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A result of instability for two-temperatures Cosserat bodies

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Abstract

In our study we consider a generalized thermoelasticity theory based on a heat conduction equation in micropolar bodies. Specifically, the heat conduction depends on two distinct temperatures, the conductive temperature and the thermodynamic temperature. In our analysis, the difference between the two temperatures is clear and is highlighted by the heat supply. After we formulate the mixed initial boundary value problem defined in this context, we prove the uniqueness of a solution corresponding some specific initial data and boundary conditions. Also, if the initial energy is negative or null, we prove that the solutions of the mixed problem are exponentially instable.

1. Introduction

Many studies dedicated to classical thermoelasticity used a heat conduction equation which are based on the classical Fourier law. As a consequence, the heat flux vector is depending on the gradient of temperatures and, as a consequence, the thermal signals will propagate with an infinite speed. But this contradicts the causality principle. To avoid this contradiction, a series of new theories of thermoelasticity have emerged that propose different alternatives to the classical heat conduction equation. This is how various models appeared, of which the best known in the literature are Green and Lindsay [1], Lord and Shulman [2], Green and Naghdi [3-5], More-Gibson-Thompson [6],

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or [7]. In all these models, the thermal waves propagate with finite speeds and all results from thes generalized theorie are more general and physically more realistic than in the classical theory. In our study we propose a new temperature rate, which is depending on two temperatures, by changing the relation between the two temperatures, namely, the thermodynamic temperature and the conductive temperature.
There are many studies that take into account the two temperatures, of which we mention [8-11]. Other generalizations of the heat conduction equation can be found in many articles, of which we list [12-14]. Our uniqueness result is obtained by assuming the initial energy is not strictly positive. Other uniqueness results are based on the assumption that the elastic tensor is a positively defined one. But there are concrete thermoelastic situations in which the positive definition of the elastic tensor cannot be guaranteed. And our result on exponential instability is also obtained on the assumption that the initial energy is not strictly positive. We must emphasize that our mixed problem is considered both in the theory in which it is considered dependent on the rate of both temperature, and the theory does not depend on the rate of the conductive temperature, but depends on the rate of thermodynamic temperature. However, the calculations are quite similar in both situations, which is why the demonstrations are made in detail only in the case of dependence on the rate of thermodynamic temperature.
We must also say what is the motivation that we took into account the effect due to the dipolar structure. In opinion of many researchers, it is known that this effect makes an important contribution to the general deformations of the media. It is enough to refer to media that have a granular structure, for instance, polymers, human bones or graphite. Also, other concrete usefulness of this effect are for the various materials with pores or composite materials which are reinforced with chopped fibers. From the large number of studies dedicated to media with dipole structure we have selected a few: [15-25].

2. The mixed initial-boundary value problem

Consider that the thermoelastic micropolar body occupies the three-dimensional domain Ω from the Euclidian space R^3. The closure of Ω is denoted by $\bar{\Omega}$ and we have $\bar{\Omega} = \Omega \cup \partial\Omega$, where $\partial\Omega$ is the border of the domain Ω and is considered regular enough to allow the application of the divergence theorem. The outward unit normal to $\partial\Omega$ has the components marked with n_i. The vector and tensors fields are denoted by letters in boldface. The notation v_i is used for the components of a vector field $\mathbf{v}$, the notation u_{ij} is used for the components of a tensor field $\mathbf{u}$ of second order, and so on. For the material time derivative we will use a superposed dot. By convention, the subscripts are understood to range over integers $(1, 2, 3)$. The summation rule regarding repeated subscripts is also implied. For the partial differentiation of a function f regarding the spatial variables x_j we will use the notation f_j, to simplify the writing. When there are no possibilities of confusion, the time variable and/or the spatial variables of a function may not be highlighted. A fixed system of Cartesian axes $Ox_i,\ i = 1, 2, 3$ will be used to refer the motion of the thermoelastic body.

In order to characterize the evolution of our media we will consider the set of variables $(v_m, \varphi_m, \phi, \vartheta)$, where we denoted by v_m the components of the vector of displacement , by φ_m the components of the couple vector, ϕ the conductive temperature and by ϑ the thermodynamic temperature measured from the constant absolute temperature ϑ_0 of the body in its reference state. By using the internal variables (v_m, φ_m) we can introduce the kinematic characteristics of the body, that is, the strain tensors, through the following geometrical equations:

$$e_{mn} = v_{n,m} + \varepsilon_{knm}\varphi_k,\ \sigma_{mn} = \varphi_{n,m}. \tag{1}$$

As usual, the notation t_{mn} is used for the elements of the stress tensor, τ_{mn} for the elements of the tensor of couple stress, all over Ω.

For a homogeneous thermoelastic body, which have in each point of its reference state a point of symmetry, and the rest is non-isotropic, we can define the stress tensors by means of the following constitutive equations:

$$\begin{aligned}
t_{mn} &= A_{mnkl}e_{kl} + B_{mnkl}\sigma_{kl} - \alpha_{mn}\left(\vartheta + a\dot{\vartheta}\right),\\
\tau_{mn} &= B_{klmn}e_{kl} + C_{mnkl}\sigma_{kl} - \beta_{mn}\left(\vartheta + a\dot{\vartheta}\right),\\
\eta &= \alpha_{mn}e_{mn} + \beta_{mn}\varepsilon_{mn} + d\vartheta + h\dot{\vartheta},\\
q_m &= \kappa_{mn}\phi_{,n}.
\end{aligned} \tag{2}$$

In the absence of body force, of body couple and of heat supply fields, the field of basic equations for the two-temperature thermoelasticity of dipolar bodies are:

- the motion equations:

$$t_{mn,n} = \rho\ddot{v}_m, \tag{3}$$

$$\tau_{nm,n} + \varepsilon_{kmn}t_{kn} = I_{mn}\ddot{\phi}_n; \tag{4}$$

- the energy equation:

$$q_{m,m} = \dot{\eta}; \tag{5}$$

- two type of the two-temperatures equations:

$$\phi - c\left(\kappa_{mn}\phi_{,m}\right)_{,n} = \vartheta + a\dot{\vartheta}; \tag{6}$$

$$a\dot{\phi} + \phi - c\left(\kappa_{mn}\phi_{,m}\right)_{,n} = \vartheta + a\dot{\vartheta}. \tag{7}$$

It is easy to see that the equation (6) will be considered in the theory dependent on rate of conductive temperature and on rate of thermodynamic temperature, where (1) and (2) are the equations of motion and (3) is the equation of energy. In the above equations we have used the following notations: ρ is the reference constant mass density, S is the specific entropy per unit mass, q_i are the components of heat flux vector.
The above coefficients A_{mnkl}, B_{mnrs}, C_{mnrs}, α_{mn}, ..., κ_{mn}, c, d, h and a are used to describe the material structure.
The following properties of symmetry are satisfied:

$$A_{mnkl} = A_{klmn},\ C_{mnkl} = C_{klmn},\ I_{mn} = I_{nm},\ \kappa_{mn} = \kappa_{nm}. \tag{8}$$

We wish to outline that the coefficients a, c, d and h are specific constants of the heat.
The temperature ϑ_0 and density ρ are given strict positive constants. From the entropy production inequality (see Gren and Lindsay (1972)) we obtain the following conditions

$$c > 0,\ h > 0,\ da - h \geq 0, \tag{9}$$

and, according to the same entropy inequality, we assume that A_{mnkl}, C_{mnkl} and κ_{mn} are positive definite tensors, i.e.

$$\begin{aligned} A_{mnkl}\xi_{mn}\xi_{kl} &\geq k_1\xi_{mn}\xi_{mn},\ k_1 > 0,\ \forall\ \xi_{mn} = \xi_{nm}, \\ C_{mnkl}\xi_{mn}\xi_{kl} &\geq k_2\xi_{mn}\xi_{mn},\ k_2 > 0,\ \forall\ \xi_{mn} = \xi_{nm}, \\ \kappa_{mn}\xi_m\xi_n &\geq k_3\xi_m\xi_m,\ k_3 > 0,\ \forall\ \xi_m. \end{aligned} \tag{10}$$

Along with the above basic equations (3)-(7), we consider the following homogeneous boundary conditions of Dirichlet type:

$$v_m = 0,\ \varphi_m = 0, \vartheta = 0,\ \phi = 0 \text{ on } \partial\Omega \times [0, \infty). \tag{11}$$

To this system of equations we adjoin the following initial conditions:

$$\begin{aligned} &v_m(x,0) = v_m^0(x),\ \dot{v}_m(x,0) = v_m^1(x), \\ &\varphi_m(x,0) = \varphi_m^0(x),\ \dot{\varphi}_m(x,0) = \varphi_m^1(x), \\ &\phi(x,0) = \phi^0(x),\ \vartheta(x,0) = \vartheta^0(x),\ \dot{\vartheta}(x,0) = \vartheta^1(x), \end{aligned} \tag{12}$$

which are satisfied for any $x \in \Omega$.
Considering the geometric equations and the constitutive equations (2), which are introduced in the basic equations (3)-(5), we are led the following system of partial differential equations:

$$A_{ijmn}\left(v_{n,mj}+\varepsilon_{kmn}\varphi_{k,j}\right)+B_{ijmn}\varphi_{n,mj}-\alpha_{ij}\left(\vartheta_{,j}+a\dot{\vartheta}_{,j}\right)=\rho\ddot{v}_i,$$

$$B_{ijmn}\left(v_{n,mj}+\varepsilon_{kmn}\varphi_{k,j}\right)+C_{ijmn}\varphi_{n,mj}-\beta_{ij}\left(\vartheta_{,j}+a\dot{\vartheta}_{,j}\right)+$$
$$+\varepsilon_{ijk}\left[A_{jkmn}\left(v_{n,m}+\varepsilon_{lmn}\varphi_l\right)+B_{jkmn}\varphi_{n,m}-\alpha_{ij}\left(\vartheta_{,j}+a\dot{\vartheta}_{,j}\right)\right]=I_{ij}\ddot{\varphi}_j, \quad (13)$$
$$\kappa_{ij}\phi_{,ij}-\alpha_{ij}\left(\dot{v}_{j,i}+\varepsilon_{ijk}\dot{\varphi}_k\right)-\beta_{ij}\dot{\varphi}_{j,i}=h\ddot{\vartheta}+d\dot{\vartheta},$$

which are satisfied for any $(x) \in \Omega \times (0,\infty)$.
By a solution of the mixed initial boundary value problem in the two temperatures thermoelastic theory of dipolar bodies in the cylinder $\Omega \times [0,\infty)$ we mean an ordered array $(v_m,\ \varphi_m,\ \phi,\ \vartheta)$ which satisfies the system of equations (13), the boundary conditions (11) and the initial conditions (12).

3. Main results

We will start this section by specifying a conservation law of energy, considering the rate of the conductive temperature, that is, by taking into account the two-temperatures relation (7). This law has the following form:

$$E_1(t)=E_1(0),\ t\in[0,\infty), \quad (14)$$

where

$$E_1(t)=\frac{1}{2}\int_\Omega\left[\rho\dot{v}_m(t)\dot{v}_m(t)+I_{mn}\dot{\varphi}_m(t)\dot{\varphi}_n(t)+\right.$$
$$+A_{mnkl}e_{mn}(t)e_{kl}(t)+2B_{mnkl}e_{mn}(t)\sigma_{kl}(t)+C_{mnkl}\sigma_{mn}(t)\sigma_{kl}(t)+$$
$$\left.+c\kappa_{mn}\varphi_{,m}(t)\,\varphi_{,n}(t)+d\left(\vartheta(t)+\frac{h}{d}\dot{\vartheta}(t)\right)^2+h\left(a-\frac{h}{d}\right)\dot{\vartheta}^2(t)\right]dV+$$
$$+\int_0^t\int_\Omega\left[\kappa_{mn}\phi_{,m}(s)\phi_{,n}(s)+c\left(\left(\kappa_{mn}\phi_{,m}(s)\right)_{,n}\right)^2+(ad-h)\dot{\vartheta}^2(s)\right]dV\,ds.$$

In the case that we consider the relation (6), that is, we don't take into account the rate of conductivity temperature, the conservation law of energy receives the following form:

$$E_2(t)=E_2(0),\ t\in[0,\infty), \quad (15)$$

where

$$E_2(t) = \frac{1}{2}\int_\Omega [\rho \dot{v}_m(t)\dot{v}_m(t) + I_{mn}\dot{\varphi}_m(t)\dot{\varphi}_n(t) + A_{mnkl}e_{mn}(t)e_{kl}(t) +$$
$$+2B_{mnkl}e_{mn}(t)\sigma_{kl}(t) + C_{mnkl}\sigma_{mn}(t)\sigma_{kl}(t) +$$
$$+d\left(\vartheta(t) + \frac{h}{d}\dot{\vartheta}(t)\right)^2 + h\left(a - \frac{h}{d}\right)\dot{\vartheta}^2(t)\Big] dV +$$
$$+\int_0^t\int_\Omega\Big[\kappa_{mn}\phi_{,m}(s)\phi_{,n}(s) + c\Big(\left(\kappa_{mn}\varphi_{,m}(s)\right)_{,n}\Big)^2 + (ad-h)\dot{\vartheta}^2(s)\Big] dV ds.$$

We will denote by $\mathcal{P}$ the problem consists in system of equations (13), the boundary conditions (11) and the initial conditions (12).
Our two main results, namely, an uniqueness and an instability results for the solution of our mixed problem, will be obtained in the simpler case in which we consider the equation (6). For uniqueness we use the usual procedure.
Consider two solutions $\left(v_m^{(\nu)},\ \varphi^{(\nu)},\ \phi^{(\nu)},\ \vartheta^{(\nu)}\right)$, $\nu = 1, 2$ of the problem $\mathcal{P}$ and denote by $(v_m,\ \varphi_m,\ \phi,\ \vartheta)$ the difference between the two solutions. Due to the linearity of the problem $\mathcal{P}$, the difference is also a solution to the problem, but this corresponds to zero initial data. It remains to be shown that if the problem $\mathcal{P}$ is considered in the case of null initial data, then it admits only the identical null solution.

Theorem 1.. *In the case of null initial data, the mixed problem $\mathcal{P}$, admits only the identical null solution.*

Proof. Consider the difference $(v_i,\ \varphi_{ij},\ \phi,\ \vartheta)$ of two solutions of the problem $\mathcal{P}$. For the demonstration we need the function $F(t)$, which has a logarithmic convexity and, if we consider the equation (6), it is defined by:

$$F(t) = \frac{1}{2}\int_\Omega [\rho v_m v_m + I_{mn}\varphi_m\varphi_n]\, dV +$$
$$+\frac{1}{2}\int_0^t\int_\Omega\Big[\kappa_{mn}\zeta_{,m}(s)\zeta_{,n}(s) + c\Big(\left(\kappa_{mn}\zeta_{,m}(s)\right)_{,n}\Big) + (ad-h)\vartheta^2(s)\Big] dV ds, \tag{16}$$

where the function ζ is defined by:

$$\zeta(x,t) = \int_0^t \phi(x,s)ds.$$

Considering the expression in (16) of the function F, we calculate its first two derivatives:

$$\begin{aligned}\dot{F}(t) = \int_\Omega \left[\rho v_m \dot{v}_m + I_{mn}\varphi_m\dot{\varphi}_n\right] dV + \\ +\frac{1}{2}\int_\Omega\Big[\kappa_{mn}\zeta_{,m}\zeta_{,n} + c\Big(\left(\kappa_{mn}\zeta_{,m}\right)_{,n}\Big) + (ad-h)\vartheta^2\Big] dV, \\ \ddot{F}(t) = \int_\Omega \left[\rho\left(\dot{v}_m\dot{v}_m + v_m\ddot{v}_m\right) + I_{mn}\left(\dot{\varphi}_m\dot{\varphi}_n + \varphi_m\ddot{\varphi}_n\right)\right] dV + \\ +\int_\Omega\Big[\kappa_{mn}\zeta_{,m}\phi_{,n} + (ad-h)\vartheta\dot{\vartheta} + c\Big(\left(\kappa_{mn}\zeta_{,m}\right)_{,n}\Big)\Big(\left(\kappa_{mn}\varphi_{,m}\right)_{,n}\Big)\Big] dV.\end{aligned} \tag{17}$$

If we consider the constitutive relations (2), we deduce the following equality:

$$\begin{aligned}\int_\Omega \left[\rho v_m \ddot{v}_m + I_{mn}\varphi_m\ddot{\varphi}_n + A_{mnkl}e_{mn}e_{kl} + \right. \\ \left. +2B_{mnkl}e_{mn}\sigma_{kl} + C_{mnkl}\sigma_{mn}\sigma_{kl}\right] dV = \\ = \int_\Omega \left(\alpha_{mn}e_{mn} + \beta_{mn}\sigma_{mn}\right)\left(\vartheta + a\dot{\vartheta}\right) dV.\end{aligned} \tag{18}$$

Now, if we take into account the equation (6), we are led to the equality:

$$\begin{aligned}\int_\Omega \Big[\kappa_{mn}\zeta_{,m}\phi_{,n} + \left(\vartheta + a\dot{\vartheta}\right)\left(d\vartheta + h\dot{\vartheta}\right) + \\ +c\left(\left(\kappa_{mn}\zeta_{,m}\right)_{,n}\right)\left(\left(\kappa_{mn}\varphi_{,m}\right)_{,n}\right)\Big] dV = \\ = -\int_\Omega \left(\alpha_{mn}e_{mn} + \beta_{mn}\sigma_{mn}\right)\left(\vartheta + a\dot{\vartheta}\right) dV.\end{aligned} \tag{19}$$

Comparing the equalities (18) and (19), we obtain:

$$\begin{aligned}\int_\Omega \left[\rho v_m \ddot{v}_m + I_{mn}\varphi_m\ddot{\varphi}_n + A_{mnkl}e_{mn}e_{kl} + \right. \\ \left. +2B_{mnkl}e_{mn}\sigma_{kl} + C_{mnkl}\sigma_{mn}\sigma_{kl}\right] dV + \\ +\int_\Omega\Big[\kappa_{mn}\zeta_{,m}\phi_{,n} + \left(\vartheta + a\dot{\vartheta}\right)\left(d\vartheta + h\dot{\vartheta}\right) + \\ +c\left(\left(\kappa_{mn}\zeta_{,m}\right)_{,n}\right)\left(\left(\kappa_{mn}\varphi_{,m}\right)_{,n}\right)\Big] dV = 0.\end{aligned} \tag{20}$$

By direct calculations we obtain the equality:

$$\begin{aligned}\left(d\vartheta(x) + h\dot{\vartheta}(x)\right)\left(\vartheta(x) + a\dot{\vartheta}(x)\right) = (ad-h)\vartheta(x)\dot{\vartheta}(x) + \\ +\frac{h}{d}(ad-h)\left(\dot{\vartheta}(x)\right)^2 + \frac{1}{d}\left(d\vartheta(x) + h\dot{\vartheta}(x)\right)^2.\end{aligned}$$

If we take into account this relationship, equality (20) becomes:

$$\begin{aligned}
&\int_\Omega [\rho v_m \ddot{v}_m + I_{mn}\varphi_m\ddot{\varphi}_n + A_{mnkl}e_{mn}e_{kl}+ \\
&\qquad +2B_{mnkl}e_{mn}\sigma_{kl} + C_{mnkl}\sigma_{ij}\sigma_{kl}]\, dV + \\
&+\int_\Omega\Big[\kappa_{mn}\zeta_{,m}\phi_{,n} + c\left(\left(\kappa_{mn}\zeta_{,m}\right)_{,n}\right)\left(\left(\kappa_{mn}\phi_{,m}\right)_{,n}\right)\Big]dV + \\
&\qquad + \int_\Omega \left[\frac{h}{d}(ad-h)\left(\dot{\vartheta}\right)^2 + (ad-h)\vartheta\dot{\vartheta}\right] dV + \\
&\qquad\qquad + \int_\Omega \left[\frac{1}{d}\left(d\vartheta + h\dot{\vartheta}\right)^2\right] dV = 0.
\end{aligned} \tag{21}$$

An equivalent form of equality (21) is the following:

$$\begin{aligned}
&\int_\Omega \left[\rho v_m \ddot{v}_m + I_{mn}\varphi_m\ddot{\varphi}_n + (ad-h)\vartheta\dot{\vartheta}\right] dV + \\
&\qquad +\int_\Omega\Big[\kappa_{mn}\zeta_{,m}\phi_{,n} + c\left(\left(\kappa_{mn}\zeta_{,m}\right)_{,n}\right)\left(\left(\kappa_{mn}\phi_{,m}\right)_{,n}\right)\Big]dV = \\
&= -\int_\Omega [A_{mnkl}e_{mn}e_{kl} + 2B_{mnkl}e_{mn}\sigma_{kl} + C_{mnkl}\sigma_{mn}\sigma_{kl}]\, dV - \\
&\qquad - \int_\Omega \left[\frac{h}{d}(ad-h)\left(\dot{\vartheta}\right)^2 + \frac{1}{d}\left(d\vartheta + h\dot{\vartheta}\right)^2\right] dV.
\end{aligned} \tag{22}$$

Given (22), we can write the second order derivative of the function $F(t)$, from $(17)_2$, in the following form:

$$\begin{aligned}
\ddot{F}(t) &= \int_\Omega [\rho v_m \ddot{v}_m + I_{mn}\varphi_m\ddot{\varphi}_n]\, dV - \\
&- \int_\Omega [A_{mnkl}e_{mn}e_{kl} + 2B_{mnkl}e_{mn}\sigma_{kl} + C_{mnkl}\sigma_{mn}\sigma_{kl}]\, dV - \\
&\qquad - \int_\Omega \left[\frac{h}{d}(ad-h)\left(\dot{\vartheta}\right)^2 + \frac{1}{d}\left(d\vartheta + h\dot{\vartheta}\right)^2\right] dV.
\end{aligned} \tag{23}$$

On the other hand, based on the law of energy conservation (15), the expression of $\ddot{F}(t)$ is simplified in form:

$$\begin{aligned}
\ddot{F}(t) &= 2\int_\Omega [\rho v_m \ddot{v}_m + I_{mn}\varphi_m\ddot{\varphi}_n]\, dV + \\
&+2\int_0^t\int_\Omega\Big[(ad-h)\left(\dot{\vartheta}\right)^2 + \kappa_{mn}\phi_{,m}\phi_{,n} + c\left(\left(\kappa_{mn}\phi_{,m}\right)_{,n}\right)^2\Big]\, dV ds.
\end{aligned} \tag{24}$$

Taking into account the expression of $\dot{F}(t)$ from from $(17)_1$ and the expression of $\ddot{F}(t)$ from (24), we can deduce that:

$$\frac{\ddot{F}(t)}{\dot{F}(t)} \geq \frac{\dot{F}(t)}{F(t)},\ \forall t \geq 0,$$

and from here it follows that:

$$\frac{d^2}{dt^2}(\ln F(t)) \geq 0.$$

The last inequality ensures that the function $\ln F(t)$ is a convex one, regarding the time variable t.
Considering that the domain of definition of the solution $(v_i,\ \varphi_i,\ \phi,\ \vartheta)$ is the interval $[0, t_0]$, we can integrate the previous inequality to deduce that:

$$F(t) \leq (F(t_0))^{t/t_0}\,(F(0))^{(t_0-t)/t_0},\ 0 \leq t \leq t_0. \tag{25}$$

But the solution $(v_m,\ \varphi_m,\ \phi,\ \vartheta)$ corresponds to null initial, as such from (16) we deduce that $F(0) = 0$ and then from (25) it follows that:

$$F(t) = 0,\ \forall t \in [0, t_0] \Rightarrow v_m(t) = 0,\ \varphi_m(t) = 0,\ \phi(t) = 0,\ \vartheta(t) = 0,\ \forall t \in [0, t_0],$$

and the proof of Theorem 1 is completed. ■
In order to obtain second main result, regarding the exponential instability for the solution of the mixed problem $\mathcal{P}$, we will have to assume that an additional condition is satisfied. Namely, we need to suppose that the energy of the system, in its initial state, is not positive, that is, $E_2(0) \leq 0$.
We will start with some useful auxiliary considerations.
Let us consider a boundary value problem of the following form:

$$\begin{aligned} &(\kappa_{mn} u_{,m}(x))_{,n} = d\vartheta^1 + h\vartheta^0 - \left(\alpha_{mn} e^0_{mn} + \beta_{mn}\sigma^0_{mn}\right),\ x \in \Omega, \\ &\nu(x) = 0,\ x \in \partial\Omega, \end{aligned} \tag{26}$$

where $u = u(x)$ is the unknown function and the constants ϑ^0, ϑ^1, e^0_{mn} and σ^0_{mn} are the initial data from (12).
Based on the usual properties of the boundary value problems, defined in the context of elliptical equations, we can deduce that the boundary value problem (24) admits a solution $u(x)$, defined on the domain Ω.
With the help of the function:

$$\zeta(x, t) = \int_0^t \phi(x, s)ds,$$

from (26) we can observe that the function $u(x)$ is a solution of the equation

$$d\vartheta(x) + h\dot{\vartheta}(x) - \left[\kappa_{mn}\left(u_{,m}(x) + \zeta_{,m}(x)\right)\right]_{,n} = \\ = \alpha_{mn}e_{mn}(x) + \beta_{mn}\sigma_{mn}(x).$$

We can now formulate and demonstrate the second important result of our study. For this we will consider the mixed problem $\mathcal{P}$ in the case the initial data have the general form of (12). Boundary data have the homogeneous form from (11).

Theorem 2.. *We assume that conditions (9) and (10) are met.*
If in the mixed problem $\mathcal{P}$ the boundary data are zero, then any of its solutions, for which the condition $W_2(0) \leq 0$ takes place, is exponentially unstable.

Proof. As in the proof of Theorem 1 the proof was facilitated by the introduction of a function with logarithmic convexity, and in the case of the present theorem we will introduce a function, inspired by papers [26] and [27]. From the study of this function we will deduce that any solution of the problem $\mathcal{P}$ increases exponentially. Therefore, consider the function $G(t)$ defined by:

$$G(t) = \frac{1}{2}\int_\Omega \left(\rho v_m v_m + I_{mn}\varphi_m\varphi_n\right) dV + w\left(t+t_0\right)^2 + \\ + \frac{1}{2}\int_0^t\int_\Omega\left[(ad-h)\vartheta^2 + \kappa_{mn}\left(u_{,m}+\zeta_{,m}\right)\zeta_{,n} + c\left(\kappa_{mn}\left(u_{,m}+\zeta_{,m}\right)_{,n}\right)^2\right] dV ds, \quad (27)$$

where ζ is defined above, and the function u is the solution of the problem (26). The positive constants w and t_0 will be determined later.
Derivatives $\dot{G}(t)$ and $\ddot{G}(t)$ can be obtained by direct derivation in (27):

$$\dot{G}(t) = \int_\Omega \left(\rho v_m \dot{v}_m + I_{mn}\varphi_m\dot{\varphi}_n\right) dV + 2w\left(t+t_0\right) + \\ + \int_0^t\int_\Omega\left[(ad-h)\vartheta\dot{\vartheta} + \kappa_{ij}\left(u_{,m}+\zeta_{,m}\right)\varphi_{,n} + c\left(\left(\kappa_{mn}\phi_{,m}\right)_{,n}\right)\left(\left(\kappa_{mn}\left(u_{,m}+\zeta_{,m}\right)\right)_{,n}\right)\right] dV ds \\ + \frac{1}{2}\int_\Omega\left[(ad-h)\left(\vartheta^0\right)^2 + \kappa_{mn}\phi^0_{,m}\phi^0_{,n} + c\left(\left(\kappa_{mn}\phi^0_{,m}\right)_{,n}\right)^2\right] dV, \quad (28)$$

$$\ddot{G}(t) = \int_\Omega \left[\rho\left(v_m\ddot{v}_m + \dot{v}_m\dot{v}_m\right) + I_{mn}\left(\varphi_m\ddot{\varphi}_n + \dot{\varphi}_m\dot{\varphi}_n\right)\right] dV + 2w + \\ + \int_\Omega\left[(ad-h)\vartheta\dot{\vartheta} + \kappa_{mn}\left(u_{,m}+\zeta_{,m}\right)\phi_{,n} + c\left(\left(\kappa_{mn}\phi_{,m}\right)_{,n}\right)\left(\left(\kappa_{mn}\left(u_{,m}+\zeta_{,m}\right)\right)_{,n}\right)\right] dV.$$

Now, we will use the form of the initial energy E_2, defined after (15), in order to obtain a simplified form of the derivative of second order of the function

$G(t)$, namely:

$$\ddot{G}(t)=2\int_{\Omega}\left(\rho\dot{v}_m\dot{v}_m+I_{mn}\dot{\varphi}_m\dot{\varphi}_n\right)dV+2\left(w-E_2(0)\right)+$$
$$+2\int_0^t\int_{\Omega}\left[(ad-h)\left(\dot{\vartheta}\right)^2+c\left(\left(\kappa_{mn}\phi_{,m}\right)_{,n}\right)^2+\kappa_{mn}\phi_{,m}\phi_{,n}\right]dV\,ds. \tag{29}$$

In order to simplify the writing, we now introduce the notation:

$$I=\int_{\Omega}\left[(ad-h)\left(\vartheta^0\right)^2+c\left(\left(\kappa_{mn}\phi^0_{,m}\right)_{,n}\right)^2+\kappa_{mn}\phi^0_{,m}\phi^0_{,n}\right]dV.$$

Taking into account the expression of the first derivative $\dot{G}(t)$ of $(28)_1$ and of the second order derivative $\ddot{G}(t)$ of (29), with the help of the previous notation we deduce:

$$G(t)\ddot{G}(t)-\left(\dot{G}(t)-\frac{1}{2}I\right)^2\geq 2\left(E_2(0)+w\right)G(t). \tag{30}$$

We have the possibility to choose t_0 large enough so that we have fulfilled condition $\dot{G}(0)>M$. Also, by hypothesis, we supposed that $E_2(0)\leq 0$, so that we can choose $w=-E_2(0)$. Then, from (30) it follows that:

$$G(t)\ddot{G}(t)-\dot{G}(t)\left(\dot{G}(t)-M\right)\geq 0,$$

and so we are led to the conclusion that that the function

$$\frac{\dot{G}(t)-M}{G(t)},$$

is an increasing function which growth, regarding the time variable t. As a a consequence, we can obtain that:

$$\frac{\dot{G}(t)-M}{G(t)}\geq\frac{\dot{G}(0)-M}{G(0)},\quad\forall t\geq 0,$$

from which we deduce that:

$$\dot{G}(t)\geq\frac{\dot{G}(t)-M}{G(0)}G(t)+M.$$

Finally, if we integrate this inequality on the interval $[0,t]$ we are led to the conclusion that:

$$G(t)\geq\frac{G(0)}{\dot{G}(0)-I}\left(\dot{G}(0)e^{\frac{\dot{G}(0)-M}{G(0)}t}-1\right),$$

and this inequality assures us that the solutions of the problem $\mathcal{P}$ are exponential growth functions, that is, are exponentially unstable.
Thus we have concluded the proof of Theorem 2. $\blacksquare$

4. Conclusions

It is almost obvious that if we consider the rate of the conductive temperature, that is, we take into account the two-temperatures relation (7), then we can prove both the above main results, that is, the uniqueness result and the exponentially instability results. For this, after replacing equation (6) with equation (7), we must substitute the initial energy of system $W_2(t)$ with the energy $W_1(t)$.

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