Ellipses surrounding convex bodies

Tudor Zamfirescu

Abstract

If, for a double normal xx^* of a convex body K, an ellipse $E \ni x, x^*$ is included in K, we say that E is *surrounded* by the boundary of K. If, instead, in the plane of E, K is included in the convex hull of E, then we say that E is *surrounding* K. In this paper we investigate surrounding and surrounded ellipses, particularly circles. We do this for arbitrary convex bodies, for polytopes, for convex bodies of constant width, and for most convex bodies (in the sense of Baire categories).

Introduction

We work in $\mathbb{R}^3$, and start with some definitions and notation.

For distinct points x, $y \in \mathbb{R}^3$, let xy denote the line-segment from x to y, and $\overline{xy}$ the line through x, y.

For any compact set $M \subset \mathbb{R}^3$, let $\overline{M}$ mean the affine hull of M, conv$M$ the convex hull of M, int$M$ the relative interior of M (i.e., in the topology of $\overline{M}$), bd$M$ the relative boundary of M, and $\rho(x, M) = \min_{y \in M} \|x - y\|$ the distance from $x \in \mathbb{R}^3$ to M.

Denote by $\mathbb{S}_n$ the n-dimensional unit sphere, by $\mathbb{B}_n$ the n-dimensional compact unit ball, and by $B_r(x)$ the closed ball of radius r and centre $x \in \mathbb{R}^3$.

A *polytope* is the convex hull of a finite set. Its extreme points, i.e. boundary points not in the relative interior of any line-segment included in the

polytope, are called *vertices*. A polytope is called *inscribable* if all its vertices lie on some sphere.

For convex surfaces, a lower and an upper curvature can be defined, as follows (see [4]). Consider a smooth, strictly convex body K, a point x on its boundary and a tangent direction (unit vector) τ at x. Take the 2-dimensional half-plane H containing $x+\tau$ with the normal at x as relative boundary. Then, for any point $z \in H \cap \mathrm{bd}K$ different from x, there is exactly one circle with its centre on the normal line, containing both x and z. Let $r(z)$ be the radius of this circle. Then

$$\gamma_i^\tau(x) = 1/\limsup_{z \to x} r(z), \qquad \gamma_s^\tau(x) = 1/\liminf_{z \to x} r(z)$$

are called the *lower and upper curvature* of $\mathrm{bd}K$ at x in direction τ.

If they are equal, the common value $\gamma^\tau(x)$ is the *curvature* of $\mathrm{bd}K$ at x in direction τ.

The space $\mathcal{K}$ of all convex bodies equipped with the Pompeiu-Hausdorff metric h is Baire.

In a Baire space, we say that *most* of its elements have property **P** if those not enjoying **P** form a set of first Baire category. For example, most convex bodies are smooth and strictly convex, as Klee proved in [6]. Surveys about properties of most convex bodies are [5], [10].

For results in Convexity, many books are good. One of the best is Schneider's [8].

Let K be a convex body, and $a, b \in \mathrm{bd}K$. The line-segment ab is called a *chord* of K; ab is a *double normal* of K (or of its boundary $\mathrm{bd}K$), if the planes through a, b orthogonal to $\overline{ab}$ are supporting K. A circle or an ellipsoid C with ab as a diameter or axis is said to *surround* K if $K \cap \overline{C} \subset \mathrm{conv}C$. If, moreover, $K \cap C = \{a, b\}$, then we say that it is a *simple surrounding circle* or *ellipsoid*.

So, if C is a sphere, it is circumscribed to K. But, obviously, not for every K, the sphere circumscribed to K surrounds K. By a theorem in [9], for most convex bodies K, the circumscribed sphere has exactly 4 contact points with K. From its proof it can be easily deduced that those circumscribed spheres are not surrounding K.

The points $x, x^* \in \mathrm{bd}K$ are called *opposite* if some supporting plane through x is parallel to and distinct from some supporting plane through x^*. Generalizing the notion of a surrounding circle, we say that E is a *surrounding ellipse* (or a *weakly surrounding ellipse*) of K, if there exists a double normal xx^* (respectively opposite points $x, x^* \in \mathrm{bd}K$), such that $x, x^* \in E$ and $K \cap \overline{E} \subset \mathrm{conv}E$.

The circle C is said to *fix* K if each continuous and rigid move of C during which C does not meet int K is a rotation keeping C fixed as a set. For results on circles and other frames fixing convex bodies, see [3].

If the convex body K admits a surrounding sphere, then it also admits many surrounding circles. But the converse is not true. The regular tetrahedron has many surrounding circles, but no surrounding sphere. So, is it true that most convex bodies admit no surrounding circle? This question remains open.

In case, for a double normal xx^* (or for two opposite points $x, x^* \in \mathrm{bd}K$), an ellipse $E \ni x, x^*$ is included in K, we say that E is *surrounded* (respectively *weakly surrounded*) by $\mathrm{bd}K$. Surrounded and surrounding ellipses are treated in the following sections.

The subspace $\mathcal{K}'$ of $\mathcal{K}$ of all convex bodies admitting a surrounding circle is Baire. Indeed, its closure $\mathrm{cl}\mathcal{K}'$ is complete, and $(\mathrm{cl}\mathcal{K}') \setminus \mathcal{K}'$ is nowhere dense. We investigate in a further section the behaviour of most elements of $\mathcal{K}'$.

Surrounded ellipses

Theorem 1. *If P is a point-symmetric polytope, then*

1) it has a surrounding sphere and infinitely many simple surrounding circles,

2) it has a surrounded sphere,

3) every point which is interior to a facet is an endpoint of the long axis of a weakly surrounded ellipse,

4) there exists a weakly surrounded ellipse the long axis of which has an endpoint on an edge of P.

Proof. Let P be a polytope symmetric about the origin $\mathbf{0}$.

1) Choose a vertex v of P with largest $\|v\|$. Then the sphere S with diameter $v(-v)$ surrounds P. There might be more vertices of P on S, but all circles with diameter $v(-v)$ avoiding those vertices are simple surrounding circles.

2) The largest sphere of centre $\mathbf{0}$ included in P is a surrounded sphere.

3) Take an arbitrary facet F of P, and $x \in \mathrm{int}F$. The facets F and $-F$ are parallel. Consider the plane $\Pi \ni x$ orthogonal to x, and the line $L = \Pi \cap \overline{F}$. In the plane determined by L and $-L$ we obviously have an ellipse weakly surrounded by $\mathrm{bd}P$ with $x(-x)$ as long axis.

4) Let x be a point on the boundary 1-skeleton of P closest from $\mathbf{0}$. This point is interior to some edge J of P. Moreover, $-x \in \mathrm{int} - J$, and the points

$x, -x$ are opposite. There obviously exists a weakly surrounded ellipse with long axis $x(-x)$ in the plane $\overline{J \cup -J}$. □

Theorem 2. *Most convex bodies admit no weakly surrounded ellipse.*

Proof. Assume that a convex body K has a weakly surrounded ellipse E through the opposite points x, x^*. Then, in both tangent directions $\tau, -\tau$ of E at x, the surface bd$K$ satisfies the inequalities

$$\gamma_s^{\pm\tau}(x) < \infty, \quad \gamma_s^{\pm\tau}(x^*) < \infty,$$

by Meusnier's Theorem (see [4]).

However, on most convex bodies there is no pair of opposite points x, x^* and no tangent direction τ at x, such that both inequalities

$$\gamma_s^{\tau}(x) < \infty, \quad \gamma_s^{\tau}(x^*) < \infty,$$

hold, by Theorem 4.1 in [1]. □

Despite of Theorem 2, the following holds.

Theorem 3. *The subspace $\mathcal{K}^\flat$ of $\mathcal{K}$ consisting of all convex bodies admitting a surrounded ellipsoid is dense.*

Proof. Let $K \in \mathcal{K}$. We approximate K by a polytope P in $\mathcal{K}^\flat$.

Consider the points a, b realizing the diameter of K. Approximate K by a polytope P having two facets F_a, F_b such that $a \in \text{int} F_a$, $b \in \text{int} F_b$, and both $\overline{F_a}$ and $\overline{F_b}$ are orthogonal to $\overline{ab}$. This polytope obviously admits a surrounded ellipsoid with ab as long axis. □

Surrounding ellipses

Theorem 4. *Every inscribable polytope admits surrounding circles.*

Proof. Take a polytope P with all vertices on $\mathbb{S}_2$.

Let v, v' be vertices of P, with $\|v + v'\|$ minimal. Assume $v' \neq -v$. Then vv' is a double normal of P. Indeed, consider the planes $H \ni v$, $H' \ni v'$ orthogonal to $\overline{vv'}$. Both components of $\mathbb{S}_2 \setminus \text{conv}(H \cup H')$ have diameter equal to $\|v + v'\|$, and contain therefore no vertices of P.

Among the circles on $\mathbb{S}_2$ through v, v', one has vv' as diameter (or all of them if $v' = -v$). These are surrounding circles of P. □

It is rather evident that point-symmetric convex bodies, like polytopes, have a surrounding sphere. This remark can be strengthened for most point-symmetric convex bodies.

Theorem 5. *Most point-symmetric convex bodies admit a simple surrounding sphere.*

Proof. Let $\mathcal{K}^\circ$ be the space of all point-symmetric convex bodies.

Take $K \in \mathcal{K}^\circ$, and choose $x, y \in K$ realizing the diameter of K. Clearly, the sphere S of diameter xy surrounds K. For some $\varepsilon > 0$, $S \cap K \subset B_\varepsilon(x) \cup B_\varepsilon(y)$.

Let $\mathcal{K}_n$ be the set of all $K \in \mathcal{K}^\circ$, for which the above inclusion is not true with $\varepsilon = 1/n$. We show that $\mathcal{K}_n$ is nowhere dense.

Let $\mathcal{O} \subset \mathcal{K}^\circ$ be open, and consider $K \in \mathcal{O}$. Let S be the sphere surrounding K. Approximate K by a polytope $P \subset \mathrm{conv}\, S$, which meets S at x and y only, these points realizing the diameter of K, such that $P \in \mathcal{O}$. For a whole neighbourhood $\mathcal{N}$ of P in $\mathcal{O}$, $S \cap K \subset B_{1/n}(x) \cup B_{1/n}(y)$ for all $K \in \mathcal{N}$. Hence, $\mathcal{N} \cap \mathcal{K}_n = \emptyset$, whence $\mathcal{K}_n$ is nowhere dense.

The set of those point-symmetric convex bodies which do not admit a simple surrounding sphere equals $\cup_{n=1}^\infty \mathcal{K}_n$, which is of first category. $\square$

Theorem 6. *Every polytope admits a surrounding ellipsoid. Also, for every vertex of a polytope, there exists a weakly surrounding ellipse passing through it.*

Proof. Let $\mathcal{K}^\circ$ be the space of all point-symmetric convex bodies.

Take $K \in \mathcal{K}^\circ$, and choose $x, x' \in K$ realizing the diameter of K. Denote by S_K the sphere circumscribed to K. Clearly, the sphere of diameter xx' surrounds K and equals S_K.

Let $\mathcal{K}_n$ be the set of all $K \in \mathcal{K}^\circ$, for which there exist two pairs of points x, x' and y, y' both realizing the diameter of K, with $\|x - y\| \geq 1/n$ and $\|x' - y'\| \geq 1/n$. We show that $\mathcal{K}_n$ is nowhere dense.

Let $\mathcal{O} \subset \mathcal{K}^\circ$ be open, and consider $K' \in \mathcal{O}$. Approximate K' by a polytope P admitting a single diameter xx', such that $P \in \mathcal{O}$. For a whole neighbourhood $\mathcal{N}$ of P in $\mathcal{O}$, $S_P \cap K'' \subset B_{1/(2n)}(x) \cup B_{1/(2n)}(x')$ for all $K'' \in \mathcal{N}$. Thus, if K'' admits a second diameter yy', then $\|x - y\| \leq 1/(2n)$ or $\|x - y'\| \leq 1/(2n)$. Hence, $\mathcal{N} \cap \mathcal{K}_n = \emptyset$, whence $\mathcal{K}_n$ is nowhere dense.

The set of those point-symmetric convex bodies which do not admit a simple surrounding sphere equals $\cup_{n=1}^\infty \mathcal{K}_n$, which is of first category. $\square$

It looks very probable that many, perhaps most, convex bodies admit no surrounding circles. Good candidates appear to be convex bodies of constant

width. However, our efforts of finding such convex bodies using Baire cate-
gories remained unsuccessful, as we detail in the next section. Even finding a
single convex body without surrounding circles is less trivial than we initially
thought.

Theorem 7. *There exist convex bodies without any surrounding circles.*

Proof. We construct a convex body without any surrounding circle.

Let $K \subset \mathbb{S}_2$ be compact. There exists a convex body B_K with differentiable
boundary, such that $B_K \subset \mathbb{B}_3$ and $B_K \cap \mathbb{S}_2 = K$. This can be shown, for
example, by taking

$$S = \{y \in \mathbf{0}x : x \in \mathbb{S}_2 \ \wedge \ \|x - y\| = \rho(x, K)^2\},$$

and then $B_K = \mathrm{conv}S$. Let now M' be the union of n equidistant meridians
on $\mathbb{S}_2$, joining $(0, 0, 1)$ with $(0, 0, -1)$, where n is odd. Let M equal M' minus
a small open spherical disc on $\mathbb{S}_2$ about $(0, 0, -1)$. Let E be the equator of $\mathbb{S}_2$
with $\overline{E}$ orthogonal to the x_3-axis. Take a large number κ. Each point x of $\mathbb{S}_2$
will be rotated (in the same sense) about the x_3-axis by an angle $\kappa\rho(x, E)$. Let
r_κ be this mapping from $\mathbb{S}_2$ to $\mathbb{S}_2$. If n and κ are large enough, $r_\kappa(M)$ meets
every circular arc on $\mathbb{S}_2$ of length at least $9\pi/10$, and the width of $\mathrm{conv}r_\kappa(M)$
is larger than $9/5$.

We now take K to be $r_\kappa(M)$. We show that B_K has no surrounding circle.
Assume, it has such a circle C. Let ab be the double normal of B_K, which
is a diameter of C. We observe that a, b cannot be both on $\mathbb{S}_2$. Indeed, if
they were, they would be antipodal on $\mathbb{S}_2$. But M has no pair of antipodal
points, and this remains true for K. So, at least one half of C, say C^*, lies
inside of $\mathbb{S}_2$ (possibly except for one endpoint). The width of B_K is larger than
the width of $\mathrm{conv}r_\kappa(M)$, so it is larger than $9/5$. This yields $\|a - b\| > 9/5$,
whence the length of C^* is larger than $9\pi/10$. The projection of C^* onto $\mathbb{S}_2$
from the centre of C has a fortiori length larger than $9\pi/10$, and therefore
meets $r_\kappa(M)$. But then, C cannot be a surrounding circle of B_K. $\square$

About convex bodies of constant width

Let us turn now to convex bodies of constant width. We remark that,
in the case of convex bodies of constant width, "surrounding" and "weakly
surrounding" are equivalent notions.

Those convex bodies of constant width obtained by rotating a Reuleaux
polygon admit of course a surrounding circle. But what happens for most of
them? We do not know.

One can strongly believe that most convex bodies of constant width contain no surrounding circles. To prove this, we thought that known curvature results about the boundaries of most convex bodies of constant width could help.

By Theorem 6.2 in [2], most convex bodies K of constant width 1 have the following property. At any point $x \in \text{bd}K$,

(i) $\gamma_i^\tau(x) = 1$ for all tangent vectors τ at x, or

(ii) $\gamma_s^\tau(x) = \infty$ for some tangent vector τ at x.

In case (i) there is no surrounding circle with diameter xx^*, since the circle has radius $1/2$.

Suppose now that case (ii) holds. Since the lower Dupin indicatrix at x is convex, it follows that it contains $\mathbf{0}$ as a boundary point, and $\gamma_s^\tau(x) = \infty$ for all τ in a whole open half-circle $C^+ \subset \mathbb{S}_1$ of tangent directions at x.

By projecting orthogonally K onto a plane parallel to τ and to the diametral chord xx^*, we obtain a planar convex body of constant width 1. By Theorem 1 in [11],
$$\gamma_i^\tau(x)^{-1} + \gamma_s^{-\tau}(x^*)^{-1} = 1,$$
for any convex body of constant width 1 and for any x and τ.

It follows that, in our case, $\gamma_i^{-\tau}(x^*) = 1$ for all $\tau \in C^+$.

If $\gamma_i^{-\tau}(x^*) = 1$ for all τ, then there is no surrounding circle with diameter xx^*, as in case (i). If not, then the same conclusion holds, for every $\tau \in C^+ \cup -C^+$, hence for every $\tau \in \mathbb{S}_1$ except for a single direction and its opposite.

Unfortunately, this leaves open the existence of a surrounding circle in those directions. But we dare to formulate the following.

Conjecture. Most convex bodies of constant width admit no surrounding circle.

Convex bodies with a given line-segment as double normal

To simplify matters, we shall consider the space $\mathcal{K}'$ restricted to those convex bodies admitting a surrounding circle with a fixed double normal ab as diameter, the same double normal for all convex bodies.

We also consider the space $\mathcal{K}^*$ of all convex bodies with $ab \in \mathbb{R}^3$ as double normal. Thus, $\mathcal{K}' \subset \mathcal{K}^*$.

A first step would be, of course, to detect convex bodies of constant width without surrounding circles.

Theorem 8. *Most convex bodies from $\mathcal{K}^*$ admit no surrounding ellipse with ab as an axis.*

Proof. Assume that a convex body $K \in \mathcal{K}^*$ has a surrounding circle C of diameter ab. Then, in both tangent directions $\tau, -\tau$ of C at a and b, the surface bd$K$ satisfies the inequalities

$$\gamma_i^{\pm\tau}(a) \geq 2/\|a - b\|, \quad \gamma_i^{\pm\tau}(b) \geq 2/\|a - b\|.$$

If K has just a surrounding ellipse, then still

$$\gamma_i^{\pm\tau}(a) > 0, \quad \gamma_i^{\pm\tau}(b) > 0.$$

However, on most convex bodies from $\mathcal{K}^*$, for every tangent direction τ at a, we have

$$\gamma_i^{\tau}(a) = \gamma_i^{\tau}(b) = 0,$$

by Theorem 9 in [7]. $\qquad\qquad\square$

Now, we turn our attention to the space $\mathcal{K}'$.

Take, for example, the regular tetrahedron. It belongs to $\mathcal{K}'$, and possesses both simple (infinitely many) and non-simple surrounding circles. What is the situation for other convex bodies in $\mathcal{K}'$?

Theorem 9. *Most convex bodies from $\mathcal{K}'$ have a single surrounding circle of diameter ab, which is simple.*

Proof. Let $\mathcal{K}_n$ be the subspace of all $K \in \mathcal{K}'$ which admit two surrounding circles C_1, C_2 of diameter ab with $h(C_1, C_2) \geq 1/n$ or a surrounding circle C with $h(K \cap C, \{a, b\}) \geq 1/n$. We prove that $\mathcal{K}_n$ is nowhere dense.

Let $\mathcal{O} \subset \mathcal{K}'$ be open, and consider $K \in \mathcal{O}$. Let C be a circle surrounding K having as a diameter the double normal ab of K. By the definition, $K \cap \overline{C} \subset$ conv$C$. Approximate K by a polytope $P \subset ab \cup \text{int}K$, with $a, b \in P$. Let H^+, H^- be the two closed half-spaces determined by $\overline{C}$ in $\mathbb{R}^3$.

Put $P^+ = P \cap H^+$ and $P^- = P \cap H^-$. Let $\widetilde{x_1 x_n}$ be a small arc of C with a as midpoint. Let $x_2, ..., x_{n-1}$ divide $\widetilde{x_1 x_n}$ in $n - 1$ congruent subarcs. The tangent lines at x_i and x_{i+1} meet at y_i $(i = 1, ..., n - 1)$. Consider the broken lines $A^+ = y_1 y_3 y_5, ...$ and $A^- = y_2 y_4 y_6...$ Also, consider the symmetric broken lines B^+ and B^- at b. Take $Q^+ = \text{conv}(P^+ \cup A^+ \cup B^+)$ and $Q^- = \text{conv}(P^- \cup A^- \cup B^-)$. If $\|x_1 - x_n\| < 1/2n$ is small enough, $Q^+, Q^- \in \mathcal{O}$.

Now, let τ be orthogonal to $\overline{C}$ and with $\|\tau\|$ small. We translate Q^+, Q^-, obtaining $R^+ = Q^+ + \tau$, $R^- = Q^- - \tau$. Take $P' = \text{conv}(R^+ \cup R^-)$. Still $P' \cap \overline{C} \subset \text{conv}C$. Denoting by Π_a, Π_b the planes through a, b orthogonal to $\overline{ab}$, we consider

$$P^* = P' \cap \text{conv}(\Pi_a \cup \Pi_b),$$

which means taking off P' the parts not between Π_a and Π_b. Now, C surrounds P^*. If $\|\tau\|$ is small enough, then still $P^* \in \mathcal{O}$.

The polytope P^* fixes C. Since $\|x_1 - x_n\| < 1/2n$, there exists a neighbourhood $\mathcal{N} \subset \mathcal{O}$ of P^*, such that $h(K \cap C', \{a, b\}) < 1/n$ for all $K \in \mathcal{N}$ and any circle C' surrounding K. Moreover, $h(C_1, C_2) < 1/n$ for any pair of circles C_1, C_2 surrounding K, with ab as diameter. This yields $\mathcal{N} \cap \mathcal{K}_n = \emptyset$, whence $\mathcal{K}_n$ is nowhere dense.

Hence, the set of convex bodies from the statement has the set $\cup_{n=1}^{\infty} \mathcal{K}_n$ of first category as complement. $\qquad\square$

Acknowledgements.

References

[1] K. Adiprasito, T. Zamfirescu, Large curvature on typical convex surfaces, *J. Convex Analysis* **19** (2012), No. 2, 385-391.

[2] I. Bárány, R. Schneider, Typical curvature behaviour of bodies of constant width, *Advances in Mathematics* **272** (2015) 308-329.

[3] I. Bárány, T. Zamfirescu, Holding Circles and Fixing Frames, *Discrete Comput. Geometry* **50** (2013) 1101-1111.

[4] H. Busemann, *Convex Surfaces*, New York: Interscience (1958).

[5] P. Gruber, Baire categories in convexity. In: *Handbook of Convex Geometry* (Eds. Gruber, P., Wills, J.). Amsterdam: Elsevier Science (1993).

[6] V. Klee, Some new results on smoothness and rotundity in normed linear spaces, *Math. Annalen* **139** (1959) 51-63.

[7] A. Rivière, J. Rouyer, C. Vîlcu, T. Zamfirescu, Double normals of most convex bodies, *Advances in Mathematics* **343** (2019) 245-272.

[8] R. Schneider, *Convex Bodies – The Brunn-Minkowski Theory* (second ed.), Cambridge University Press, Cambridge (2014).

[9] T. Zamfirescu, Inscribed and circumscribed circles to convex curves, *Proc. Amer. Math. Soc.* **80** (1980) 455-457.

[10] T. Zamfirescu, Baire categories in convexity, *Atti Sem. Mat. Fis. Univ. Modena* **39** (1991) 139-164.

[11] T. Zamfirescu, On the curvatures of convex curves of constant width, *Atti Sem. Mat. Fis. Univ. Modena* **42** (1994) 253-256.

Tudor ZAMFIRESCU,
Fakultät für Mathematik,
Technische Universität Dortmund,
44221 Dortmund, Germany
and
Roumanian Academy, Bucharest, Roumania
and
College of Mathematics and Information Science,
Hebei Normal University,
050024 Shijiazhuang, P.R. China.
Email: tudor.zamfirescu@mathematik.uni-dortmund.de