(s)ds+\frac{\vartheta\mathbb{S}^2}{B(\vartheta)}\int_0^\varsigma\widehat{\mathscr{B}}_\vartheta(\varsigma-s)p(s)ds.$$

This is obvious that $\overline{\Upsilon}$ is a bounded linear operator. With the use of $\overline{\Upsilon}$, we may develop the set-valued map $\widetilde{\Upsilon}:Y_\varphi([0,\xi],E)\to\nu(\mathscr{C}([0,\xi];E))$ given by

$$\widetilde{\Upsilon}(\kappa) = \overline{\Upsilon}\left(\Upsilon_{\mathscr{F},\sigma,\kappa}\right).$$

Based on the above discussions, now, we state the following crucial lemma from [34, Lemma 3.2].

Lemma 3.1. *Consider a set-valued map $\mathscr{F} : [0,\xi] \times \mathscr{B} \to \mathcal{M}\nu(E)$ and suppose $\mathscr{F}$ satisfies $\mathscr{F}(i)$–$\mathscr{F}(iii)$. Then $\widetilde{\Upsilon}$ is a upper semi-continuous map with convex compact values.*

First, we characterize the solution map for the system (1.1)-(1.2) as below.

Let $\kappa \in Y_\varphi([0,\xi], E)$. For simplicity, we described $\kappa(\cdot)$ with its extension to $(-\infty, \xi]$ supplied by $\kappa(y) = \varphi(y)$ for all $y \leq 0$. Utilizing this terminology, $\Upsilon^*(\xi)$ is defined as the set generated by all z functions given by

$$\begin{aligned} z(\varsigma) = \mathbb{S}\mathscr{B}_\vartheta(\varsigma)\varphi(0) &+ \frac{\mathbb{S}\mathbb{T}(1-\vartheta)}{B(\vartheta)\Gamma(\vartheta)} \int_0^\varsigma (\varsigma - s)^{\vartheta-1} p(s) ds \\ &+ \frac{\vartheta \mathbb{S}^2}{B(\vartheta)} \int_0^\varsigma \widehat{\mathscr{B}}_\vartheta(\varsigma - s) p(s) ds, \quad p \in \Upsilon_{\mathscr{F},\sigma,\kappa}. \end{aligned} \tag{3.4}$$

Through our assumptions, it indicates that $z \in Y_\varphi([0,\xi], E)$. Hence, $\Upsilon^* : Y_\varphi([0,\xi], E) \to \mathcal{P}\left(Y_\varphi([0,\xi], E)\right)$. Moreover, it is obvious that $\kappa(\cdot)$ is a mild solution of the inclusions (1.1)-(1.2) if and only if $\kappa(\cdot)$ is a fixed point of Υ^*.

Our existence result is based on the contractive maps. $\mathcal{P}_{cb}(E)$ denotes the listing of closed bounded subsets of E and by $d_{\mathcal{H}}$ the Hausdorff metric in $\mathcal{P}_{cb}(E)$.

Theorem 3.1. *Assume that assumptions $\mathscr{F}(i), \mathscr{F}(iii)$ are satisfied. Further, suppose that the subsequent assumptions hold:*

(A1) We can find a constant $\mathscr{L}_{\mathscr{F}} > 0$ in a way that

$$d_{\mathcal{H}}\left(\mathscr{F}\left(\varsigma, \omega_1\right), \mathscr{F}\left(\varsigma, \omega_2\right)\right) \leq \mathscr{L}_{\mathscr{F}} \left\|\omega_1 - \omega_2\right\|_{\mathscr{B}}$$

for all $\varsigma \in [0,\xi]$ and all $\omega_1, \omega_2 \in \mathscr{B}$.

(A2) We can find a positive function $\widetilde{\sigma} \in L^1([0,\xi])$ and every $Q > 0$, there exists a positive constant $\widetilde{\mathscr{L}_{\mathscr{F}}}(Q) > 0$ in a way that

$$d_{\mathcal{H}}\left(\mathscr{F}\left(\varsigma, \kappa_{\varsigma_2}\right), \mathscr{F}\left(\varsigma, \kappa_{\varsigma_1}\right)\right) \leq \widetilde{\mathscr{L}_{\mathscr{F}}}(q) \widetilde{\sigma}(\varsigma) \left|\varsigma_2 - \varsigma_1\right|,$$

for all $\varsigma_1, \varsigma_2 \in [0,\xi]$ and $\kappa : (-\infty, \xi] \to E$ in a way that $\kappa_0 \in \mathscr{B}, \kappa : [0,\xi] \to E$ is continuous, and $\max\limits_{0 \leq \varsigma \leq \xi} \|\kappa_\varsigma\|_{\mathscr{B}} \leq Q$.

(A3) There is a constant $\mathscr{L}_\sigma > 0$ such that

$$|\sigma(\varsigma, \omega_1) - \sigma(\varsigma, \omega_2)| \leq \mathscr{L}_\sigma \|\omega_1 - \omega_2\|_{\mathscr{B}}, \omega_1, \omega_2 \in \mathscr{B}.$$

(A4) There exist constants $\mu, \overline{\mu}$ in a way that $\|\mathbb{S}\| \leq \mu$ and $\|\mathbb{T}\| \leq \overline{\mu}$ for the bounded linear operators $\mathbb{S}$ and $\mathbb{T}$.

If there exists $Q > 0$ in ways that

$$(Q_1^* \mu \widehat{M}_{\mathscr{B}} \overline{W} + Q_2^*)\|\varphi\|_{\mathscr{B}} + Q_1^* \gamma^* \Phi(Q) \left(\frac{\mu\overline{\mu}(1-\vartheta)}{B(\vartheta)\Gamma(\vartheta+1)} + \frac{\widehat{M}_{\widehat{\mathscr{B}}}\mu^2}{B(\vartheta)} \right) \xi^\vartheta \leq Q, \tag{3.5}$$

and

$$\left[Q_1^* \left[\mathscr{L}_{\mathscr{F}} + \widetilde{\mathscr{L}_{\mathscr{F}}}(Q)\mathscr{L}_\sigma \widetilde{\sigma}^* \right] \left(\frac{\mu\overline{\mu}(1-\vartheta)}{B(\vartheta)\Gamma(\vartheta+1)} + \frac{\widehat{M}_{\widehat{\mathscr{B}}}\mu^2}{B(\vartheta)} \right) \xi^\vartheta \right] < 1, \tag{3.6}$$

where $Q_1^* = \sup_{x\in[0,\xi]} Q_1(x)$ and $Q_2^* = \sup_{x\in[0,\xi]} Q_2(x)$, then there exists a mild solution of the inclusions (1.1)-(1.2).

Proof. In view of Lemma 3.1 and our assumptions that Υ^* is upper semicontinuous set-valued map with convex compact values.

Let $B_Q = \{\kappa \in Y_\varphi([0,\xi], E) : \|\kappa_\varsigma\|_{\mathscr{B}} \leq Q, 0 \leq \varsigma \leq \xi\}$. It is obvious that B_Q is a complete metric space. Furthermore, $\Upsilon^*\kappa \subseteq B_Q$ for all $\kappa \in B_Q$. As a matter of truth, if $z(\varsigma) = \mathbb{S}\mathscr{B}_\vartheta(\varsigma)\varphi(0) + \overline{\Upsilon}p(\varsigma)$, for $p \in \Upsilon_{\mathscr{F},\sigma,\kappa}$, and denote $\gamma^* = \sup_{0<s<\xi} \gamma(s)$, it follows from $\mathscr{F}(iii)$ that

$$\begin{aligned}
\|z(\alpha)\| = &\Bigg\| \mathbb{S}\mathscr{B}_\vartheta(\varsigma)\varphi(0) + \frac{\mathbb{S}\mathbb{T}(1-\vartheta)}{B(\vartheta)\Gamma(\vartheta)} \int_0^\varsigma (\varsigma - s)^{\vartheta-1} p(s) ds \\
&+ \frac{\vartheta\mathbb{S}^2}{B(\vartheta)} \int_0^\varsigma \widehat{\mathscr{B}}_\vartheta(\varsigma - s) p(s) ds \Bigg\| \\
&\leq \mu \widehat{M}_{\mathscr{B}} \overline{W} \|\varphi\|_{\mathscr{B}} + \left(\frac{\mu\overline{\mu}(1-\vartheta)}{B(\vartheta)\Gamma(\vartheta)} + \frac{\widehat{M}_{\widehat{\mathscr{B}}}\vartheta\mu^2}{B(\vartheta)} \right) \int_0^\varsigma (\varsigma - s)^{\vartheta-1} \|p(s)\| ds \\
&\leq \mu \widehat{M}_{\mathscr{B}} \overline{W} \|\varphi\|_{\mathscr{B}} \\
&\quad + \left(\frac{\mu\overline{\mu}(1-\vartheta)}{B(\vartheta)\Gamma(\vartheta)} + \frac{\widehat{M}_{\widehat{\mathscr{B}}}\vartheta\mu^2}{B(\vartheta)} \right) \int_0^\varsigma (\varsigma - s)^{\vartheta-1} \gamma(s) \Phi(\|\kappa_{\sigma(s,\kappa_s)}\|_{\mathscr{B}}) ds \\
&\leq \mu \widehat{M}_{\mathscr{B}} \overline{W} \|\varphi\|_{\mathscr{B}} + \gamma^* \Phi(Q) \left(\frac{\mu\overline{\mu}(1-\vartheta)}{B(\vartheta)\Gamma(\vartheta)} + \frac{\widehat{M}_{\widehat{\mathscr{B}}}\vartheta\mu^2}{B(\vartheta)} \right) \int_0^\varsigma (\varsigma - s)^{\vartheta-1} ds
\end{aligned}$$

$$\leq \mu\widehat{M}_{\mathscr{B}}\overline{W}\|\varphi\|_{\mathscr{B}} + \gamma^*\Phi(Q)\left(\frac{\mu\overline{\mu}(1-\vartheta)}{B(\vartheta)\Gamma(\vartheta)} + \frac{\widehat{M}_{\widehat{\mathscr{B}}}\vartheta\mu^2}{B(\vartheta)}\right)\frac{\xi^\vartheta}{\vartheta}$$
$$= \mu\widehat{M}_{\mathscr{B}}\overline{W}\|\varphi\|_{\mathscr{B}} + \gamma^*\Phi(Q)\left(\frac{\mu\overline{\mu}(1-\vartheta)}{B(\vartheta)\Gamma(\vartheta+1)} + \frac{\widehat{M}_{\widehat{\mathscr{B}}}\mu^2}{B(\vartheta)}\right)\xi^\vartheta$$

which suggest that

$$\begin{aligned}\|z_\varsigma\|_{\mathscr{B}} &\leq Q_1^*\left[\mu\widehat{M}_{\mathscr{B}}\overline{W}\|\varphi\|_{\mathscr{B}} + \gamma^*\Phi(Q)\left(\frac{\mu\overline{\mu}(1-\vartheta)}{B(\vartheta)\Gamma(\vartheta+1)} + \frac{\widehat{M}_{\widehat{\mathscr{B}}}\mu^2}{B(\vartheta)}\right)\xi^\vartheta\right] \\ &\quad + Q_2^*\|\varphi\|_{\mathscr{B}} \\ &\leq (Q_1^*\mu\widehat{M}_{\mathscr{B}}\overline{W} + Q_2^*)\|\varphi\|_{\mathscr{B}} + Q_1^*\gamma^*\Phi(Q)\left(\frac{\mu\overline{\mu}(1-\vartheta)}{B(\vartheta)\Gamma(\vartheta+1)} + \frac{\widehat{M}_{\widehat{\mathscr{B}}}\mu^2}{B(\vartheta)}\right)\xi^\vartheta\end{aligned}$$

and utilizing (3.5), we get the affirmation.

Next we demonstrate that a contraction map is defined by Υ^* described on B_Q.

Indeed, let $\kappa^\ell \in B_Q, z^\ell \in \Upsilon^*\kappa^\ell, z^\ell(\varsigma) = \mathbb{S}\mathscr{B}_\vartheta(\varsigma)\varphi(0) + \overline{\Upsilon}p^\ell(\varsigma)$, for $\ell = 1, 2$. We can use [34, Lemma 3.3] to find out if there are any $p^2 \in \Upsilon_{\mathscr{F},\sigma,\kappa^2}$ that fulfills

$$\left\|p^1(s) - p^2(s)\right\| = d\left(p^1(s), \mathscr{F}\left(s, \kappa^2_{\sigma(s,\kappa^2_s)}\right)\right), \quad \text{a. e.}$$

This means that

$$\begin{aligned}&d\left(z^1, \Upsilon^*\kappa^2\right) \\ &= \inf_{p^2\in\Upsilon_{\mathscr{F},\sigma,\kappa^2}}\left\|\overline{\Upsilon}\left(p^1 - p^2\right)\right\| \\ &\leq \left(\frac{\mu\overline{\mu}(1-\vartheta)}{B(\vartheta)\Gamma(\vartheta)} + \frac{\widehat{M}_{\widehat{\mathscr{B}}}\vartheta\mu^2}{B(\vartheta)}\right)\inf_{p^2\in\Upsilon_{\mathscr{F},\sigma,\kappa^2}}\sup_{0\leq\varsigma\leq\xi}\int_0^\varsigma(\varsigma - s)^{\vartheta-1}\left\|p^1(s) - p^2(s)\right\|ds \\ &\leq \left(\frac{\mu\overline{\mu}(1-\vartheta)}{B(\vartheta)\Gamma(\vartheta)} + \frac{\widehat{M}_{\widehat{\mathscr{B}}}\vartheta\mu^2}{B(\vartheta)}\right)\sup_{0\leq\varsigma\leq\xi}\int_0^\varsigma(\varsigma - s)^{\vartheta-1}\left\|p^1(s) - p^2(s)\right\|ds. \qquad (3.7)\end{aligned}$$

Now,

$$\begin{aligned}d\left(p^1(s), \mathscr{F}\left(s, \kappa^2_{\sigma(s,\kappa^2_s)}\right)\right) &\leq d_{\mathcal{H}}\left(\mathscr{F}\left(s, \kappa^1_{\sigma(s,\kappa^1_s)}\right), \mathscr{F}\left(s, \kappa^2_{\sigma(s,\kappa^1_s)}\right)\right) \\ &\quad + d_{\mathcal{H}}\left(\mathscr{F}\left(s, \kappa^2_{\sigma(s,\kappa^1_s)}\right), \mathscr{F}\left(s, \kappa^2_{\sigma(s,\kappa^2_s)}\right)\right) \\ &\leq \sum_{i=1}^2 I_i, \qquad (3.8)\end{aligned}$$

where

$$I_1 = d_{\mathcal{H}}\left(\mathscr{F}\left(s, \kappa^1_{\sigma(s,\kappa^1_s)}\right), \mathscr{F}\left(s, \kappa^2_{\sigma(s,\kappa^1_s)}\right)\right);$$
$$I_2 = d_{\mathcal{H}}\left(\mathscr{F}\left(s, \kappa^2_{\sigma(s,\kappa^1_s)}\right), \mathscr{F}\left(s, \kappa^2_{\sigma(s,\kappa^2_s)}\right)\right).$$

By using the assumption (A1), we can find the estimation of I_1 as

$$\begin{aligned} I_1 &\leq \mathscr{L}_{\mathscr{F}}\left\|\kappa^1_{\sigma(s,\kappa^1_s)} - \kappa^2_{\sigma(s,\kappa^1_s)}\right\|_{\mathscr{B}} \\ &\leq \mathscr{L}_{\mathscr{F}} Q_1\left(\sigma\left(s,\kappa^1_s\right)\right) \max_{0\leq\delta\leq\sigma(s,\kappa^1_s)} \left\|\kappa^1(\delta) - \kappa^2(\delta)\right\| \\ &\leq \mathscr{L}_{\mathscr{F}} Q_1^* \max_{0\leq\delta\leq s}\left\|\kappa^1(\delta) - \kappa^2(\delta)\right\|. \end{aligned} \tag{3.9}$$

By utilizing the assumptions (A2)-(A3), we can find the estimation of I_2 as

$$\begin{aligned} I_2 &\leq \widetilde{\mathscr{L}_{\mathscr{F}}}(Q)\widetilde{\sigma}(s)|\sigma(s,\kappa^1_s) - \sigma(s,\kappa^2_s)| \\ &\leq \widetilde{\mathscr{L}_{\mathscr{F}}}(Q)\widetilde{\sigma}(s)\mathscr{L}_\sigma\|\kappa^1_s - \kappa^2_s\|_{\mathscr{B}} \\ &\leq \widetilde{\mathscr{L}_{\mathscr{F}}}(Q)\widetilde{\sigma}(s)\mathscr{L}_\sigma Q_1^* \max_{0\leq\delta\leq s}\|\kappa^1(\delta) - \kappa^2(\delta)\| \\ &\leq \widetilde{\mathscr{L}_{\mathscr{F}}}(Q)\widetilde{\sigma}^*\mathscr{L}_\sigma Q_1^* \max_{0\leq\delta\leq s}\|\kappa^1(\delta) - \kappa^2(\delta)\|, \end{aligned} \tag{3.10}$$

where $\widetilde{\sigma}^* = \sup\limits_{0<s<\xi} \widetilde{\sigma}(s)$.

By substituting the equations (3.9) and (3.10) in (3.8), we obtain

$$\left\|p^1(s) - p^2(s)\right\| \leq Q_1^*\left[\mathscr{L}_{\mathscr{F}} + \widetilde{\mathscr{L}_{\mathscr{F}}}(Q)\mathscr{L}_\sigma\widetilde{\sigma}^*\right] \max_{0\leq\delta\leq s}\left\|\kappa^1(\delta) - \kappa^2(\delta)\right\|. \tag{3.11}$$

By substituting the equation (3.11) in (3.7), we get

$$\begin{aligned} d\left(z^1, \Upsilon^*\kappa^2\right) \leq &\left[Q_1^*\left[\mathscr{L}_{\mathscr{F}} + \widetilde{\mathscr{L}_{\mathscr{F}}}(Q)\mathscr{L}_\sigma\widetilde{\sigma}^*\right]\left(\frac{\mu\overline{\mu}(1-\vartheta)}{B(\vartheta)\Gamma(\vartheta+1)} + \frac{\widehat{M}_{\widehat{\mathscr{B}}}\mu^2}{B(\vartheta)}\right)\xi^\vartheta\right] \\ &(\times)\|\kappa^1 - \kappa^2\|_\infty. \end{aligned}$$

Continuing to estimate $d\left(z^2, \Upsilon^*\kappa^1\right)$ as above, and applying that

$$d_{\mathcal{H}}\left(\Upsilon^*\kappa^1, \Upsilon^*\kappa^2\right) = \max\left\{\sup_{z^1\in\Upsilon^*\kappa^1} d\left(z^1, \Upsilon^*\kappa^2\right), \sup_{z^2\in\Upsilon^*\kappa^2} d\left(z^2, \Upsilon^*\kappa^1\right)\right\},$$

we come to the conclusion that

$$d\left(\Upsilon^*\kappa^1,\Upsilon^*\kappa^2\right)\leq\left[Q_1^*\left[\mathscr{L}_{\mathscr{F}}+\widetilde{\mathscr{L}_{\mathscr{F}}}(Q)\mathscr{L}_\sigma\widetilde{\sigma}^*\right]\left(\frac{\mu\overline{\mu}(1-\vartheta)}{B(\vartheta)\Gamma(\vartheta+1)}+\frac{\widehat{M}_{\widehat{\mathscr{B}}}\mu^2}{B(\vartheta)}\right)\xi^\vartheta\right]$$
$$(\times)d(\kappa^1,\kappa^2).$$

As a result, by (3.6), the map Υ^* is a contraction on B_Q. From [17, Theorem 1.2.3.1], we realize that the operator Υ^* has a fixed point $\kappa\in B_Q$. □

Example 3.1.

We develop a class of functions $\mathscr{F}$ in this illustration that fulfill all of the previous assumptions. Consider the phase space $\mathscr{B}=C_0\times L^q(g,E)$, $1<q<\infty$ as described in [34, Example 2.1]. Let $\widetilde{q}$ be the conjugate exponent of q.

Define a function $\mathscr{F}_0:[0,\xi]\times\mathscr{B}\to E$ by

$$\mathscr{F}_0(\varsigma,\omega)=\widetilde{\sigma}(\varsigma)\widetilde{W}\left(\int_{-\infty}^0 e(\eta)\omega(\eta)d\eta\right),\ \varsigma\in[0,\xi],\ \omega\in\mathscr{B},$$

where $\widetilde{\sigma}\in L^1([0,\xi]),\widetilde{W}:E\to E$ is a map that fulfills the Lipschitz assumption

$$\|\widetilde{W}(u)-\widetilde{W}(v)\|\leq\mathscr{L}_{\widetilde{W}}\|u-v\|,\ \mathscr{L}_{\widetilde{W}}\geq 0;\ u,v\in E$$

and $e:(-\infty,0]\to(-\infty,0)$ is a function that fulfills the assumptions:

(i) Let $e\in\mathscr{C}^1$ and $\mathscr{L}_e^1:(-\infty,0]\to[0,\infty)$ is a continuous function in a way that $|e'(\eta)|\leq\mathscr{L}_e^1(s)$, $\eta\leq s\leq 0$, and

$$\nu^1=\left(\int_{-\infty}^0\frac{\left|\mathscr{L}_e^1(\eta)\right|^{\widetilde{q}}}{g(\eta)^{\widetilde{q}-1}}d\eta\right)^{1/\widetilde{q}}<\infty.$$

(ii) $\nu=\left(\int_{-\infty}^0\frac{|e(\eta)|^{\widetilde{q}}}{g(\eta)^{\widetilde{q}-1}}d\eta\right)^{1/\widetilde{q}}<\infty.$

We describe $\mathcal{G}:\mathscr{B}\to E$ by

$$\mathcal{G}(\omega)=\int_{-\infty}^0 e(\eta)\omega(\eta)d\eta.$$

Given (ii), we may deduce that $\mathcal{G}(\omega)$ is properly described and

$$\begin{aligned}\|\mathcal{G}(\omega)\| &\leq \int_{-\infty}^{0} |e(\eta)|\|\omega(\eta)\|d\eta \\ &\leq \int_{-\infty}^{0} \frac{|e(\eta)|}{g(\eta)^{1/q}} g(\eta)^{1/q}\|\omega(\eta)\|d\eta \\ &\leq \left(\int_{-\infty}^{0} \frac{|e(\eta)|^{\widetilde{q}}}{g(\eta)^{\widetilde{q}/q}} d\eta\right)^{1/\widetilde{q}} \left(\int_{-\infty}^{0} g(\eta)\|\omega(\eta)\|^{q} d\eta\right)^{1/q} \\ &\leq \nu\|\omega\|_{\mathscr{B}}.\end{aligned}$$

Assume that $\mathscr{C} \subseteq E$ is a convex compact set with $0 \in \mathscr{C}$. Further, we describe

$$\mathscr{F}(z,\omega) = \mathscr{F}_0(\varsigma,\omega) + \mathscr{C}, \varsigma \in [0,\xi], \omega \in \mathscr{B}.$$

From this, we notice that $\mathscr{F}$ fulfills $\mathscr{F}(i)$. Utilizing [14, Proposition 1.1], we can see that $\mathscr{F}$ is upper semi-continuous implying that $\mathscr{F}$ satisfies $\mathscr{F}(ii)$. Further,

$$\begin{aligned}\|\mathscr{F}(\varsigma,\omega)\| = \sup\{\|\nu\| : \nu \in \mathscr{F}(\varsigma,\omega)\} &\leq \|\mathscr{F}_0(\varsigma,\omega)\| + \sup_{\kappa\in\mathscr{C}} \|\kappa\| \\ &\leq |\widetilde{\sigma}(\varsigma)| \left[\mathscr{L}_{\widetilde{W}}\nu\|\omega\|_{\mathscr{B}} + \|\widetilde{W}(0)\|\right] + \sup_{\kappa\in\mathscr{C}} \|\kappa\| \\ &\leq \gamma(\varsigma)\Phi\left(\|\omega\|_{\mathscr{B}}\right), \text{ a.e. } \varsigma \in [0,\xi], \omega \in \mathscr{B},\end{aligned}$$

thinking $\gamma = |\widetilde{\sigma}(\varsigma)| \in L^1([0,\xi])$ and a continuous function $\Phi : [0,\infty) \to [0,\infty)$, described by

$$\Phi(\varsigma) = \left[\mathscr{L}_{\widetilde{W}}\nu\|\omega\|_{\mathscr{B}} + \|\widetilde{W}(0)\|\right] + |\widetilde{\sigma}(\varsigma)|^{-1} \sup_{\kappa\in\mathscr{C}} \|\kappa\|.$$

This delivers that $\mathscr{F}$ fulfills the assumption $\mathscr{F}(iii)$.

Next, we show that $\mathscr{F}$ fulfills the assumptions (A1) and (A2). Since

$$\begin{aligned}d_{\mathcal{H}}\left(\mathscr{F}\left(\varsigma,\omega_1\right), \mathscr{F}\left(\varsigma,\omega_2\right)\right) &\leq \left\|\mathscr{F}\left(\varsigma,\omega_1\right) - \mathscr{F}\left(\varsigma,\omega_2\right)\right\| \\ &\leq |\widetilde{\sigma}(\varsigma)|\mathscr{L}_{\widetilde{W}}\nu\left\|\omega_1 - \omega_2\right\|_{\mathscr{B}}.\end{aligned}$$

From this, we conclude that $\mathscr{F}$ fulfills (A1). Let $\kappa : (-\infty,\xi] \to E$ be a function in a way that $\kappa_0 = \varphi, \kappa : [0,\xi] \to E$ is continuous, and $\|\kappa_\varsigma\|_{\mathscr{B}} \leq Q$ for all $0 \leq \varsigma \leq \xi$. Taking $0 \leq \varsigma_1 < \varsigma_2 \leq \tau$, we obtain

$$\begin{aligned}&d_{\mathcal{H}}\left(\mathscr{F}\left(\varsigma,\kappa_{\varsigma_2}\right), \mathscr{F}\left(\varsigma,\kappa_{\varsigma_1}\right)\right) \\ &\leq |\widetilde{\sigma}(\varsigma)|\left\|\mathscr{F}_0\left(\varsigma,\kappa_{\varsigma_2}\right) - \mathscr{F}_0\left(\varsigma,\kappa_{\varsigma_1}\right)\right\| \\ &\leq |\widetilde{\sigma}(\varsigma)|\mathscr{L}_{\widetilde{W}}\left\|\int_{-\infty}^{0} e(\eta)\kappa\left(\varsigma_2+\eta\right)d\eta - \int_{-\infty}^{0} e(\eta)\kappa\left(\varsigma_1+\eta\right)d\eta\right\|. \qquad (3.12)\end{aligned}$$

Moreover, since

$$\begin{aligned}
&\int_{-\infty}^{0} e(\eta)\kappa(\varsigma_2+\eta)\,d\eta-\int_{-\infty}^{0} e(\eta)\kappa(\varsigma_1+\eta)\,d\eta \\
&=\int_{-\infty}^{\varsigma_2} e(s-\varsigma_2)\kappa(s)ds-\int_{-\infty}^{\varsigma_1} e(s-\varsigma_1)\kappa(s)ds \\
&=\int_{-\infty}^{0}[e(s-\varsigma_2)-e(s-\varsigma_1)]\varphi(s)ds+\int_{0}^{\varsigma_1}[e(s-\varsigma_2)-e(s-\varsigma_1)]\kappa(s)ds \\
&+\int_{\varsigma_1}^{\varsigma_2} e(s-\varsigma_2)\kappa(s)ds
\end{aligned}$$

and using (i) we sustain

$$\begin{aligned}
&\left\|\int_{-\infty}^{0} e(\eta)\kappa(\varsigma_2+\eta)\,d\eta-\int_{-\infty}^{0} e(\eta)\kappa(\varsigma_1+\eta)\,d\eta\right\| \\
&\leq\left(\nu^1+\mathscr{L}_e^1(0)\xi+\overline{W}\sup_{-\xi\leq\eta\leq 0}|e(\eta)|\right)(\varsigma_2-\varsigma_1)\,Q. \qquad (3.13)
\end{aligned}$$

From (3.12) and (3.13) we infer that (A2) is fulfilled.

Further, in this section, as an application on Theorem 3.1, we consider the following system

$$\mathscr{D}^{\vartheta}_{ABC}p(\varsigma)=Ap(\varsigma)+\mathscr{F}(\varsigma,p_{\sigma(\varsigma,p_\varsigma)}),\quad \varsigma\in[0,\xi] \qquad (3.14)$$

$$p(\alpha)=\varphi(\varsigma)\in\mathscr{B}. \qquad (3.15)$$

Here, the function $\mathscr{F}:[0,\xi]\times\mathscr{B}\to E$ is single-valued. The other functions are identical to those specified in (1.1)-(1.2).

The subsequent conclusion can be drawn as a result of Theorem 3.1.

Corollary 3.1. *Let assumptions (A3)-(A4) be hold. In addition, we assume that the following assumptions are fulfilled:*

(i) For every $\omega\in\mathscr{B}$, $\mathscr{F}(\cdot,\omega):[0,\xi]\to E$ is strongly measurable.

(ii) We can find a constant $\mathscr{L}_{\mathscr{F}}>0$ in a way that

$$\|\mathscr{F}(\varsigma,\omega_1)-\mathscr{F}(\varsigma,\omega_2)\|\leq\mathscr{L}_{\mathscr{F}}\|\omega_1-\omega_2\|_{\mathscr{B}}$$

for all $\varsigma\in[0,\xi]$ and all $\omega_1,\omega_2\in\mathscr{B}$.

(iii) We can find a positive function $\widetilde{\sigma}\in L^1([0,\xi])$ and every $Q>0$, there exists a positive constant $\widetilde{\mathscr{L}_{\mathscr{F}}}(Q)>0$ in a way that

$$\|\mathscr{F}(\varsigma,\kappa_{\varsigma_2})-\mathscr{F}(\varsigma,\kappa_{\varsigma_1})\|\leq\widetilde{\mathscr{L}_{\mathscr{F}}}(Q)\widetilde{\sigma}(\varsigma)|\varsigma_2-\varsigma_1|$$

for all $\varsigma_1, \varsigma_2 \in [0,\xi]$ *and* $\kappa : (-\infty, \xi] \to E$ *in a way that* $\kappa_0 \in \mathscr{B}, \kappa : [0,\xi] \to E$ *is continuous, and* $\max\limits_{0\le\varsigma\le\xi} \|\kappa_\varsigma\|_{\mathscr{B}} \le Q$.

(iv) *We can find a function* $\gamma \in L^1([0,\xi])$ *in a way that*

$$\|\mathscr{F}(\varsigma,\omega)\| \le \gamma(\varsigma)\Phi\left(\|\omega\|_{\mathscr{B}}\right), \quad a.e. \ \varsigma \in [0,\xi], \omega \in \mathscr{B},$$

where $\Phi : [0,\infty) \to [0,\infty)$ *is continuous non-decreasing function.*

If assumptions (3.5) *and* (3.6) *are fulfilled, a unique mild solution to the system* (3.14)-(3.15) *is available.*

We can now prove the section's main result without the assumptions (3.5) and (3.6).

Theorem 3.2. *Assume that assumptions* $\mathscr{F}(i) - \mathscr{F}(iii)$ *are satisfied. In addition, we also assume that the subsequent assumption holds.*

$\mathscr{F}(iv)$ *Let* $u(\cdot)$ *be a positive integrable function on* $[0,\xi]$ *in ways that*

$$\chi\left(\mathscr{F}\left(\varsigma, V_s\right)\right) \le u(\varsigma) \sup_{0\le x\le s} \chi(\{\kappa(x) : \kappa \in Q\}), \ a.e. \ \varsigma \in [0,\xi]$$

for all bounded sets $V \subseteq Y_\varphi([0,\xi],E)$, *where* $V_x = \{\kappa_x : \kappa \in V\}$.

If

$$\left[2\left(\frac{\mu\overline{\mu}(1-\vartheta)}{B(\vartheta)\Gamma(\vartheta)} + \frac{\widehat{M}_{\mathscr{B}}\vartheta\mu^2}{B(\vartheta)}\right)\left(\int_0^\xi (\xi - s)^{\vartheta-1} u(s)ds\right)\right] < 1, \qquad (3.16)$$

then $\Upsilon^* : Y_\varphi([0,\xi],E) \to \mathcal{M}\nu\left(Y_\varphi([0,\xi],E)\right)$ *given by* (3.4) *is upper semi-continuous and* β*-condensing.*

Proof. According to our hypothesis, the operator Υ^* is a upper semi-continuous set-valued map with convex compact values. Demonstrating that Υ^* is β-condensing is still required. Let $K \subset Y_\varphi([0,\xi],E)$ be a bounded set in a way that $\beta(\Upsilon^*(K)) \ge \beta(K)$. In view of [21, Lemma 2.9], we can find a sequence $(z^n)_n$ in $\Upsilon^*(K)$ in a way that $\beta(\Upsilon^*(K)) = \beta\left(\{z^n : n \in \mathbb{N}\}\right)$. We may write $z^n \in \Upsilon^*\kappa^n$, for some $\kappa^n \in K$.

Utilizing (3.4), we can calculate $\beta\left(\{z^n : n \in \mathbb{N}\}\right)$, we have

$$z^n(\varsigma) = \mathbb{S}\mathscr{B}_\vartheta(\varsigma)\varphi(0) + \Xi(p^n)(\varsigma),$$

where

$$\Xi(p^n)(\varsigma) = \frac{\mathbb{ST}(1-\vartheta)}{B(\vartheta)\Gamma(\vartheta)}\int_0^\varsigma (\varsigma - s)^{\vartheta-1} p^n(s)ds + \frac{\vartheta\mathbb{S}^2}{B(\vartheta)}\int_0^\varsigma \widehat{\mathscr{B}}_\vartheta(\varsigma - s)p^n(s)ds$$

for $p^n \in \Upsilon^*_{\mathscr{F},\sigma,\kappa^n}$.

Then, we obtain

$$\beta\left(\{z^n(\cdot) : n \in \mathbb{N}\}\right) \leq \beta\left(\{\Xi\left(p^n\right)(\cdot) : n \in \mathbb{N}\}\right).$$

Since $p^n \in \Upsilon^*_{\mathscr{F},\sigma,\kappa^n}$, for $\varsigma \in [0,\xi]$, we sustain $p^n(\varsigma) \in \mathscr{F}\left(\varsigma, \kappa^n_{\sigma\left(\varsigma,\kappa^n_\varsigma\right)}\right)$. Thus $\{p^n : n \in \mathbb{N}\}$ is uniformly integrable and from $\mathscr{F}(iv)$, we have

$$\begin{aligned}\chi\left(\{p^n(\varsigma) : n \in \mathbb{N}\}\right) &\leq u(\varsigma) \sup_{0\leq x\leq\sigma\left(\varsigma,\kappa^n_\varsigma\right)} \chi\left(\{\kappa^n(x) : n \in \mathbb{N}\}\right) \\ &\leq u(\varsigma)\beta\left(\{\kappa^n : n \in \mathbb{N}\}\right).\end{aligned}$$

We may deduce from this estimation and from [24] that

$$\begin{aligned}&\beta\left(\{\Xi\left(p^n\right)(\cdot) : n \in \mathbb{N}\}\right) \\ &\leq 2\left(\frac{\mu\overline{\mu}(1-\vartheta)}{B(\vartheta)\Gamma(\vartheta)} + \frac{\widehat{M}_{\widehat{\mathscr{B}}}\vartheta\mu^2}{B(\vartheta)}\right)\beta\left(\{\kappa^n : n \in \mathbb{N}\}\right)\int_0^\xi (\xi - s)^{\vartheta-1}u(s)ds.\end{aligned}$$

Consequently, after compiling these information, we arrive at

$$\begin{aligned}\beta(K) &\leq \beta(\Upsilon^*(K)) \\ &= \beta\left(\{z^n : n \in \mathbb{N}\}\right) \\ &\leq 2\left(\frac{\mu\overline{\mu}(1-\vartheta)}{B(\vartheta)\Gamma(\vartheta)} + \frac{\widehat{M}_{\widehat{\mathscr{B}}}\vartheta\mu^2}{B(\vartheta)}\right)\left(\int_0^\xi (\xi - s)^{\vartheta-1}u(s)ds\right)\beta\left(\{\kappa^n : n \in \mathbb{N}\}\right) \\ &= \left[2\left(\frac{\mu\overline{\mu}(1-\vartheta)}{B(\vartheta)\Gamma(\vartheta)} + \frac{\widehat{M}_{\widehat{\mathscr{B}}}\vartheta\mu^2}{B(\vartheta)}\right)\left(\int_0^\xi (\xi - s)^{\vartheta-1}u(s)ds\right)\right]\beta(K).\end{aligned}$$

Thus

$$\beta(K)\left[1 - 2\left(\frac{\mu\overline{\mu}(1-\vartheta)}{B(\vartheta)\Gamma(\vartheta)} + \frac{\widehat{M}_{\widehat{\mathscr{B}}}\vartheta\mu^2}{B(\vartheta)}\right)\left(\int_0^\xi (\xi - s)^{\vartheta-1}u(s)ds\right)\right] \leq 0.$$

From (3.16), we conclude that $\beta(K) = 0$ and hence Υ^* is a β-condensing map. The proof is now completed. □

Example 3.2.

Now, we need to prove that the function $\mathscr{F}$ fulfills $\mathscr{F}(iv)$. For this, by thinking of Example 3.1, assume $V \subseteq Y_\varphi([0,\xi], E)$ be a bounded set and $V_s = \{\kappa_s : \kappa \in V\}$. Using the characteristics stated in preliminaries, it is easy to demonstrate that

$$\chi\left(\mathscr{F}\left(\varsigma, V_s\right)\right) \leq \nu|\widetilde{\sigma}(\varsigma)|\mathscr{L}_{\widetilde{W}}\beta(K).$$

Thus the function $\mathscr{F}$ fulfills $\mathscr{F}(iv)$.

Conclusion

The power law has been used to produce fractional order derivatives if Caputo and Riemann-Liouville are considered to be convolutions. It is not always possible to find power law behaviour in nature. In this study, we used their [7] new result to our differential inclusions (1.1)-(1.2). Theorem 3.1 is proved to investigate the existence of the addressing model (1.1)-(1.2) by means of contractive and condensing map. Next, in Example 3.1, we show that the function $\mathscr{F}$ fulfill the assumptions $\mathscr{F}(i) - \mathscr{F}(iii), (A1)$ and $(A2)$. We establish the existence result for the equation (3.14)-(3.15) as a result of Theorem 3.1. The main outcome of this manuscript is derived in Theorem 3.2 and further, we also verified the assumption $\mathscr{F}(iv)$ in Example 3.2. With an appropriate fixed point theorem, the effectiveness of such existing research may be developed to approximate controllability with non-instantaneous impulses for different models.

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