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A Study on Commutative Elliptic Octonion Matrices

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Abstract

In this study, firstly notions of similarity and consimilarity are given for commutative elliptic octonion matrices. Then the Kalman-Yakubovich s-conjugate equation is solved for the first conjugate of commutative elliptic octonions. Also, the notions of eigenvalue and eigenvector are studied for commutative elliptic octonion matrices. In this regard, the fundamental theorem of algebra and Gershgorin's Theorem are proved for commutative elliptic octonion matrices. Finally, some examples related to our theorems are provided.

1 Introduction

The octonion algebra is an eight-dimensional division algebra by the Cayley-Dickson method, [17]. Since these numbers do not provide the properties of commutative law and linear combination, their applications have been limited. Therefore, to solve the difficulties encountered in the equation of solutions, studies have recently been carried out in the field of octonion matrices, [2, 4, 6, 14, 16].

The notions such as eigenvalue and eigenvector, which have an important place in matrix theory, are used in the solution of many equations and one of the most important of them is the Gershgorin Theorem, which is used to determine the eigenvalues of a matrix, [1, 3, 7, 8, 9, 10, 11, 13, 15, 18].

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In this study, the similarity and consimilarity notions are given together with the isomorphism defined between commutative elliptic octonions and their matrices. Then the real-valued representation of a commutative elliptic octonion matrix is defined and related theorems are given. Considering these theorems and definitions, Kalman Yakubovich s-conjugate linear equation is solved. Finally, the fundamental theorem of algebra and the Gershgorin Theorem for commutative elliptic octonions are proved, and then examples related to them are given.

2 Algebraic Properties of Commutative Elliptic Octonions

In this section, we will give the algebraic properties of the commutative elliptic octonion set based on elliptic numbers and commutative octonions, which have widely considered in the literature.

The set of commutative elliptic octonion is defined as

$$CO_p = \{a = a_0e_0 + a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4 + a_5e_5 + a_6e_6 + a_7e_7 \mid a_i \in R,\ 0 \leq i \leq 7\}$$

where $\{e_i;\ 0 \leq i \leq 7\}$ is a base of the commutative elliptic octonion.

Let a be a commutative elliptic octonion which is expressed as

$$a = a' + a''e. \tag{1}$$

Since $a' = a_0 + a_1i + a_2j + a_3k \in H_p$, $a'' = a_4 + a_5i + a_6j + a_7k \in H_p$, the base vectors of a commutative elliptic octonion are defined by

$$\begin{array}{llll} e_0 = 1, & e_4 = e, & {e_0}^2 = 1, & {e_4}^2 = 1, \\ e_1 = i, & e_5 = ie = ei, & {e_1}^2 = \alpha, & {e_5}^2 = \alpha, \\ e_2 = j, & e_6 = je = ej, & {e_2}^2 = 1, & {e_6}^2 = 1, \\ e_3 = k, & e_7 = ke = ek, & {e_3}^2 = \alpha, & {e_7}^2 = \alpha, \end{array} \tag{2}$$

[5].

Considering the equation (1) for a commutative elliptic octonion, the conjugate definition for a commutative elliptic octonion is defined by the following equations:

$$\begin{aligned} a^{o_1} &= a'^{(1)} + a''^{(1)}e, \\ a^{o_2} &= a'^{(2)} + a''^{(2)}e, \\ a^{o_3} &= a'^{(3)} + a''^{(3)}e, \\ a^{o_4} &= a' - a''e, \\ a^{o_5} &= a'^{(1)} - a''^{(1)}e, \\ a^{o_6} &= a'^{(2)} - a''^{(2)}e, \\ a^{o_7} &= a'^{(3)} - a''^{(3)}e, \end{aligned} \tag{3}$$

[5]. The expressions (1), (2) and (3) correspond to the conjugates definition for the elliptic quaternions, [12].

Considering (3), the norm of a commutative elliptic octonion is defined as

$$\begin{aligned} \|a\|^8 &= a \times a^{o_1} \times a^{o_2} \times a^{o_3} \times a^{o_4} \times a^{o_5} \times a^{o_6} \times a^{o_7} \\ &= \left[(a_0 + a_2 - a_4 - a_6)^2 - \alpha(a_1 + a_3 - a_5 - a_7)^2\right] \\ &\times \left[(a_0 - a_2 + a_4 - a_6)^2 - \alpha(a_1 - a_3 + a_5 - a_7)^2\right] \\ &\times \left[(a_0 - a_2 - a_4 + a_6)^2 - \alpha(a_1 - a_3 - a_5 + a_7)^2\right] \\ &\times \left[(a_0 + a_2 + a_4 + a_6)^2 - \alpha(a_1 + a_3 + a_5 + a_7)^2\right] \geq 0, \end{aligned} \tag{4}$$

[5].

Let $a = \sum_{i=0}^{7} a_i e_i$ and $b = \sum_{i=0}^{7} b_i e_i$ be two commutative elliptic octonions, then the multiplication of two commutative elliptic octonions is defined by the following equation

$$\begin{aligned} a \times b = {} & (a_0b_0 + \alpha a_1b_1 + a_2b_2 + \alpha a_3b_3 + a_4b_4 + \alpha a_5b_5 + a_6b_6 + \alpha a_7b_7)\, e_0 \\ & + (a_0b_1 + a_1b_0 + a_2b_3 + a_3b_2 + a_4b_5 + a_5b_4 + a_6b_7 + a_7b_6)\, e_1 \\ & + (a_0b_2 + \alpha a_1b_3 + a_2b_0 + \alpha a_3b_1 + a_4b_6 + \alpha a_5b_7 + a_6b_4 + \alpha a_7b_5)\, e_2 \\ & + (a_0b_3 + a_1b_2 + a_2b_1 + a_3b_0 + a_4b_7 + a_5b_6 + a_6b_5 + a_7b_4)\, e_3 \\ & + (a_0b_4 + \alpha a_1b_5 + a_2b_6 + \alpha a_3b_7 + a_4b_0 + \alpha a_5b_1 + a_6b_2 + \alpha a_7b_3)\, e_4 \\ & + (a_0b_5 + a_1b_4 + a_2b_7 + a_3b_6 + a_4b_1 + a_5b_0 + a_6b_3 + a_7b_2)\, e_5 \\ & + (a_0b_6 + \alpha a_1b_7 + a_2b_4 + \alpha a_3b_5 + a_4b_2 + \alpha a_5b_3 + a_6b_0 + \alpha a_7b_1)\, e_6 \\ & + (a_0b_7 + a_1b_6 + a_2b_2 + a_3b_4 + a_4b_3 + a_5b_2 + a_6b_1 + a_7b_0)\, e_7, \end{aligned} \tag{5}$$

[5].

The expression of $a \in CO_p$ in terms of an 8×1 dimensional matrix is given by

$$a = \sum_{i=0}^{7} a_i e_i \cong \mathbf{a} = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 \end{bmatrix}^T \in R^{8\times 1} \qquad (6)$$

[5].

On the other hand, considering equations (5) and (6), the multiplication of two commutative elliptic octonions a and b is defined as

$$a\times b = b\times a \cong \varphi(a)\,\mathbf{b} = \begin{bmatrix} a_0 & \alpha a_1 & a_2 & \alpha a_3 & a_4 & \alpha a_5 & a_6 & \alpha a_7 \\ a_1 & a_0 & a_3 & a_2 & a_5 & a_4 & a_7 & a_6 \\ a_2 & \alpha a_3 & a_0 & \alpha a_1 & a_6 & \alpha a_7 & a_4 & \alpha a_5 \\ a_3 & a_2 & a_1 & a_0 & a_7 & a_6 & a_5 & a_4 \\ a_4 & \alpha a_5 & a_6 & \alpha a_7 & a_0 & \alpha a_1 & a_2 & \alpha a_3 \\ a_5 & a_4 & a_7 & a_6 & a_1 & a_0 & a_3 & a_2 \\ a_6 & \alpha a_7 & a_4 & \alpha a_5 & a_2 & \alpha a_3 & a_0 & \alpha a_1 \\ a_7 & a_6 & a_5 & a_4 & a_3 & a_2 & a_1 & a_0 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \\ b_6 \\ b_7 \end{bmatrix}$$

where $\varphi(a)$ is the basic matrix of the commutative elliptic octonion a. The function φ determines an isomorphism as $\varphi : CO_p \to M$, where M is the set of elementary matrices of commutative elliptic octonions. Accordingly the following theorem is given.

Theorem 2.1. *Let a and b be two commutative elliptic octonions and β_1, β_2 be any real numbers. Then the following identities are held:*

1) $a = b \Leftrightarrow \varphi(a) = \varphi(b)$,
2) $\varphi(a + b) = \varphi(a) + \varphi(b)$,
   $\varphi(a \times b) = \varphi(a)\varphi(b)$,
3) $\varphi(\beta_1 a + \beta_2 b) = \beta_1\varphi(a) + \beta_2\varphi(b)$,
4) $\|a\|^8 = |\det(\varphi(a))|$,
5) $Trace(\varphi(a)) = 8a_0$,

[5].

On the other hand, since a commutative elliptic octonion $a = \sum_{i=0}^{7} a_i e_i$ can be expressed as a hyperbolic number

$$a = a' + a''e \qquad (7)$$

here are $a',\ a'' \in H_p$ and $e^2 = 1$.

Taking (7), the function

$$\begin{array}{l} \psi_a : CO_p \to CO_p \\ \qquad b \quad \to \psi_a\,(b) = a \times b \end{array}$$

is defined for any $b \in CO_p$, and if this transformation is considered

$$N = \left\{ \left(\begin{array}{cc} a' & a'' \\ a'' & a' \end{array} \right) : \ a', a'' \in H_p \right\}$$

can be given. In that case, an isomorphism between a commutative elliptic octonion and a 2×2 type matrix is defined by

$$\begin{array}{l} \psi : CO_p \to N \\ \qquad a = a' + a''e \to \psi\,(a) = \left(\begin{array}{cc} a' & a'' \\ a'' & a' \end{array} \right), \end{array}$$

[5]. Along with this isomorphism, the following theorem is given.

Theorem 2.2. *Let be $a \in CO_p$, then there is 2×2 type of the elliptic quaternion matrix corresponding the matrix a,* [5].

Since there is an isomorphism between commutative elliptic octonions and matrices, similarity and consimilarity definitions defined on matrices can be given for commutative elliptic octonions. Now, let us give definitions of similarity and consimilarity.

Definition 2.1. *Let $a, a_1, a_2 \in CO_p$, if there is a $(\|a\| \neq 0)$ that provides $a^{-1}a_1a = a_2$, a_1 and a_2 are called similar. This state is denoted by $a_1 \sim a_2$.*

Definition 2.2. *Let $a_1, a_2 \in CO_p$, if there is $a \in CO_p$ $(\|a\| \neq 0)$ providing $a^{o_i}a_1a^{-1} = a_2$ $(1 \leq i \leq 7)$, a_1 and a_2 are called consimilar. This state is denoted by $a_1 \overset{c_i}{\sim} a_2$.*

Theorem 2.3. *The consimilarity relation in commutative elliptic octonions is an equivalence relation.*

Proof. Let a, a_1, a_2, a_3 be commutative elliptic octonions. Let us show that the relation $\overset{c_i}{\sim}$ $(1 \leq i \leq 7)$ satisfies the following properties:

i. Reflection : $a_1 \overset{c_i}{\sim} a_1$,
ii. Symmetry : $a_1 \overset{c_i}{\sim} a_2$ if and only if $a_2 \overset{c_i}{\sim} a_1$,
iii. Transitive : If $a_1 \overset{c_i}{\sim} a_2$ and $a_2 \overset{c_i}{\sim} a_3$ then $a_1 \overset{c_i}{\sim} a_3$.

i. Since $1a1^{-1} = a$, $a \overset{c_i}{\sim} a$ is provided. Therefore, the reflection property is provided for $\overset{c_i}{\sim}$ $(1 \leq i \leq 7)$.

ii. Let $a_1 \overset{c_i}{\sim} a_2$ be satisfied. In other words, there is a $(\|a\| \neq 0)$ providing $a^{o_i} a_1 a^{-1} = a_2$. Since

$$(a^{o_i})^{-1} a_2 a = (a^{o_i})^{-1} a^{o_i} a_1 a^{-1} a = a_1$$

is provided, $a_2 \overset{c_i}{\sim} a_1$ can be written. In this case, the relation $\overset{c_i}{\sim}$ $(1 \leq i \leq 7)$ provides the symmetry property.

iii. Let the relations $a_1 \overset{c_i}{\sim} a_2$ and $a_2 \overset{c_i}{\sim} a_3$ be provided. Thus, there are commutative elliptic octonions a and b $(\|a\| \neq 0,\ \|b\| \neq 0)$ that satisfy, the equations $a^{o_i} a_1 a^{-1} = a_2$ and $b^{o_i} a_2 b^{-1} = a_3$. In this case, since

$$a_3 = b^{o_i} a_2 b^{-1} = b^{o_i} a^{o_i} a_1 a^{-1} a b^{-1} = (ba)^{o_i} a_1 (ab)^{-1}$$

is provided, it becomes $a_1 \overset{c_i}{\sim} a_3$, that is the property of transitive law is satisfied for $\overset{c_i}{\sim}$ $(1 \leq i \leq 7)$.
Since conditions i, ii and iii are provided, $\overset{c_i}{\sim}$ $(1 \leq i \leq 7)$ is an equivalence relation. □

As a result of this theorem, it can be asserted that the norms of two adjoint similarity commutative elliptic octonions are equal to each other.

3 Consimilarity of Commutative Elliptic Octonion Matrices

The set of $m \times n$ matrices whose members are commutative elliptic octonions is a ring with addition and multiplication operations in matrices, and it is denoted by $U_{m\times n}(CO_p)$. Considering the conjugate definitions of commutative elliptic octonions, the conjugates and transposition of the matrix $A \in U_{m\times n}(CO_p)$ are denoted by $A^{o_k} = ({a_{ij}}^{o_k})$ $(1 \leq k \leq 7)$ and $A^T \in U_{n\times m}(CO_p)$, respectively, [5].

Theorem 3.1. *Let $A \in U_{m\times n}(CO_p)$ and $B \in U_{n\times s}(CO_p)$. Then the following properties are provided for A, B*

i. $\left(A^{o_k}\right)^T = \left(A^T\right)^{o_k} \quad (1 \le k \le 7)$,
ii. $(AB)^T = B^T A^T$,
iii. $(AB)^{o_k} = A^{o_k} B^{o_k} \quad (1 \le k \le 7)$,
iv. If A and B are invertible, $(AB)^{-1} = B^{-1}A^{-1}$,
[5].

Definition 3.1. *Let A and B be $n \times n$ type commutative elliptic octonion matrices. In that case, the matrices A and B are similar but there is an invertible matrix $P \in U_{n\times n}(CO_p)$ that provides the equation $P^{-1}AP = B$. The similarities of the matrices A and B are expressed as $A \sim B$. $\sim$ expression is an equivalence relation on the set $U_{n\times n}(CO_p)$.*

Definition 3.2. *Let A and B be $n \times n$ type commutative elliptic octonion matrices. In that case, the matrices A and B are consimilarity but there is an invertible matrix $P \in U_{n\times n}(CO_p)$ that provides the equation $P^{o_i}AP^{-1} = B$ $(1 \le i \le 7)$. The consimilarity of matrix A and B is expressed as $A \overset{c_i}{\sim} B$. $\overset{c_i}{\sim}$ $(1 \le i \le 7)$ is an equivalence relation on the set $U_{n\times n}(CO_p)$.*

Definition 3.3. *Let $A \in U_{n\times n}(CO_p)$ and $\lambda \in CO_p$. If there is a nonzero matrix $x \in U_{n\times 1}(CO_p)$ that provides the equation $Ax^{o_i} = x\lambda$ $(1 \le i \le 7)$, λ is called the commutative elliptic octonion, the coneigenvalue of the matrix A, and the matrix x is called coneigenvector corresponding to the commutative elliptic octonion λ. The set of coneigenvalues of the matrix A is defined by*

$$\xi^{o_i}(A) = \left\{\lambda \in CO_p : \ Ax^{o_i} = x\lambda, \ x \neq 0 \ \text{ and } 1 \le \mathrm{i} \le 7\right\}.$$

Theorem 3.2. *Let A* and *$B \in U_{n\times n}(CO_p)$. A and B are consimilarity of matrices whereas the matrices A and B have the same coneigenvalues.*

Proof. Let A and $B \in U_{n\times n}(CO_p)$ be consimilarity matrices. Then there is an invertible matrix $P \in U_{n\times n}(CO_p)$ that provided $B = P^{o_i}AP^{-1}$ $(1 \le i \le 7)$. Let λ be the coneigenvalues of the matrix A and $x \in U_{n\times 1}(CO_p)$ be the eigenvector corresponding to the coneigenvalue λ. In this case, $Ax^{o_i} = x\lambda$ $(1 \le i \le 7)$ is provided. If we consider the equation $Y = Px^{o_i}$ $(1 \le i \le 7)$,

$$BY = P^{o_i}AP^{-1}Y = P^{o_i}AP^{-1}Px^{o_i} = P^{o_i}x\lambda = Y^{o_i}\lambda$$

is found. Thus, the proof is completed. □

Theorem 3.3. *If the coneigenvalue of the matrix A is λ, then $\beta^{o_i}\lambda\beta^{-1}$ $(1 \le i \le 7)$ is the coneigenvalue of the A matrix where is $\beta \in CO_p$ $(\beta \neq 0)$.*

Proof. If the coneigenvalue of the matrix A is λ, then the equality $Ax^{o_i} = x\lambda$ $(1 \le i \le 7)$ is provided and $0 \neq x \in U_{n\times 1}(CO_p)$ corresponding to commutative elliptic octonion λ exists. So, since the equations $Ax^{o_i}\beta^{-1} = x\lambda\beta^{-1} =$

$x(\beta^{o_i})^{-1}\beta^{o_i}\lambda\beta^{-1}$ are provided, $\beta^{o_i}\lambda\beta^{-1}$ $(1 \leq i \leq 7)$ is also a coneigenvalue for matrix A. The proof of necessary condition is easily seen and the proof is concluded. □

Definition 3.4. *Let $A = A_1 + A_2 e \in U_{n\times n}(CO_p)$ and $\eta(A)$. Then the 2×2 dimensional matrix*

$$\eta(A) = \begin{pmatrix} A_1 & A_2 \\ A_2 & A_1 \end{pmatrix}.$$

is adjoint matrix of A and is denoted by $\eta(A)$, [5].

Theorem 3.4. *Let $A, B \in U_{n\times n}(CO_p)$. Then the following properties are held;*

i. $\eta(I_n) = I_{2n}$,
ii. $\eta(A+B) = \eta(A) + \eta(B)$,
iii. $\eta(AB) = \eta(A)\eta(B)$,
iv. If $A^{-1} \neq 0$, $\eta(A^{-1}) = (\eta(A))^{-1}$,

[5].

Theorem 3.5. *Let $A \in U_{n\times n}(CO_p)$ and A be the adjoint matrix of $\eta(A)$. So the set of coneigenvalues of $\eta(A)$ is*

$$\xi^{o_i}(A) \cap H_p = \xi^{o_i}(\eta(A)) \qquad (1 \leq i \leq 7)$$

where $\xi^{o_i}(\eta(A)) = \{\lambda \in H_p : \ \eta(A) X^{o_i} = X\lambda, \ 0 \neq X \in U_{n\times 1}(CO_p), \ 1 \leq i \leq 7\}$.

Proof. Let $A = A_1 + A_2 e \in U_{n\times n}(CO_p)$ and $A_1, A_2 \in H_p^{n\times n}$. There is $0 \neq X = X_1 + X_2 e \in U_{n\times 1}(CO_p)$ that satisfies $AX^{o_i} = X\lambda$ $(1 \leq i \leq 7)$, where $\lambda \in H_p$ is the coneigenvalue of A. Then

$$(A_1 + A_2 e)({X_1}^{o_i} + {X_2}^{o_i} e) = (X_1 + X_2 e)\lambda,$$
$$A_1 {X_1}^{o_i} + A_2 {X_2}^{o_i} = X_1\lambda \quad \text{and} \quad A_2 {X_1}^{o_i} + A_1 {X_2}^{o_i} = X_2\lambda$$

and

$$\begin{bmatrix} A_1 - \lambda_1 I & A_2 - \lambda_2 I \\ A_2 - \lambda_2 I & A_1 - \lambda_1 I \end{bmatrix} \begin{bmatrix} {X_1}^{o_i} \\ {X_2}^{o_i} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

are written. As can be seen from the above equations, the elliptic quaternion coneigenvalue of A is equal to the coneigenvalue of $\eta(A)$. So

$$\xi^{o_i}(\eta(A)) = \{\lambda \in H_p : \ \eta(A) X^{o_i} = X\lambda, \ 0 \neq X \in U_{n\times 1}(CO_p), \ 1 \leq i \leq 7\}$$

is provided. □

4 Real Representations of Commutative Elliptic Octonion Matrices

Let $A = A_0 + A_1 i + A_2 j + A_3 k + A_4 e + A_5 ei + A_6 ej + A_7 ek \in U_{m\times n}(CO_p)$ and ϕ be linear isomorphism such that $\phi_A(X) = AX^{o_1}$ where X is any $m \times n$ dimensional commutative elliptic octonion matrix. With this isomorphism, the real matrix

$$\phi_A = \begin{pmatrix} A_0 & -\alpha A_1 & A_2 & -\alpha A_3 & A_4 & -\alpha A_5 & A_6 & -\alpha A_7 \\ A_1 & -A_0 & A_3 & -A_2 & A_5 & -A_4 & A_7 & -A_6 \\ A_2 & -\alpha A_3 & A_0 & -\alpha A_1 & A_6 & -\alpha A_7 & A_4 & -\alpha A_5 \\ A_3 & -A_2 & A_1 & -A_0 & A_7 & -A_6 & A_5 & -A_4 \\ A_4 & -\alpha A_5 & A_6 & -\alpha A_7 & A_0 & -\alpha A_1 & A_2 & -\alpha A_3 \\ A_5 & -A_4 & A_7 & -A_6 & A_1 & -A_0 & A_3 & -A_2 \\ A_6 & -\alpha A_7 & A_4 & -\alpha A_5 & A_2 & -\alpha A_3 & A_0 & -\alpha A_1 \\ A_7 & -A_6 & A_5 & -A_4 & A_3 & -A_2 & A_1 & -A_0 \end{pmatrix} \in R^{8m\times 8n} \quad (8)$$

corresponding to the base $\{1, i, j, k, e, ei, ej, ek\}$ is obtained. Here, ϕ_A corresponds to the real representation of A.

Commutative elliptic octonion matrix A is isomorphic to $\mathbf{A} \in R^{8m\times n}$. This situation is denoted by $\cong$ and written by

$$A = A_0 + A_1 i + A_2 j + A_3 k + A_4 e + A_5 ei + A_6 ej + A_7 ek \cong \mathbf{A} = \begin{bmatrix} A_0 \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \\ A_6 \\ A_7 \end{bmatrix} \in R^{8m\times n}.$$

In that case, considering the multiplication process defined on matrices,

$$AB^{o_1} \cong \phi_A \mathbf{B}$$

is provided for matrices $A \in U_{m\times n}(CO_p)$ and $B \in U_{n\times k}(CO_p)$.

Theorem 4.1. *Let A be a $m \times n$ type commutative elliptic octonion matrix. Then the following properties are provided for A:*

i. $(P_m^{o_1})^{-1}\phi_A(P_n^{o_1}) = \phi_{A^{o_1}}$, $\quad (Q_m)^{-1}\phi_A(Q_n) = -\phi_A$,
$(R_m)^{-1}\phi_A(R_n) = \phi_A$, $\quad (S_m)^{-1}\phi_A(S_n) = -\phi_A$,
$(T_m)^{-1}\phi_A(T_n) = \phi_A$, $\quad (U_m)^{-1}\phi_A(U_n) = -\phi_A$,
$(V_m)^{-1}\phi_A(V_n) = \phi_A$, $\quad (W_m)^{-1}\phi_A(W_n) = -\phi_A$,

where

$$P_m{}^{o1} = \begin{bmatrix} I_m & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -I_m & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I_m & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -I_m & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_m & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -I_m & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I_m & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -I_m \end{bmatrix}$$

$$Q_m = \begin{bmatrix} 0 & \alpha I_m & 0 & 0 & 0 & 0 & 0 & 0 \\ I_m & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha I_m & 0 & 0 & 0 & 0 \\ 0 & 0 & I_m & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha I_m & 0 & 0 \\ 0 & 0 & 0 & 0 & I_m & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha I_m \\ 0 & 0 & 0 & 0 & 0 & 0 & I_m & 0 \end{bmatrix}$$

$$R_m = \begin{bmatrix} 0 & 0 & I_m & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_m & 0 & 0 & 0 & 0 \\ I_m & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I_m & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I_m & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_m \\ 0 & 0 & 0 & 0 & I_m & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_m & 0 & 0 \end{bmatrix}$$

$$S_m = \begin{bmatrix} 0 & 0 & 0 & \alpha I_m & 0 & 0 & 0 & 0 \\ 0 & 0 & I_m & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha I_m & 0 & 0 & 0 & 0 & 0 & 0 \\ I_m & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha I_m \\ 0 & 0 & 0 & 0 & 0 & 0 & I_m & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha I_m & 0 & 0 \\ 0 & 0 & 0 & 0 & I_m & 0 & 0 & 0 \end{bmatrix}$$

$$T_m = \begin{bmatrix} 0 & 0 & 0 & 0 & I_m & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_m & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I_m & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_m \\ I_m & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I_m & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I_m & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_m & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$U_m = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \alpha I_m & 0 & 0 \\ 0 & 0 & 0 & 0 & I_m & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha I_m \\ 0 & 0 & 0 & 0 & 0 & 0 & I_m & 0 \\ 0 & \alpha I_m & 0 & 0 & 0 & 0 & 0 & 0 \\ I_m & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha I_m & 0 & 0 & 0 & 0 \\ 0 & 0 & I_m & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$V_m = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & I_m & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_m \\ 0 & 0 & 0 & 0 & I_m & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_m & 0 & 0 \\ 0 & 0 & I_m & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_m & 0 & 0 & 0 & 0 \\ I_m & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I_m & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$W_m = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha I_m \\ 0 & 0 & 0 & 0 & 0 & 0 & I_m & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha I_m & 0 & 0 \\ 0 & 0 & 0 & 0 & I_m & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha I_m & 0 & 0 & 0 & 0 \\ 0 & 0 & I_m & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha I_m & 0 & 0 & 0 & 0 & 0 & 0 \\ I_m & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

ii. For $A, B \in U_{m\times n}(CO_p)$, $\phi_{A+B} = \phi_A + \phi_B$ *is provided.*

iii. For $A \in U_{m\times n}(CO_p)$ *and* $B \in U_{n\times r}(CO_p)$,

$$\phi_{A\times B} = \phi_A \left({P_n}^{o_1}\right) \phi_B = \phi_A \phi_{B^{o_1}} \left({P_r}^{o_1}\right)$$

is provided.

iv. If the matrix $A \in U_{m\times m}(CO_p)$ *and* A *is invertible, then* ϕ_A *can be inverted and* $(\phi_A)^{-1} = \left({P_m}^{o_1}\right)\phi_{A^{-1}}\left({P_m}^{o_1}\right)$ *is provided.*

v.

$$\xi^{o_1}(A) \cap H_p = \xi(\phi_A),$$

including $A \in U_{m\times m}(CO_p)$, *is provided. Here the set* $\xi(\phi_A) = \{\lambda \in H_p : \ \phi(A)Y = \lambda Y, \ \ 0 \neq Y \in U_{m\times 1}(CO_p)$ *becomes the eigenvalue set of* ϕ_A.

Proof. Proofs of i, ii and iii are easily seen. Let's check at the proof of the cases iv and v.

iv. Let $A \in U_{m\times m}(CO_p)$ be an invertible matrix. In that case,
$$\phi_{AA^{-1}} = \phi_A (P_m)^{o_1} \phi_{A^{-1}} = \phi_{I_8} \quad \text{and} \quad \phi_A (P_m)^{o_1} \phi_{A^{-1}} (P_m)^{o_1} = \phi_{I_{8m}}$$
are written from $AA^{-1} = I_8$. From here it can be seen that ϕ_A is an invertible matrix and $(\phi_A)^{-1} = (P_m)^{o_1}\phi_{A^{-1}}(P_m)^{o_1}$ is found.

v. Let $A = \sum\limits_{i=0}^{7} A_i e_i \in U_{m\times m}(CO_p)$ and $A_s \in R^{m\times m} \quad (0 \leq s \leq 7)$. $\lambda \in H_p$ is the conjugate eigenvalue of A, and there is conjugate eigenvalue $0 \neq X \in U_{m\times 1}(CO_p)$ corresponding to λ and satisfying $AX^{o_i} = X\lambda \quad (1 \leq i \leq 7)$. Here, $\phi_A X = X\lambda$ is written. So, the eigenvalue of matrix ϕ_A corresponds to A. As a result, $\xi^{o_1}(A) \cap H_p = \xi(\phi_A)$ is obtained.

□

Now let's consider the solution of

$$X - AX^{o_1}B = C \tag{9}$$

which the Kalman-Yakubovich s-conjugate equation for commutative elliptic octonion matrices is. Here is $A \in U_{m\times m}(CO_p)$, $B \in U_{n\times n}(CO_p)$ and $C \in U_{m\times n}(CO_p)$. In addition, the real representation of (9) is expressed by

$$Y - \phi_A Y \phi_B = \phi_C. \tag{10}$$

On the other hand, considering the equation $\phi_A X = A X^{o_1}$ and the Theorem 4.1,

$$\begin{aligned} X - AX^{o_1}B = C \quad &\Leftrightarrow \quad X - \phi_A X B = C \\ &\Leftrightarrow \quad (X - \phi_A X B)\, X^{o_1} = C X^{o_1} \\ &\Leftrightarrow \quad \phi_X - \phi_A X \phi_B = \phi_C \end{aligned}$$

is written. As can be seen here, if X is a solution in (9) , $\phi_X = Y$ is a solution in (10). So $X - AX^{o_1}B = C$ has a solution if and only if for $\phi_X = Y$, $Y - \phi_A Y \phi_B = \phi_C$ is a solution.

Theorem 4.2. *Let $A \in U_{m\times m}(CO_p)$, $B \in U_{n\times n}(CO_p)$ and $C \in U_{m\times n}(CO_p)$. If $Y \in R^{8m\times 8n}$ is the solution of $Y - \phi_A Y \phi_B = \phi_C$, the solution of the $X - AX^{o_1}B = C$ is*

$$X = \frac{1}{32 - 32\alpha} \begin{bmatrix} I_m \\ iI_m \\ jI_m \\ kI_m \\ eI_m \\ eiI_m \\ ejI_m \\ ekI_m \end{bmatrix}^T \left(\begin{array}{l} Y - Q_m^{-1}\phi_X Q_n + R_m^{-1}\phi_X R_n \\ -S_m^{-1}\phi_X S_n + T_m^{-1}\phi_X T_n \\ -U_m^{-1}\phi_X U_n + V_m^{-1}\phi_X V_n - W_m^{-1}\phi_X W_n \end{array} \right) \begin{bmatrix} I_n \\ iI_n \\ jI_n \\ kI_n \\ eI_n \\ eiI_n \\ ejI_n \\ ekI_n \end{bmatrix}. \tag{11}$$

Proof. Let

$$Y = \begin{bmatrix} Y_{11} & Y_{12} & Y_{13} & Y_{14} & Y_{15} & Y_{16} & Y_{17} & Y_{18} \\ Y_{21} & Y_{22} & Y_{23} & Y_{24} & Y_{25} & Y_{26} & Y_{27} & Y_{28} \\ Y_{31} & Y_{32} & Y_{33} & Y_{34} & Y_{35} & Y_{36} & Y_{37} & Y_{38} \\ Y_{41} & Y_{42} & Y_{43} & Y_{44} & Y_{45} & Y_{46} & Y_{47} & Y_{48} \\ Y_{51} & Y_{52} & Y_{53} & Y_{54} & Y_{55} & Y_{56} & Y_{57} & Y_{58} \\ Y_{61} & Y_{62} & Y_{63} & Y_{64} & Y_{65} & Y_{66} & Y_{67} & Y_{68} \\ Y_{71} & Y_{72} & Y_{73} & Y_{74} & Y_{75} & Y_{76} & Y_{77} & Y_{78} \\ Y_{81} & Y_{82} & Y_{83} & Y_{84} & Y_{85} & Y_{86} & Y_{87} & Y_{88} \end{bmatrix} \tag{12}$$

be a solution to (10) and where $Y_{uv} \in R^{m\times n} \quad (1 \le u,\ v \le 8)$ is. In that case, since $\phi_X = Y$, the following equations are provided;

$$\begin{array}{ll} Q_m^{-1}\phi_X Q_n = -Y, & U_m^{-1}\phi_X U_n = -Y, \\ R_m^{-1}\phi_X R_n = Y, & V_m^{-1}\phi_X V_n = Y, \\ S_m^{-1}\phi_X S_n = -Y, & W_m^{-1}\phi_X W_n = -Y, \\ T_m^{-1}\phi_X T_n = Y, & \end{array} \tag{13}$$

$$\begin{array}{l} -Q_m^{-1}YQ_n - \phi_A\left(-Q_m^{-1}YQ_n\right)\phi_B = \phi_C, \\ R_m^{-1}YR_n - \phi_A\left(R_m^{-1}YR_n\right)\phi_B = \phi_C, \\ -S_m^{-1}YS_n - \phi_A\left(-S_m^{-1}YS_n\right)\phi_B = \phi_C, \\ \mathrm{T}_m^{-1}YT_n - \phi_A\left(T_m^{-1}YT_n\right)\phi_B = \phi_C, \\ -U_m^{-1}YU_n - \phi_A\left(-U_m^{-1}YU_n\right)\phi_B = \phi_C, \\ V_m^{-1}YV_n - \phi_A\left(V_m^{-1}YV_n\right)\phi_B = \phi_C, \\ -W_m^{-1}YW_n - \phi_A\left(-W_m^{-1}YW_n\right)\phi_B = \phi_C, \end{array} \tag{14}$$

are written. As a result, if Y is a solution for (9), (13) are also a solution. So,

$$Y^{'}=\frac{1}{8}\left(\begin{array}{l} Y-Q_m^{-1}\phi_X Q_n+R_m^{-1}\phi_X R_n-S_m^{-1}\phi_X S_n \\ +T_m^{-1}\phi_X T_n-U_m^{-1}\phi_X U_n+V_m^{-1}\phi_X V_n-W_m^{-1}\phi_X W_n \end{array}\right) \tag{15}$$

is also a solution to (10).

$$Y^{'}=\begin{pmatrix} Z_0 & -\alpha Z_1 & Z_2 & -\alpha Z_3 & Z_4 & -\alpha Z_5 & Z_6 & -\alpha Z_7 \\ Z_1 & -Z_0 & Z_3 & -Z_2 & Z_5 & -Z_4 & Z_7 & -Z_6 \\ Z_2 & -\alpha Z_3 & Z_0 & -\alpha Z_1 & Z_6 & -\alpha Z_7 & Z_4 & -\alpha Z_5 \\ Z_3 & -Z_2 & Z_1 & -Z_0 & Z_7 & -Z_6 & Z_5 & -Z_4 \\ Z_4 & -\alpha Z_5 & Z_6 & -\alpha Z_7 & Z_0 & -\alpha Z_1 & Z_2 & -\alpha Z_3 \\ Z_5 & -Z_4 & Z_7 & -Z_6 & Z_1 & -Z_0 & Z_3 & -Z_2 \\ Z_6 & -\alpha Z_7 & Z_4 & -\alpha Z_5 & Z_2 & -\alpha Z_3 & Z_0 & -\alpha Z_1 \\ Z_7 & -Z_6 & Z_5 & -Z_4 & Z_3 & -Z_2 & Z_1 & -Z_0 \end{pmatrix} \tag{16}$$

is obtained with the equality of (15) where

$$\begin{array}{l} Z_0=\frac{1}{8}\left(Y_{11}-Y_{22}+Y_{33}-Y_{44}+Y_{55}-Y_{66}+Y_{77}-Y_{88}\right) \\ Z_1=\frac{1}{8}\left(-\frac{Y_{12}}{\alpha}+Y_{21}-\frac{Y_{34}}{\alpha}+Y_{43}-\frac{Y_{56}}{\alpha}+Y_{65}-\frac{Y_{78}}{\alpha}+Y_{87}\right) \\ Z_2=\frac{1}{8}\left(Y_{13}-Y_{24}+Y_{31}-Y_{42}+Y_{57}-Y_{68}+Y_{75}-Y_{86}\right) \\ Z_3=\frac{1}{8}\left(-\frac{Y_{14}}{\alpha}+Y_{23}-\frac{Y_{32}}{\alpha}+Y_{41}-\frac{Y_{58}}{\alpha}+Y_{67}-\frac{Y_{76}}{\alpha}+Y_{85}\right) \\ Z_4=\frac{1}{8}\left(Y_{15}-Y_{26}+Y_{37}-Y_{48}+Y_{51}-Y_{62}+Y_{73}-Y_{84}\right) \\ Z_5=\frac{1}{8}\left(-\frac{Y_{16}}{\alpha}+Y_{25}-\frac{Y_{38}}{\alpha}+Y_{47}-\frac{Y_{52}}{\alpha}+Y_{61}-\frac{Y_{74}}{\alpha}+Y_{83}\right) \\ Z_6=\frac{1}{8}\left(Y_{17}-Y_{28}+Y_{35}-Y_{46}+Y_{53}-Y_{64}+Y_{71}-Y_{82}\right) \\ Z_7=\frac{1}{8}\left(-\frac{Y_{18}}{\alpha}+Y_{27}-\frac{Y_{36}}{\alpha}+Y_{45}-\frac{Y_{54}}{\alpha}+Y_{63}-\frac{Y_{72}}{\alpha}+Y_{81}\right). \end{array} \tag{17}$$

Since there is $\phi_X=Y$, the solution to (9) is

$$X=Z_0+Z_1i+Z_2j+Z_3k+Z_4e+Z_5ei+Z_6ej+Z_7ek=\frac{1}{4-4\alpha}\begin{bmatrix} I_m \\ iI_m \\ jI_m \\ kI_m \\ eI_m \\ eiI_m \\ ejI_m \\ ekI_m \end{bmatrix}^T Y^{'}\begin{bmatrix} I_n \\ iI_n \\ jI_n \\ kI_n \\ eI_n \\ eiI_n \\ ejI_n \\ ekI_n \end{bmatrix}. \tag{18}$$

□

Example 4.1. *Let's find $X\in U_{2\times 2}(CO_p)$ satisfying the equality*

$$X-\begin{bmatrix} 0 & k \\ j & 1 \end{bmatrix}X^{o_1}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}=\begin{bmatrix} i & j \\ k & e \end{bmatrix}.$$

Considering the equation (10)*, the real representation of the above equivalence is* *denoted by the following equation*

$$
Y - \left[\begin{array}{cccccccccccccccc}
0&0&0&0&0&0&0&-\alpha&0&0&0&0&0&0&0&0\\
0&1&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&-1&0&0&-1&0&0&0&0&0&0&0&0&0\\
0&0&0&-\alpha&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&-1&0&0&0&0&-1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-\alpha\\
0&0&0&0&0&0&0&0&0&1&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&-1&0&0&-1&0\\
0&0&0&0&0&0&0&0&0&0&0&-\alpha&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&-1
\end{array}\right] Y
$$

$$
\left[\begin{array}{cccccccccccccccc}
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{array}\right]
$$

$$
= \left[\begin{array}{cccccccccccccccc}
0&0&-\alpha&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&-\alpha&0&0&1&0&0&0&0&0&0\\
1&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&-1&0&0&0&0\\
0&1&0&0&0&0&-\alpha&0&0&0&0&0&0&0&0&0\\
0&0&-\alpha&0&0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&-1&1&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1\\
0&0&0&0&0&0&0&0&0&0&-\alpha&0&0&1&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&-\alpha&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&-1\\
0&0&0&-1&0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&-\alpha&0\\
0&0&0&0&0&1&0&0&0&0&-\alpha&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&-1&1&0&0&0\\
0&0&0&0&0&0&0&-1&1&0&0&0&0&0&0&0
\end{array}\right].
$$

If the found real matrix equation is solved, then

$$
Y = \left[\begin{array}{cccccccccccccccc}
0&0&-\alpha&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
1&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0\\
0&1&0&0&0&0&-\alpha&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&-1&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1\\
0&0&0&0&0&0&0&0&0&0&0&-\alpha&0&1&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&-1\\
0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&-\alpha&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&-1&1&0&0&0\\
0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0
\end{array}\right]
$$

is obtained. Considering

$$
\phi_X = \frac{1}{8}\left(Y - Q_m^{-1}\phi_X Q_n + R_m^{-1}\phi_X R_n - S_m^{-1}\phi_X S_n + T_m^{-1}\phi_X T_n - U_m^{-1}\phi_X U_n + V_m^{-1}\phi_X V_n - W_m^{-1}\phi_X W_n \right)
$$

then $Y = \phi_X$. *So if Theorem* 4.2 *is taken into consideration,*

$$X = \frac{1}{4-4\alpha}\begin{bmatrix} I_2 \\ iI_2 \\ jI_2 \\ kI_2 \\ eI_2 \\ eiI_2 \\ ejI_2 \\ ekI_2 \end{bmatrix}^T \phi_X \begin{bmatrix} I_2 \\ iI_2 \\ jI_2 \\ kI_2 \\ eI_2 \\ eiI_2 \\ ejI_2 \\ ekI_2 \end{bmatrix} = \begin{bmatrix} i & j \\ 0 & e \end{bmatrix}$$

is found.

5 Gershgorin's Theorem in Commutative Elliptic Octonion Matrices

A way to locate the roots of a polynomial is to indicate the location of the eigenvalues of the matrix corresponding to the polynomial. For this, Gershgorin disks that contain these eigenvalues are defined. The Gershgorin theorem ensures that the combination of these disks includes all eigenvalues.

Considering the definition of the adjoint matrix given in Definition 3.4 and the properties of adjoint matrix given in Theorem 3.4, the eigenvalue set of $A = A_1 + eA_2 \in U_{n\times n}(CO_p)$ can be defined. In that case, the eigenvalue of A is $\lambda \in CO_p$ and the eigenvector $0 \neq x \in U_{n\times 1}(CO_p)$ corresponding to the eigenvalue λ at the same time providing of $Ax = \lambda x$ is available. Then, the eigenvalue set of A is defined by

$$\xi(A) = \{\lambda \in CO_p : \quad Ax = \lambda x \;\; \exists x \neq 0\},$$

[5]. Now let's give the fundamental theorem of algebra, which is the basis of the Gershgorin Theorem.

Theorem 5.1. *A is $n \times n$ type commutative elliptic octonion matrix, has at most $2n$ elliptic quaternion eigenvalues and $4n$ elliptic eigenvalues.*

Proof. Let $A = A_1 + A_2e \in U_{n\times n}(CO_p)$ and $\lambda \in H_p$ be an eigenvalue of A. Since there exist $0 \neq x = x_1 + x_2e \in U_{n\times 1}(CO_p)$ and the column vector x, $Ax = \lambda x$ is provided. From here

$$(A_1 + A_2e)(x_1 + x_2e) = \lambda x_1 + \lambda x_2 e,$$
$$A_1x_1 + A_2x_2 = \lambda x_1 \quad \text{and} \quad A_1x_2 + A_2x_1 = \lambda x_2$$

can be written and from the above equations

$$\begin{pmatrix} A_1 & A_2 \\ A_2 & A_1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

is found. In that case, it is seen that the commutative elliptic octonion matrix A has at most $2n$ elliptic quaternion eigenvalues. In addition, if a commutative elliptic octonion matrix has at most $2n$ elliptic quaternion eigenvalues, it can also have at most $4n$ elliptic eigenvalues and the proof is complete. □

Corollary 5.1. *Let $A \in U_{n\times n}(CO_p)$ and $\xi(\eta(A)) = \{\lambda \in H_p : \ \eta(A)\, y = \lambda y \ , \exists y \neq 0\}$ be the set of eigenvalues of adjoint matrix $\eta(A)$, then*

$$\xi(A) \cap H_p = \xi(\eta(A))$$

is provided.

Theorem 5.2. *Let $A = A_1 + A_2 e \in U_{n\times n}(CO_p)$ and $\lambda = \lambda_1 + \lambda_2 e$ be an eigenvalue of A. Then λ is an eigenvalue of A if and only if there exist $x_1, x_2 \in H_p^{n\times 1} \quad (x_1 \neq 0, \ x_2 \neq 0)$ such that*

$$\begin{bmatrix} A_1 - \lambda_1 I & A_2 - \lambda_2 I \\ A_2 - \lambda_2 I & A_1 - \lambda_1 I \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Proof. Let $A = A_1 + A_2 e \in U_{n\times n}(CO_p)$ and $\lambda = \lambda_1 + \lambda_2 e$ be the eigenvalue of A. $\lambda = \lambda_1 + \lambda_2 e$ is an eigenvalue of A if and only if there exists $x_1, x_2 \in H_p^{n\times 1} \quad (x_1 \neq 0, \ x_2 \neq 0)$ such that

$$(A_1 + A_2 e)(x_1 + x_2 e) = (\lambda_1 + \lambda_2 e)(x_1 + x_2 e)\,.$$

Hence

$$\begin{array}{l} (A_1 - \lambda_1 I_n)\, x_1 + (A_2 - \lambda_2 I_n)\, x_2 = 0 \\ (A_1 - \lambda_1 I_n)\, x_2 + (A_2 - \lambda_2 I_n)\, x_1 = 0. \end{array}$$

can be written and using these obtained equations, we may write

$$\begin{bmatrix} A_1 - \lambda_1 I & A_2 - \lambda_2 I \\ A_2 - \lambda_2 I & A_1 - \lambda_1 I \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

□

Theorem 5.3 (Gershgorin Theorem)**.** *Let $A = (a_{ij}) \in U_{n\times n}(CO_p)$. Then Gershgorin set for commutative elliptic octonion matrices is given as follows*

$$\xi(A) \subseteq \bigcup_{i=1}^{n} \{a \in CO_p : \ \ \|a - a_{ii}\| \leq R_i\}$$

where $R_i = \sum\limits_{j=1,\, i\neq j}^{n} \|a_{ij}\|$.

Proof. Let $A \in M_{n\times n}(CO_p)$, λ be the eigenvalue of $A = (a_{ij})$ and $x \neq 0$ be the corresponding eigenvector then $Ax = \lambda x$. Also x_i is component of x such that $\|x_i\| \geq \|x_j\|$ for all j then we have $\|x_i\| > 0$ and λx_i corresponds to the i^{th} component of vector Ax which means that

$$\lambda x_i = \sum_{j=1}^{n} a_{ij} x_j.$$

For this reason, we may write

$$\lambda x_i - a_{ii}x_i = \sum_{j=1,\mathrm{i}\neq\mathrm{j}}^{n} a_{ij}x_j \Rightarrow \left(\lambda - a_{ii}\right)x_i = \sum_{j=1,\mathrm{i}\neq\mathrm{j}}^{n} a_{ij}x_j.$$

Taking the norm of both sides in the above equation

$$\left\|\left(\lambda - a_{ii}\right)x_i\right\| = \left\|\sum_{j=1,\mathrm{i}\neq\mathrm{j}}^{n} a_{ij}x_j\right\|$$

is obtained. Then make use of triangle inequality is written the following inequalities

$$\begin{aligned}
\left\|\left(\lambda - a_{ii}\right)x_i\right\| &\leq \sum_{j=1,\mathrm{i}\neq\mathrm{j}}^{n} \|a_{ij}x_j\|, \\
\left\|\left(\lambda - a_{ii}\right)\right\|\left\|x_i\right\| &\leq \sum_{j=1,\mathrm{i}\neq\mathrm{j}}^{n} \left\|a_{ij}\right\|\left\|x_j\right\|, \\
\left\|\left(\lambda - a_{ii}\right)\right\| &\leq \sum_{j=1,\mathrm{i}\neq\mathrm{j}}^{n} \left\|a_{ij}\right\| = R_i.
\end{aligned}$$

So, we have

$$\xi\left(A\right) \subseteq \bigcup_{i=1}^{n} \{a \in CO_p : \ \ \|a - a_{ii}\| \leq R_i\}.$$

□

Example 5.1. *Let* $A = \begin{bmatrix} 1+i+j+k+e & ei \\ ej & ek \end{bmatrix}$; *A is a commutative elliptic octonion matrix. Then the adjoint matrix of A is*

$$\eta\left(A\right) = \begin{bmatrix} 1+i+j+k & 0 & 1 & i \\ 0 & 0 & j & k \\ 1 & i & 1+i+j+k & 0 \\ j & k & 0 & 0 \end{bmatrix}$$

The set of eigenvalues of $\eta\left(A\right)$ *is*

$$\xi\left(\eta\left(A\right)\right) = \left\{\begin{array}{ll} \frac{1}{2}\left(2+i+j-\sqrt{5+\alpha+4i+4j+6k}+2k\right), & \frac{1}{2}\left(2+i+j+\sqrt{5+\alpha+4i+4j+6k}+2k\right), \\ \frac{1}{2}\left(i+j-\sqrt{1+5\alpha+4i+4\alpha j+6k}\right), & \frac{1}{2}\left(i+j+\sqrt{1+5\alpha+4i+4\alpha j+6k}\right)\} \end{array}\right\}.$$

The Gershgorin disks are

$$\begin{aligned}
D_1 &= \left\{q \in H_p : \ \|q - (1+i+j+k)\| \leq \sqrt{\|\alpha\|} + 1 \right\}, \\
D_2 &= \left\{q \in H_p : \ \|q\| \leq \sqrt{\|\alpha\|} + 1 \right\}.
\end{aligned}$$

Thus, we obtain

$$\xi\left(A\right) \ \cap \ H_p \subseteq D_1 \cup D_2.$$

6 Conclusions

In this article, firstly, the notions of similarity and conjugate similar are given for commutative elliptic octonion matrices with an isomorphism defined between commutative elliptic octonions and matrices. Then, using linear transformation $\phi_A(X) = AX^{o_1}$ for the first conjugate of the commutative elliptic octonions, ϕ_A is obtained. With this matrix, equivalence $AB^{o_1} \cong \phi_A \mathbf{B}$ has been defined. In addition, the solution of $X - AX^{o_1}B = C$, which is the Kalman-Yakubovich s-conjugate equation for commutative elliptic octonion matrices, is given and this solution is illustrated with examples. The solution of Kalman-Yakubovich s-conjugate equation for the conjugates $'o_i'$ $(2 \leq i \leq 7)$ can be easily obtained by applying similar steps.

On the other hand, the fundamental theorem of algebra is studied for commutative elliptic octonion matrices. Later, the Gershgorin Theorem that determines the location of the eigenvalues of a matrix is proved, and the application of the theorem is given with some examples.

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