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Stochastic orders of log-epsilon-skew-normal distributions

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Abstract

The log-epsilon-skew-normal distributions family is generalized class of log-normal distribution. Is widely used to model non-negative data in many areas of applied research. We give necessary and/or sufficient conditions for some stochastic orders of log-epsilon-skew-normal distributions. Also, we give sufficient conditions for orders of moments and Gini indexes. Finally, it is presented a real data application.

1 Introduction

The log-normal distribution has a long and rich history. Galton (1879) suggested the use of the log-normal distribution to analyze data for which the geometric mean is better than the arthimatic mean for estimating central tendency. McAlister (1879) derived the log-normal distribution at Galton's suggestion. For a review of the history of log-normal distribution and its applications as a generative law see Mitzenmacher (2004). Finney (1941) examined the moments, moment estimation, and efficiency of the estimation. The epsilon-skew normal (ESN) distribution was introduced as a skew extension of the normal distribution in same spirit as Azzalini's (1985) skew-normal (SN) distribution. A key advantage of the LESN family is that it includes

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as a subfamily the log-normal distributions, which are well known to be in use in a wide spectrum of disciplines. Examples include astrophysics, Gandhi (2009), environmental sciences, Benning and Barnes (2009), computer science, Doerr et al. (2013), economics, Cobb et al. (2013), biomedical, Feng et al. (2013), and radiology, Neti and Howell (2008). Limpert et al. (2001) compared the use of the log-normal distribution across several different science disciplines. Hutson et al. (2020) introduced and studied a family of distributions with non-negative reals as support and termed the log-epsilon-skew normal (LESN) which includes the log-normal distributions as a special case.

Life expectancy is one of the most meaningful and widely used summary measure of the distribution of lifespan (length of life). However, the variability of this distribution is an equally important measure to consider, since the same life expectancy value can derive from very different shapes of this distribution. Gini coefficient is the most common statistical index of diversity or inequality in social sciences (Kendall and Stuart (1966), Allison (1978)). It is widely used in econometrics as a standard measure of inter-individual or inter-household inequality in income and wealth (Atkinson (1970, 1980), Sen and Amartia (1973), Anand (1983)). Illsey and Le Grand (1987) who justified the use of Gini coefficient for the analysis of inequality in health in the 1980s, stressed that the individual-based measurement of inequality in health is a way to a universal comparability of degrees of inequality over time and across countries.

Another intersting subject which we can use this distribution is survival analyzes which have been developed mainly in the medical and biological sciences, but are widely used in the social sciences such as economics and engineering (reliability and failure time analysis). The changes that characterize the changing stage of mortality are measured by variables such as age-specific death rates, life expectancy at birth, probabilities of death and survival function. Survival analysis methods depend on the distribution of survival and on the hazard function.

A classical method to compare survival, hazard rate functions or Gini index is stochastic orders which are strongly related to the insurance and risk theory. For introduction in the field, we recommend the reader more recent books (see, for instance Shaked and Shanthikumar (2007) and Levy (2015)). Recent results regarding Gini index were given by: Zbaganu (2020) analized Loenz order and Gini index for a recent model, Preda and Catana (2021) gave theoretical results for different stochastic orders of a log-scale-location family which uses Tsallis statistics functions and results which describe the inequalities of moments or Gini index according to parameters, Buffa et al. (2020) analized the inequality of the Gini coefficient in the case of a kinetic model for Wealth distribution. Also, results describing the problem of stochastic orders were given by: Ortega-Jiménez et al. (2021) provide sufficient conditions for comparing several distances between pairs of random variables (with possibly different distribution functions) in terms of various stochastic orderings;

Balakrishnan et al. (2021) analized hazard rate and reversed hazard rate orders of parallel systems with components having proportional reversed hazard rates and starting devices; Catana and Raducan (2020c) gave sufficient conditions for stochastic order of multivariate uniform distributions on closed convex sets; Catana (2021a) gave theoretical results for equivalence between di erent stochastic orders of some kind multivariate Pareto distribution family; Sfetcu et al. (2021) introduced a stochastic order on Awad–Varma residual entropy and studied some properties of this order, like closure, reversed closure and preservation in some stochastic models (the proportional hazard rate model, the proportional reversed hazard rate model, the proportional odds model and the record values model); Catana and Preda (2021b) proved that different orders between parameters vectors imply the hazard order and reverse hazard order between extremes order statistics of transmuted distributions;

Berrendero and Cárcamo (2012) provided a new meaning to the corresponding test statistics; Nadeb and Torabi (2020) analized different stochastic comparisons in the transmuted-G family with different parameters; Ahmadi and Arghami (2001) analized some univariate stochastic orders on record values; Bancescu (2018) presented the likelihood order of some classes of statistical distributions.

The purpose of this work is to analyze different stochastic orders and the order between some elements of log-epsilon-skew-normal distributions (for example Gini index or the moments) and also the usual stochastic order of the log-scale-location models. The structure of this article is: in the section 2 there are presented the preliminaries; in the section 3 there is presented the log-epsilon-skew-normal distributions family and some mathematical properties; in the section 4 we give necessary and/or sufficient conditions for different stochastic orders of log-epsilon-skew-normal distributions and some direct consequences; in the section 5 we give sufficient conditions for ususal stochastic order of log-scale-location models; in the section 6 we illustrate the theoretical results whit a real data application. In the last section we present the conclusions of this article.

2 Preliminaries

Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let $X : \Omega \to \mathbb{R}$ be a random variable. For a random variable X we consider $\mu_X(B) = P(X \in B)$ be its distribution on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, $F_X(x) = P(X \leq x)$ its distribution function and survival function $\overline{F}_X(x) = 1 - F_X(x)$, where $x \in \mathbb{R}$.

If X is absolutely continuous according to Lebesgue measure then we denote $f_X(x) = (F_X(x))'$ its density function. Also, we denote $Q_X(p) = \inf\{x \in \mathbb{R} : p \leq F_X(x)\}$ the quantile function of X.

If $\overline{F}_X$ is differentiable, we define the hazard rate function $r_X : Supp(\overline{F}_X) \to \mathbb{R}$, $r_X = \left(-\ln \overline{F}_X\right)'$, where for a function $g : \mathbb{R} \to \mathbb{R}$, $Supp(g) = \{x \in \mathbb{R} : g(x) \neq 0\}$.

Also, for a random variable X with $E(X) \neq 0$ it is defined

$$L_X(p) = \frac{\int_0^p Q_X(u)du}{EX}$$

the Lorenz curve of X and

$$G_X = \frac{\int_0^\infty \int_0^\infty |t_1 - t_2| \, dF_X(t_1) \, dF_X(t_2)}{2EX}$$

the Gini index of X (see Arnold (2007, 1987)).

Another formulas for the Gini index of X are

$$G_X = 1 - \frac{\int_0^\infty \overline{F}_X^2(x)dx}{\int_0^\infty \overline{F}_X(x)dx}$$

and

$$G_X = 1 - 2\int_0^1 L_X(p)dp.$$

For $a \geq 1$, the a-Gini generalized index

$$G_{X;a} = 1 - a(a-1)\int_0^1 (1-p)^{a-2} L_X(p) \, dp.$$

We notice that $G_{X;2} = G_X$.

A simple a simple calculation shows that

$$G_{X;a} = 1 - \frac{\int_0^\infty \overline{F}_X^a(x)dx}{\int_0^\infty \overline{F}_X(x)dt} = 1 - \frac{\int_0^\infty \overline{F}_X^a(x)dx}{EX}.$$

For a function $g : \mathbb{R} \to \mathbb{R}$, $g(x)_+ = \max(g(x), 0)$ and $g(x)_- = \min(g(x), 0)$.

Definition. *(Shaked and Shanthikumar, 2007) Let $X, Y : \Omega \to \mathbb{R}$ random variables. We say that X is said to be smaller than Y in the*

(i) stochastic order (written as $X \prec_{st} Y$) if $\overline{F}_X(x) \leq \overline{F}_Y(x)$ $\forall x \in \mathbb{R}$;

(ii) hazard rate order (written as $X \prec_{hr} Y$) if $r_X(x) \geq r_Y(x)$ $\forall x \in Supp(\overline{F}_X) \cap Supp(\overline{F}_Y)$;

(iii) Lorenz order (written as $X \prec_{Lorenz} Y$) if $L_X(p) \geq L_Y(p)$ $\forall p \in [0;1]$;

(iv) likelihood ratio order (written as $X \prec_{lr} Y$) if $\frac{f_Y(x)}{f_X(x)}$ is increasing in $x \in Supp(f_X) \cup Supp(f_Y)$.

Another equivalent definition for stochastic order is:

Definition. *(Shaked and Shanthikumar, 2007) Let $X, Y : \Omega \to \mathbb{R}$ random variables. We say that X is said to be smaller than Y in the stochastic order (written as $X \prec_{st} Y$) if $Eu(X) \leq Eu(Y)$, for all increasing functions $u : \mathbb{R} \to \mathbb{R}$, provided that the means exists.*

Definition. *(Shaked and Shanthikumar, 2007) Let $X, Y : \Omega \to \mathbb{R}$ random variables. We say that X is said to be smaller than Y in the increasing convex order (written as $X \prec_{icx} Y$) if $Eu(X) \leq Eu(Y)$, for all increasing convex functions $u : \mathbb{R} \to \mathbb{R}$, provided that the means exists.*

Definition. *(Shaked and Shanthikumar, 2007) Let $X, Y : \Omega \to \mathbb{R}$ random variables. We say that X is said to be smaller than Y in the increasing convex order (written as $X \prec_{cx} Y$) if $Eu(X) \leq Eu(Y)$, for all convex functions $u : \mathbb{R} \to \mathbb{R}$, provided that the means exists.*

It is well known:

Proposition 2.1. *(Shaked and Shanthikumar, 2007) $X \prec_{lr} Y \Rightarrow X \prec_{hr} Y \Rightarrow X \prec_{st} Y$.*

Proposition 2.2. *(Shaked and Shanthikumar, 2007) $X \prec_{cx} Y \Leftrightarrow X \prec_{icx} Y$ and $EX = EY$.*

Proposition 2.3. *(Shaked and Shanthikumar, 2007) $X \prec_{Lorenz} Y \Leftrightarrow X \prec_{cx} Y$ and $EX = EY$.*

Proposition 2.4. *(Shaked and Shanthikumar, 2007) $X \prec_{Lorenz} Y \Rightarrow G_X \leq G_Y$.*

3 Log-epsilon-skew-normal distribution

Hutson et al. (2020) introduced the following distributions family:

Definition. We say the random variable X is log-epsilon-skew-normal (LESN) distributed with the parameters $\theta \in \mathbb{R}$, $\sigma \in (0, \infty)$, $\varepsilon \in (-1, 1)$ (and we denote $X \sim LESN(\theta, \sigma, \varepsilon)$) if

$$f_X(x) = \frac{1}{x\sqrt{2\pi}\sigma} \cdot e^{-\frac{(\ln x - \theta)^2}{2\sigma^2(1-\varepsilon)^2}} \cdot 1_{(0,e^\theta)}(x) + \frac{1}{x\sqrt{2\pi}\sigma} \cdot e^{-\frac{(\ln x - \theta)^2}{2\sigma^2(1+\varepsilon)^2}} \cdot 1_{[e^\theta,\infty)}(x).$$

Also they computed:

the distribution function

$$F_X(x) = (1-\varepsilon)\Phi\left(\frac{\ln x - \theta}{\sigma(1-\varepsilon)}\right) \cdot 1_{(0,e^\theta)}(x) + \left(-\varepsilon + (1+\varepsilon)\Phi\left(\frac{\ln x - \theta}{\sigma(1+\varepsilon)}\right)\right) \cdot 1_{[e^\theta,\infty)}(x),$$

where Φ is distribution function of standard normal distribution and $\overline{\Phi} = 1 - \Phi$;

the quantile function $Q_X(x) = e^{\theta + \sigma Q_0(x)}$ for $x \in [0, 1]$, where

$Q_0(u) = (1-\varepsilon)\,\Phi^{-1}\left(\frac{u}{1-\varepsilon}\right)\cdot 1_{\left(0,\frac{1-\varepsilon}{2}\right)}(u) + \left(-\varepsilon+(1+\varepsilon)\,\Phi^{-1}\left(\frac{u+\varepsilon}{1+\varepsilon}\right)\right)\cdot 1_{\left[\frac{1-\varepsilon}{2},1\right)}(u)$,

the k-th moment ($k \in \mathbb{Z}$, $k \geq 1$)

$$E\left(X^k\right)$$

$$= e^{\theta k}\left((1-\varepsilon)\,e^{\frac{(1-\varepsilon)^2\sigma^2k^2}{2}}\cdot\Phi\left(-(1-\varepsilon)\,\sigma k\right) + (1+\varepsilon)\,e^{\frac{(1+\varepsilon)^2\sigma^2k^2}{2}}\cdot\Phi\left((1+\varepsilon)\,\sigma k\right)\right)$$

and the hazard function

$$r_X(x) = \frac{\frac{1}{x\sqrt{2\pi}\sigma}\cdot e^{-\frac{(\ln x-\theta)^2}{2\sigma^2(1-\varepsilon)^2}}}{1-(1-\varepsilon)\Phi\left(\frac{\ln x-\theta}{\sigma(1-\varepsilon)}\right)}\cdot 1_{(0,e^\theta)}(x) + \frac{\frac{1}{x\sqrt{2\pi}\sigma}\cdot e^{-\frac{(\ln x-\theta)^2}{2\sigma^2(1+\varepsilon)^2}}}{(1+\varepsilon)\overline{\Phi}\left(\frac{\ln x-\theta}{\sigma(1+\varepsilon)}\right)}\cdot 1_{[e^\theta,\infty)}(x).$$

Also in Hutson et al. (2020) there were given another properties of this distribution.

4 Stochastic orders of log-epsilon-skew-normal distribution

In this section we analize different stochastic order and as a consequence the inequalities between the moments and the a-Gini generalized index of two log-epsilon-skew-normal distributions.

Let us consider the positive real random variables X and Y.

Theorem 4.1 gives necessary conditions for usual stochastic order and theorem 4.2 gives sufficient conditions for usual stochastic order between two log-epsilon-skew-normal distributions.

Theorem 4.1. *Let the random variables $X \sim LESN\left(\theta_X, \sigma_X, \varepsilon_X\right)$ and $Y \sim LESN\left(\theta_Y, \sigma_Y, \varepsilon_Y\right)$. If $X \prec_{st} Y$ then only one of the statements is true:*

(i) $\varepsilon_X < \varepsilon_Y$, $\sigma_X\left(1-\varepsilon_X\right) > \sigma_Y\left(1-\varepsilon_Y\right)$, $\sigma_X\left(1+\varepsilon_X\right) < \sigma_Y\left(1+\varepsilon_Y\right)$;

(ii) $\theta_X \geq \theta_Y, \sigma_X \geq \sigma_Y, \varepsilon_X > \varepsilon_Y$;

(iii) $\theta_X \leq \theta_Y, \sigma_X \geq \sigma_Y, \varepsilon_X < \varepsilon_Y$;

(iv) $\theta_X = \theta_Y$, $\sigma_X = \sigma_Y$, $\varepsilon_X = \varepsilon_Y$.

Proof. If $X \prec_{st} Y$ then $F_X(x) \geq F_Y(x)\ \forall x > 0$.

Then

$$(1-\varepsilon_X)\,\Phi\left(\frac{\ln x-\theta_X}{\sigma_X\left(1-\varepsilon_X\right)}\right) \geq (1-\varepsilon_Y)\,\Phi\left(\frac{\ln x-\theta_Y}{\sigma_Y\left(1-\varepsilon_Y\right)}\right)\forall x \in \left(0, \min\left(e^{\theta_X}, e^{\theta_Y}\right)\right) \Leftrightarrow$$

$$1-\varepsilon_X \geq (1-\varepsilon_Y)\cdot \frac{\Phi\left(\frac{\ln x-\theta_Y}{\sigma_Y(1-\varepsilon_Y)}\right)}{\Phi\left(\frac{\ln x-\theta_X}{\sigma_X(1-\varepsilon_X)}\right)}\ \forall x\in\left(0,\min\left(e^{\theta_X},e^{\theta_Y}\right)\right)$$

But there exists

$$\lim_{\substack{x\to 0\\ x>0}} \frac{\Phi\left(\frac{\ln x-\theta_Y}{\sigma_Y(1-\varepsilon_Y)}\right)}{\Phi\left(\frac{\ln x-\theta_X}{\sigma_X(1-\varepsilon_X)}\right)} = \lim_{\substack{x\to 0\\ x>0}} \frac{\frac{\partial}{\partial x}\Phi\left(\frac{\ln x-\theta_Y}{\sigma_Y(1-\varepsilon_Y)}\right)}{\frac{\partial}{\partial x}\Phi\left(\frac{\ln x-\theta_X}{\sigma_X(1-\varepsilon_X)}\right)} =$$

$$\lim_{\substack{x\to 0\\ x>0}} \frac{\sigma_X(1-\varepsilon_X)}{\sigma_Y(1-\varepsilon_Y)}\cdot e^{\frac{1}{2}\left[\left(\frac{\ln x-\theta_X}{\sigma_X(1-\varepsilon_X)}\right)^2-\left(\frac{\ln x-\theta_Y}{\sigma_Y(1-\varepsilon_Y)}\right)^2\right]} =$$

$$\frac{\sigma_X(1-\varepsilon_X)}{\sigma_Y(1-\varepsilon_Y)}\cdot \lim_{\substack{x\to 0\\ x>0}} e^{\frac{1}{2}\left[A_1\cdot(\ln x)^2+A_2\cdot\ln x+A_3\right]}$$

where $A_1 = \frac{1}{(\sigma_X(1-\varepsilon_X))^2} - \frac{1}{(\sigma_Y(1-\varepsilon_Y))^2}$, $A_2 = \frac{2\theta_Y}{(\sigma_Y(1-\varepsilon_Y))^2} - \frac{2\theta_X}{(\sigma_X(1-\varepsilon_X))^2}$, $A_3 = \frac{\theta_X^2}{(\sigma_X(1-\varepsilon_X))^2} - \frac{\theta_Y^2}{(\sigma_Y(1-\varepsilon_Y))^2}$.

From

$$1-\varepsilon_X \geq (1-\varepsilon_Y)\cdot \frac{\Phi\left(\frac{\ln x-\theta_Y}{\sigma_Y(1-\varepsilon_Y)}\right)}{\Phi\left(\frac{\ln x-\theta_X}{\sigma_X(1-\varepsilon_X)}\right)}\forall x\in\left(0,\min\left(e^{\theta_X},e^{\theta_Y}\right)\right)$$

it results

$$1-\varepsilon_X \geq (1-\varepsilon_Y)\cdot \lim_{\substack{x\to 0\\ x>0}} \frac{\Phi\left(\frac{\ln x-\theta_Y}{\sigma_Y(1-\varepsilon_Y)}\right)}{\Phi\left(\frac{\ln x-\theta_X}{\sigma_X(1-\varepsilon_X)}\right)}$$

then

$$\lim_{\substack{x\to 0\\ x>0}} \left[A_1\cdot(\ln x)^2 + A_2\cdot\ln x + A_3\right] \in \mathbb{R}\cup\{-\infty\}.$$

Thus $A_1<0$ or ($A_1=0$ and $A_2\leq 0$).

Also, from $F_X(x)\geq F_Y(x)$ $\forall x>0$ we have

$$-\varepsilon_X + (1+\varepsilon_X)\,\Phi\left(\frac{\ln x-\theta_X}{\sigma_X(1+\varepsilon_X)}\right) \geq$$

$$-\varepsilon_Y + (1+\varepsilon_Y)\,\Phi\left(\frac{\ln x - \theta_Y}{\sigma_Y\,(1+\varepsilon_Y)}\right)\ \forall x \in \left(\max\left(e^{\theta_X}, e^{\theta_Y}\right), \infty\right) \Leftrightarrow$$

$$(1+\varepsilon_X)\,\overline{\Phi}\left(\frac{\ln x - \theta_X}{\sigma_X\,(1+\varepsilon_X)}\right) \leq$$

$$(1+\varepsilon_Y)\,\overline{\Phi}\left(\frac{\ln x - \theta_Y}{\sigma_Y\,(1+\varepsilon_Y)}\right)\ \forall x \in [\max\left(e^{\theta_X}, e^{\theta_Y}\right), \infty) \Leftrightarrow$$

$$(1+\varepsilon_X)\cdot\frac{\overline{\Phi}\left(\frac{\ln x-\theta_X}{\sigma_X\,(1+\varepsilon_X)}\right)}{\overline{\Phi}\left(\frac{\ln x-\theta_Y}{\sigma_Y\,(1+\varepsilon_Y)}\right)} \leq 1+\varepsilon_Y\ \forall x \in [\max\left(e^{\theta_X}, e^{\theta_Y}\right), \infty)$$

But there exists

$$\lim_{x\to\infty}\frac{\overline{\Phi}\left(\frac{\ln x-\theta_X}{\sigma_X\,(1+\varepsilon_X)}\right)}{\overline{\Phi}\left(\frac{\ln x-\theta_Y}{\sigma_Y\,(1+\varepsilon_Y)}\right)} = \lim_{x\to\infty}\frac{\frac{\partial}{\partial x}\overline{\Phi}\left(\frac{\ln x-\theta_X}{\sigma_X\,(1+\varepsilon_X)}\right)}{\frac{\partial}{\partial x}\overline{\Phi}\left(\frac{\ln x-\theta_Y}{\sigma_Y\,(1+\varepsilon_Y)}\right)} =$$

$$\lim_{x\to\infty}\frac{\sigma_Y\,(1+\varepsilon_Y)}{\sigma_X\,(1+\varepsilon_X)}\cdot e^{\frac{1}{2}\left[\left(\frac{\ln x-\theta_Y}{\sigma_Y\,(1+\varepsilon_Y)}\right)^2-\left(\frac{\ln x-\theta_X}{\sigma_X\,(1+\varepsilon_X)}\right)^2\right]} =$$

$$\frac{\sigma_Y\,(1+\varepsilon_Y)}{\sigma_X\,(1+\varepsilon_X)}\cdot\lim_{x\to\infty} e^{\frac{1}{2}\left[B_1\cdot(\ln x)^2+B_2\cdot\ln x+B_3\right]}$$

where $B_1 = \frac{1}{(\sigma_Y\,(1+\varepsilon_Y))^2} - \frac{1}{(\sigma_X\,(1+\varepsilon_X))^2}$, $B_2 = \frac{2\theta_X}{(\sigma_X\,(1-\varepsilon_X))^2} - \frac{2\theta_Y}{(\sigma_Y\,(1-\varepsilon_Y))^2}$, $B_3 = \frac{\theta_Y^2}{(\sigma_Y\,(1+\varepsilon_Y))^2} - \frac{\theta_X^2}{(\sigma_X\,(1+\varepsilon_X))^2}$.

From

$$(1+\varepsilon_X)\cdot\frac{\overline{\Phi}\left(\frac{\ln x-\theta_X}{\sigma_X\,(1+\varepsilon_X)}\right)}{\overline{\Phi}\left(\frac{\ln x-\theta_Y}{\sigma_Y\,(1+\varepsilon_Y)}\right)} \leq 1+\varepsilon_Y \forall x \in [\max\left(e^{\theta_X}, e^{\theta_Y}\right), \infty)$$

it results

$$(1+\varepsilon_X)\cdot\lim_{x\to\infty}\frac{\overline{\Phi}\left(\frac{\ln x-\theta_X}{\sigma_X\,(1+\varepsilon_X)}\right)}{\overline{\Phi}\left(\frac{\ln x-\theta_Y}{\sigma_Y\,(1+\varepsilon_Y)}\right)} \leq 1+\varepsilon_Y$$

then

$$\lim_{x\to\infty}\left[B_1\cdot(\ln x)^2+B_2\cdot\ln x+B_3\right]\in\mathbb{R}\cup\{-\infty\}.$$

Thus $B_1<0$ or ($B_1=0$ and $B_2\le 0$).

Case 1: $A_1<0$, $B_1<0$

Then $\sigma_X(1-\varepsilon_X) > \sigma_Y(1-\varepsilon_Y)$ (which is equvalent with $\frac{1}{\sigma_X(1-\varepsilon_X)} < \frac{1}{\sigma_Y(1-\varepsilon_Y)}$) and $\sigma_X(1+\varepsilon_X)<\sigma_Y(1+\varepsilon_Y)$, thus

$$\frac{1}{\sigma_X(1-\varepsilon_X)}\cdot\sigma_X(1+\varepsilon_X)<\frac{1}{\sigma_Y(1-\varepsilon_Y)}\cdot\sigma_Y(1+\varepsilon_Y)\Leftrightarrow$$

$$\frac{1+\varepsilon_X}{1-\varepsilon_X}<\frac{1+\varepsilon_Y}{1-\varepsilon_Y}\Leftrightarrow\varepsilon_X<\varepsilon_Y.$$

Therefore case 1 occurs.

Case 2: ($A_1=0$ and $A_2\le 0$) and $B_1<0$

Then $\sigma_X(1+\varepsilon_X) < \sigma_Y(1+\varepsilon_Y)$ (which is equvalent with $\frac{1}{\sigma_X(1+\varepsilon_X)} > \frac{1}{\sigma_Y(1+\varepsilon_Y)}$) and $\sigma_X(1-\varepsilon_X)=\sigma_Y(1-\varepsilon_Y)$, thus

$$\frac{1}{\sigma_X(1+\varepsilon_X)}\cdot\sigma_X(1-\varepsilon_X)<\frac{1}{\sigma_Y(1+\varepsilon_Y)}\cdot\sigma_Y(1-\varepsilon_Y)\Leftrightarrow$$

$$\frac{1-\varepsilon_X}{1+\varepsilon_X}<\frac{1-\varepsilon_Y}{1+\varepsilon_Y}\Leftrightarrow\varepsilon_X>\varepsilon_Y.$$

$A_2\le 0\Rightarrow\theta_X\ge\theta_Y$.

If $\sigma_X<\sigma_Y$ then $\sigma_X(1-\varepsilon_X)<\sigma_Y(1-\varepsilon_Y)$, contradiction!

Then $\sigma_X\ge\sigma_Y$. Therefore case 2 occurs.

Case 3: $A_1<0$ and ($B_1=0$ and $B_2\le 0$)

Then $\sigma_X(1-\varepsilon_X) > \sigma_Y(1-\varepsilon_Y)$ (which is equvalent with $\frac{1}{\sigma_X(1-\varepsilon_X)} < \frac{1}{\sigma_Y(1-\varepsilon_Y)}$) and $\sigma_X(1+\varepsilon_X)=\sigma_Y(1+\varepsilon_Y)$, thus

$$\frac{1}{\sigma_X(1-\varepsilon_X)}\cdot\sigma_X(1+\varepsilon_X)<\frac{1}{\sigma_Y(1-\varepsilon_Y)}\cdot\sigma_Y(1+\varepsilon_Y)\Leftrightarrow$$

$$\frac{1+\varepsilon_X}{1-\varepsilon_X}<\frac{1+\varepsilon_Y}{1-\varepsilon_Y}\Leftrightarrow\varepsilon_X<\varepsilon_Y.$$

$B_2\le 0\Rightarrow\theta_X\le\theta_Y$.

If $\sigma_X<\sigma_Y$ then $\sigma_X(1+\varepsilon_X)<\sigma_Y(1+\varepsilon_Y)$, contradiction!

Then $\sigma_X\ge\sigma_Y$. Therefore case 3 occurs.

Case 4: ($A_1=0$ and $A_2\le 0$) and ($B_1=0$ and $B_2\le 0$)

Then $\sigma_X(1-\varepsilon_X)=\sigma_Y(1-\varepsilon_Y)$,$\sigma_X(1+\varepsilon_X)=\sigma_Y(1+\varepsilon_Y)$, $\theta_X\le\theta_Y,\theta_Y\le\theta_X$.

$\theta_X \leq \theta_Y, \theta_Y \leq \theta_X \Rightarrow \theta_X = \theta_Y$.

If $\sigma_X (1-\varepsilon_X) = \sigma_Y (1-\varepsilon_Y)$ and $\sigma_X (1+\varepsilon_X) = \sigma_Y (1+\varepsilon_Y)$ then $\frac{1-\varepsilon_X}{1+\varepsilon_X} = \frac{1-\varepsilon_Y}{1+\varepsilon_Y}$, thus $\varepsilon_X = \varepsilon_Y$.

It results $\sigma_X = \sigma_Y$. Therefore case 4 occurs. □

Theorem 4.2. *Let the random variables* $X \sim LESN(\theta_X, \sigma_X, \varepsilon_X)$ *and* $Y \sim LESN(\theta_Y, \sigma_Y, \varepsilon_Y)$.

If

$$\varepsilon_X \leq \varepsilon_Y,\ \theta_X \leq \theta_Y,\ \sigma_X (1-\varepsilon_X) \geq \sigma_Y (1-\varepsilon_Y), \sigma_X (1+\varepsilon_X) \leq \sigma_Y (1+\varepsilon_Y)$$

then $X \prec_{st} Y$.

Proof. We have $1-\varepsilon_X \geq 1-\varepsilon_Y$ and $1+\varepsilon_X \leq 1+\varepsilon_Y$.

If $x \in \left(0, e^{\theta_X}\right)$ then $\frac{\ln x - \theta_X}{\sigma_X(1-\varepsilon_X)} \geq \frac{\ln x - \theta_Y}{\sigma_Y(1-\varepsilon_Y)}$.

Thus

$$F_X(x) = (1-\varepsilon_X)\,\Phi\left(\frac{\ln x - \theta_X}{\sigma_X(1-\varepsilon_X)}\right) \geq (1-\varepsilon_Y)\,\Phi\left(\frac{\ln x - \theta_Y}{\sigma_Y(1-\varepsilon_Y)}\right) = F_Y(x).$$

If $x \in [e^{\theta_X}, e^{\theta_Y})$ then

$$\begin{gathered} -\varepsilon_X + (1+\varepsilon_X)\,\Phi\left(\tfrac{\ln x - \theta_X}{\sigma_X(1+\varepsilon_X)}\right) = 1 - (1+\varepsilon_X)\,\overline{\Phi}\left(\tfrac{\ln x - \theta_X}{\sigma_X(1+\varepsilon_X)}\right) \geq \\ 1 - (1+\varepsilon_X)\,\overline{\Phi}(0) = \tfrac{1-\varepsilon_X}{2} \geq \tfrac{1-\varepsilon_Y}{2} = \\ (1-\varepsilon_Y)\cdot\Phi(0) \geq (1-\varepsilon_Y)\,\Phi\left(\tfrac{\ln x - \theta_Y}{\sigma_Y(1-\varepsilon_Y)}\right). \end{gathered}$$

Thus $F_X(x) \geq F_Y(x)$.

If $x \in [e^{\theta_Y}, \infty)$ then $\frac{\ln x - \theta_X}{\sigma_X(1+\varepsilon_X)} \geq \frac{\ln x - \theta_Y}{\sigma_Y(1+\varepsilon_Y)}$.

Thus

$$F_X(x) = -\varepsilon_X + (1+\varepsilon_X)\,\Phi\left(\frac{\ln x - \theta_X}{\sigma_X(1+\varepsilon_X)}\right) =$$

$$1 - (1+\varepsilon_X)\,\overline{\Phi}\left(\frac{\ln x - \theta_X}{\sigma_X(1+\varepsilon_X)}\right) \geq 1 - (1+\varepsilon_Y)\,\overline{\Phi}\left(\frac{\ln x - \theta_Y}{\sigma_Y(1+\varepsilon_Y)}\right) =$$

$$-\varepsilon_Y + (1+\varepsilon_Y)\,\Phi\left(\frac{\ln x - \theta_Y}{\sigma_Y(1+\varepsilon_Y)}\right) = F_Y(x).$$

It results $X \prec_{st} Y$. □

Theorem 4.3 gives necessary and sufficient conditions for likelihood ratio order between two log-epsilon-skew-normal distributions.

Theorem 4.3. *Let the random variables* $X \sim LESN(\theta_X, \sigma_X, \varepsilon_X)$ *and* $Y \sim LESN(\theta_Y, \sigma_Y, \varepsilon_Y)$.

Then $X \prec_{lr} Y$ *if and only if* $\theta_X = \theta_Y$, $\sigma_X(1-\varepsilon_X) \geq \sigma_Y(1-\varepsilon_Y)$ *and* $\sigma_X(1+\varepsilon_X) \geq \sigma_Y(1+\varepsilon_Y)$.

Proof. We prove the "$\Rightarrow$" implication.

For $x \in \left(0, \min\left(e^{\theta_X}, e^{\theta_Y}\right)\right)$ we have

$\frac{f_Y(x)}{f_X(x)} = \frac{\frac{1}{x\sqrt{2\pi}\sigma_Y} \cdot e^{-\frac{(\ln x - \theta_Y)^2}{2\sigma_Y^2(1-\varepsilon_Y)^2}}}{\frac{1}{x\sqrt{2\pi}\sigma_X} \cdot e^{-\frac{(\ln x - \theta_X)^2}{2\sigma_X^2(1-\varepsilon_X)^2}}} = e^{\frac{1}{2}\left[A_1 \cdot (\ln x)^2 + A_2 \cdot \ln x + A_3\right]}$, where

$A_1 = \frac{1}{(\sigma_X(1-\varepsilon_X))^2} - \frac{1}{(\sigma_Y(1-\varepsilon_Y))^2}$, $A_2 = \frac{2\theta_Y}{(\sigma_Y(1-\varepsilon_Y))^2} - \frac{2\theta_X}{(\sigma_X(1-\varepsilon_X))^2}$, $A_3 = \frac{\theta_X^2}{(\sigma_X(1-\varepsilon_X))^2} - \frac{\theta_Y^2}{(\sigma_Y(1-\varepsilon_Y))^2}$.

For $x \in \left(\max\left(e^{\theta_X}, e^{\theta_Y}\right), \infty\right)$ we have

$\frac{f_Y(x)}{f_X(x)} = \frac{\frac{1}{x\sqrt{2\pi}\sigma_Y} \cdot e^{-\frac{(\ln x - \theta_Y)^2}{2\sigma_Y^2(1+\varepsilon_Y)^2}}}{\frac{1}{x\sqrt{2\pi}\sigma_X} \cdot e^{-\frac{(\ln x - \theta_X)^2}{2\sigma_X^2(1+\varepsilon_X)^2}}} = e^{\frac{1}{2}\left[B_1 \cdot (\ln x)^2 + B_2 \cdot \ln x + B_3\right]}$, where

$B_1 = \frac{1}{(\sigma_Y(1+\varepsilon_Y))^2} - \frac{1}{(\sigma_X(1+\varepsilon_X))^2}$, $B_2 = \frac{2\theta_X}{(\sigma_X(1+\varepsilon_X))^2} - \frac{2\theta_Y}{(\sigma_Y(1+\varepsilon_Y))^2}$, $B_3 = \frac{\theta_Y^2}{(\sigma_Y(1+\varepsilon_Y))^2} - \frac{\theta_X^2}{(\sigma_X(1+\varepsilon_X))^2}$.

$X \prec_{lr} Y$ then

$$
\begin{gathered}
x \mapsto \frac{f_Y(x)}{f_X(x)} \text{ is increasing on } (0,\infty) \\
\Rightarrow x \mapsto A_1 \cdot (\ln x)^2 + A_2 \cdot \ln x + A_3 \text{ is increasing on } \left(0, \min\left(e^{\theta_X}, e^{\theta_Y}\right)\right) \text{ and} \\
x \mapsto B_1 \cdot (\ln x)^2 + B_2 \cdot \ln x + B_3 \text{ is increasing on } \left[\max\left(e^{\theta_X}, e^{\theta_Y}\right), \infty\right) \\
\Rightarrow t \mapsto A_1 \cdot t^2 + A_2 \cdot t + A_3 \text{ is increasing on } (-\infty, \min(\theta_X, \theta_Y)) \text{ and} \\
t \mapsto B_1 \cdot t^2 + B_2 \cdot t + B_3 \text{ is increasing on } [\max(\theta_X, \theta_Y), \infty) \\
\Rightarrow \frac{\partial}{\partial t}\left[A_1 \cdot t^2 + A_2 \cdot t + A_3\right] \geq 0 \ \forall t \in (-\infty, \min(\theta_X, \theta_Y)) \text{ and} \\
\frac{\partial}{\partial t}\left[B_1 \cdot t^2 + B_2 \cdot t + B_3\right] \geq 0 \ \forall t \in [\max(\theta_X, \theta_Y), \infty) \\
\Rightarrow 2A_1 t + A_2 \geq 0 \ \forall t \in (-\infty, \min(\theta_X, \theta_Y)) \text{ and} \\
2B_1 t + B_2 \geq 0 \ \forall t \in [\max(\theta_X, \theta_Y), \infty) \\
\Rightarrow A_1 \leq 0,\ 2A_1 \min(\theta_X, \theta_Y) + A_2 \geq 0,\ B_1 \geq 0 \text{ and } 2B_1 \max(\theta_X, \theta_Y) + B_2 \geq 0
\end{gathered}
$$

It results

$$
\begin{gathered}
\sigma_X(1-\varepsilon_X) \geq \sigma_Y(1-\varepsilon_Y), \\
2\left(\frac{1}{(\sigma_X(1-\varepsilon_X))^2} - \frac{1}{(\sigma_Y(1-\varepsilon_Y))^2}\right)\min(\theta_X, \theta_Y) + \frac{2\theta_Y}{(\sigma_Y(1-\varepsilon_Y))^2} - \frac{2\theta_X}{(\sigma_X(1-\varepsilon_X))^2} \geq 0, \\
\sigma_X(1+\varepsilon_X) \geq \sigma_Y(1+\varepsilon_Y),
\end{gathered}
$$

and

$2\left(\frac{1}{(\sigma_Y(1+\varepsilon_Y))^2}-\frac{1}{(\sigma_X(1+\varepsilon_X))^2}\right)\max(\theta_X,\theta_Y)+\frac{2\theta_X}{(\sigma_X(1+\varepsilon_X))^2}-\frac{2\theta_Y}{(\sigma_Y(1+\varepsilon_Y))^2}\geq 0.$

If $\theta_X>\theta_Y$ then $2\left(\frac{1}{(\sigma_X(1-\varepsilon_X))^2}-\frac{1}{(\sigma_Y(1-\varepsilon_Y))^2}\right)\min(\theta_X,\theta_Y)+\frac{2\theta_Y}{(\sigma_Y(1-\varepsilon_Y))^2}-\frac{2\theta_X}{(\sigma_X(1-\varepsilon_X))^2}<0$, contradiction!

Then $\theta_X\leq\theta_Y$. Thus

$$2\left(\frac{1}{(\sigma_Y(1+\varepsilon_Y))^2}-\frac{1}{(\sigma_X(1+\varepsilon_X))^2}\right)\max(\theta_X,\theta_Y)+\frac{2\theta_X}{(\sigma_X(1+\varepsilon_X))^2}-\frac{2\theta_Y}{(\sigma_Y(1+\varepsilon_Y))^2}=$$

$$2\left(\frac{1}{(\sigma_Y(1+\varepsilon_Y))^2}-\frac{1}{(\sigma_X(1+\varepsilon_X))^2}\right)\theta_Y+\frac{2\theta_X}{(\sigma_X(1+\varepsilon_X))^2}-\frac{2\theta_Y}{(\sigma_Y(1+\varepsilon_Y))^2}=$$

$$\frac{2(\theta_X-\theta_Y)}{(\sigma_Y(1+\varepsilon_Y))^2}\leq 0$$

But

$$2\left(\frac{1}{(\sigma_Y(1+\varepsilon_Y))^2}-\frac{1}{(\sigma_X(1+\varepsilon_X))^2}\right)\max(\theta_X,\theta_Y)+\frac{2\theta_X}{(\sigma_X(1+\varepsilon_X))^2}-\frac{2\theta_Y}{(\sigma_Y(1+\varepsilon_Y))^2}\geq 0.$$

It results $\theta_X=\theta_Y$.

We prove the "$\Leftarrow$" implication.

For $x\in\left(0,e^{\theta_X}\right)$ we have

$\frac{f_Y(x)}{f_X(x)}=\frac{\frac{1}{x\sqrt{2\pi}\sigma_Y}\cdot e^{-\frac{(\ln x-\theta_X)^2}{2\sigma_Y^2(1-\varepsilon_Y)^2}}}{\frac{1}{x\sqrt{2\pi}\sigma_X}\cdot e^{-\frac{(\ln x-\theta_X)^2}{2\sigma_X^2(1-\varepsilon_X)^2}}}=e^{\frac{1}{2}\left[A_1\cdot(\ln x)^2+A_2\cdot\ln x+A_3\right]}$, where

$A_1=\frac{1}{(\sigma_X(1-\varepsilon_X))^2}-\frac{1}{(\sigma_Y(1-\varepsilon_Y))^2}$, $A_2=\frac{2\theta_X}{(\sigma_Y(1-\varepsilon_Y))^2}-\frac{2\theta_X}{(\sigma_X(1-\varepsilon_X))^2}$, $A_3=\frac{\theta_X^2}{(\sigma_X(1-\varepsilon_X))^2}-\frac{\theta_X^2}{(\sigma_Y(1-\varepsilon_Y))^2}$.

For $x\in[e^{\theta_X},\infty)$ we have

$\frac{f_Y(x)}{f_X(x)}=\frac{\frac{1}{x\sqrt{2\pi}\sigma_Y}\cdot e^{-\frac{(\ln x-\theta_X)^2}{2\sigma_Y^2(1+\varepsilon_Y)^2}}}{\frac{1}{x\sqrt{2\pi}\sigma_X}\cdot e^{-\frac{(\ln x-\theta_X)^2}{2\sigma_X^2(1+\varepsilon_X)^2}}}=e^{\frac{1}{2}\left[B_1\cdot(\ln x)^2+B_2\cdot\ln x+B_3\right]}$, where

$B_1=\frac{1}{(\sigma_Y(1+\varepsilon_Y))^2}-\frac{1}{(\sigma_X(1+\varepsilon_X))^2}$, $B_2=\frac{2\theta_X}{(\sigma_X(1+\varepsilon_X))^2}-\frac{2\theta_X}{(\sigma_Y(1+\varepsilon_Y))^2}$, $B_3=\frac{\theta_X^2}{(\sigma_Y(1+\varepsilon_Y))^2}-\frac{\theta_X^2}{(\sigma_X(1+\varepsilon_X))^2}$.

But

$$\frac{\partial}{\partial t}\left[A_1 \cdot t^2 + A_2 \cdot t + A_3\right] \geq 0 \ \forall t \in (-\infty, \theta_X)$$

and

$$\frac{\partial}{\partial t}\left[B_1 \cdot t^2 + B_2 \cdot t + B_3\right] \geq 0 \ \forall t \in [\theta_X, \infty)$$

Then $x \mapsto \frac{f_Y(x)}{f_X(x)}$ is increasing on $\left(0, e^{\theta_X}\right)$ and on $[e^{\theta_X}, \infty)$. Thus $x \mapsto \frac{f_Y(x)}{f_X(x)}$ is increasing on $(0, \infty)$.

It results $X \prec_{lr} Y$. $\square$

For the next results we consider

$A(p) = (EX)^{-1}(1-\varepsilon_X)\, e^{\frac{\sigma_X^2(1-\varepsilon_X)^2+2\theta_X}{2}}\, \Phi\left(\Phi^{-1}\left(\frac{p}{1-\varepsilon_X}\right) - \sigma_X(1-\varepsilon_X)\right)$, $p \in \left[0, \frac{1-\varepsilon}{2}\right)$ and

$B(p) = (EX)^{-1}(1-\varepsilon_X)\, e^{\frac{\sigma_X^2(1-\varepsilon_X)^2+2\theta_X}{2}}\, \Phi\left(\sigma_X(1-\varepsilon_X)\right) +$
$(1+\varepsilon_X)\, e^{\frac{\sigma_X^2(1+\varepsilon_X)^2+2(\theta_X-\sigma_X\varepsilon_X)}{2}}\, \overline{\Phi}\left(\sigma_X(1+\varepsilon_X)\right)$, $p \in \left[\frac{1-\varepsilon}{2}, 1\right)$.

Proposition 4.4 gives a formula for Lorenz curve of log-epsilon-skew-normal distributions. Also, proposition 4.5 gives sufficient conditions for Lorenz order between two log-epsilon-skew-normal distributions.

Proposition 4.4. *The Lorenz curve of $X \sim LESN(\theta_X, \sigma_X, \varepsilon_X)$ is:*

$$L_X(p) = A(p) \cdot 1_{\left[0, \frac{1-\varepsilon}{2}\right)}(p) + B(p) \cdot 1_{\left[\frac{1-\varepsilon}{2}, 1\right)}(p).$$

Proof. Let $p \in \left(0, \frac{1-\varepsilon}{2}\right)$. Then

$$\int_0^p Q_X(u)\, du = \int_0^p e^{\theta_X + \sigma_X Q_0(u)} du = \int_0^p e^{\theta_X + \sigma_X(1-\varepsilon_X)\Phi^{-1}\left(\frac{u}{1-\varepsilon_X}\right)} du.$$

$$\Phi^{-1}\left(\frac{u}{1-\varepsilon_X}\right) = y \Rightarrow u = (1-\varepsilon_X)\,\Phi(y) \Rightarrow du = \frac{1-\varepsilon_X}{\sqrt{2\pi}} \cdot e^{-\frac{y^2}{2}} dy$$

It results

$$\int_0^p e^{\theta_X + \sigma_X(1-\varepsilon_X)\Phi^{-1}\left(\frac{u}{1-\varepsilon_X}\right)} du = \int_{-\infty}^{\Phi^{-1}\left(\frac{p}{1-\varepsilon_X}\right)} \frac{1-\varepsilon_X}{\sqrt{2\pi}} \cdot e^{\theta_X + \sigma_X(1-\varepsilon_X)y - \frac{y^2}{2}} dy =$$

$$= \frac{(1-\varepsilon_X)\, e^{\frac{\sigma_X^2(1-\varepsilon_X)^2+2\theta_X}{2}}}{\sqrt{2\pi}} \cdot \int_{-\infty}^{\Phi^{-1}\left(\frac{p}{1-\varepsilon_X}\right)} e^{-\frac{1}{2}(y-\sigma_X(1-\varepsilon_X))^2} dy =$$

$$= (1-\varepsilon_X)\, e^{\frac{\sigma_X^2(1-\varepsilon_X)^2+2\theta_X}{2}} \int_{-\infty}^{\Phi^{-1}\left(\frac{p}{1-\varepsilon_X}\right)-\sigma_X(1-\varepsilon_X)} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}\, dt =$$

$$= (1-\varepsilon_X)\, e^{\frac{\sigma_X^2(1-\varepsilon_X)^2+2\theta_X}{2}} \Phi\left(\Phi^{-1}\left(\frac{p}{1-\varepsilon_X}\right)-\sigma_X\left(1-\varepsilon_X\right)\right).$$

Thus $L_X(p) = A\,(p)$.
Now, let $p \in \left[\frac{1-\varepsilon}{2}, 1\right)$. Then

$$\int_0^p Q_X\,(u)\, du$$

$$= \int_0^p e^{\theta_X+\sigma_X Q_0(u)} du = \int_0^{\frac{1-\varepsilon_X}{2}} e^{\theta_X+\sigma_X Q_0(u)} du + \int_{\frac{1-\varepsilon_X}{2}}^1 e^{\theta_X+\sigma_X Q_0(u)} du$$

$$= (1-\varepsilon_X)\, e^{\frac{\sigma_X^2(1-\varepsilon_X)^2+2\theta_X}{2}} \Phi\left(\Phi^{-1}\left(\frac{\frac{1-\varepsilon_X}{2}}{1-\varepsilon_X}\right)-\sigma_X\left(1-\varepsilon_X\right)\right) + \int_{\frac{1-\varepsilon_X}{2}}^1 e^{\theta_X+\sigma_X Q_0(u)} du$$

$$= (1-\varepsilon_X)\, e^{\frac{\sigma_X^2(1-\varepsilon_X)^2+2\theta_X}{2}} \Phi\left(-\sigma_X\left(1-\varepsilon_X\right)\right) + \int_{\frac{1-\varepsilon_X}{2}}^1 e^{\theta_X+\sigma_X Q_0(u)} du.$$

We have $\int_{\frac{1-\varepsilon_X}{2}}^1 e^{\theta_X+\sigma_X Q_0(u)} du = \int_{\frac{1-\varepsilon_X}{2}}^1 e^{\theta_X-\sigma_X\varepsilon_X+\sigma_X(1+\varepsilon_X)\Phi^{-1}\left(\frac{u+\varepsilon_X}{1+\varepsilon_X}\right)} du$ and

$\Phi^{-1}\left(\frac{u+\varepsilon_X}{1+\varepsilon_X}\right) = y \Rightarrow \frac{u+\varepsilon_X}{1+\varepsilon_X} = \Phi\,(y) \Rightarrow u = (1+\varepsilon_X)\,\Phi\,(y) - \varepsilon_X$ and $du = \frac{1+\varepsilon_X}{\sqrt{2\pi}} \cdot e^{-\frac{y^2}{2}} dy$

It results

$$\int_{\frac{1-\varepsilon_X}{2}}^1 e^{\theta_X-\sigma_X\varepsilon_X+\sigma_X(1+\varepsilon_X)\Phi^{-1}\left(\frac{u+\varepsilon_X}{1+\varepsilon_X}\right)} du$$

$$= \int_0^\infty \frac{1+\varepsilon_X}{\sqrt{2\pi}} \cdot e^{\theta_X-\sigma_X\varepsilon_X+\sigma_X(1+\varepsilon_X)y-\frac{y^2}{2}} dy =$$

$$= \frac{1+\varepsilon_X}{\sqrt{2\pi}} \cdot e^{\frac{\sigma_X^2(1+\varepsilon_X)^2+2(\theta_X-\sigma_X\varepsilon_X)}{2}} \int_0^\infty e^{-\frac{1}{2}(y+\sigma_X(1+\varepsilon_X))^2} dy =$$

$$= (1+\varepsilon_X)\, e^{\frac{\sigma_X^2(1+\varepsilon_X)^2+2(\theta_X-\sigma_X\varepsilon_X)}{2}} \int_{\sigma_X(1+\varepsilon_X)}^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt =$$

$$= (1+\varepsilon_X)\, e^{\frac{\sigma_X^2 (1+\varepsilon_X)^2 + 2(\theta_X - \sigma_X \varepsilon_X)}{2}} \overline{\Phi}\left(\sigma_X \left(1+\varepsilon_X\right)\right).$$

Thus $L_X(p) = B(p)$. □

Proposition 4.5. *Let the random variables* $X \sim LESN\left(\theta_X, \sigma_X, \varepsilon_X\right)$ *and* $Y \sim LESN\left(\theta_Y, \sigma_Y, \varepsilon_Y\right)$ *with* $EX = EY$.

If

$$\varepsilon_X \le \varepsilon_Y,\ \theta_X \le \theta_Y,\ \sigma_X\left(1-\varepsilon_X\right) \ge \sigma_Y\left(1-\varepsilon_Y\right), \sigma_X\left(1+\varepsilon_X\right) \le \sigma_Y\left(1+\varepsilon_Y\right)$$

then $X \prec_{Lorenz} Y$.

Proof. We have $X \prec_{st} Y \Rightarrow X \prec_{icx} Y$. But $EX = EY$. Then $X \prec_{cx} Y$, thus $X \prec_{Lorenz} Y$. □

5 Consequences

Propositions 5.1 and 5.2 sufficient conditions for orders of the moments of order k and a-Gini indexes.

Proposition 5.1. *Let the random variables* $X \sim LESN\left(\theta_X, \sigma_X, \varepsilon_X\right)$ *and* $Y \sim LESN\left(\theta_Y, \sigma_Y, \varepsilon_Y\right)$.

If

$$\varepsilon_X \le \varepsilon_Y,\ \theta_X \le \theta_Y,\ \sigma_X\left(1-\varepsilon_X\right) \ge \sigma_Y\left(1-\varepsilon_Y\right), \sigma_X\left(1+\varepsilon_X\right) \le \sigma_Y\left(1+\varepsilon_Y\right)$$

then $EX^k \le EY^k\ \forall k \in \mathbb{Z},\ k \ge 1$.

Proof. It results from theorem 4.2. □

Proposition 5.2. *Let the random variables* $X \sim LESN\left(\theta_X, \sigma_X, \varepsilon_X\right)$ *and* $Y \sim LESN\left(\theta_Y, \sigma_Y, \varepsilon_Y\right)$ *with* $EX = EY$.

If

$$\varepsilon_X \le \varepsilon_Y,\ \theta_X \le \theta_Y,\ \sigma_X\left(1-\varepsilon_X\right) \ge \sigma_Y\left(1-\varepsilon_Y\right), \sigma_X\left(1+\varepsilon_X\right) \le \sigma_Y\left(1+\varepsilon_Y\right)$$

then $G_{X;a} \le G_{Y;a}$ *for all* $a \ge 1$.

Proof. It results from proposition 4.5. □

6 Real data application

In this section we illustrate the theoretical results obtained in the paper. We use estimated parameters of log-normal and LESN distribution using data from two examples in Hutson et al. (2020). The first data represents Mayo Clinic primary biliary cirrhosis data that can be downloaded from http://www.umass.edu/statdata/statdata/data/, Fleming and Harrington (1991). In particular, they examined 418 serum bilirubin (mg/dl) measurements. In Table 1 maximum likelihood estimators are presented for LESN. The second data represents data that was part of a large clinical trial carried out by the Radiation Therapy Oncology Group in the United States, Kalbfleisch and Prentice (1980). The data may also be downloaded from http://www.umass.edu/statdata/statdata/data/. In particular, they examined the survival time in days from date of diagnosis in 195 patients with squamous carcinoma in the oropharynx. There were 53 patients (27%) whose survival times were right censored. In Table 2 maximum likelihood estimators are presented for LESN.

Table 1

Parameters	**LESN**
Θ	-0.4169
Σ	0.9273
ε	0.6713

Table 2

Parameters	**LESN**
Θ	5.7489
Σ	1.3002
ε	0.2266

Let the random variables $X \sim LESN\,(-0.4169, 0.9273, 0.6713)$ and $Y \sim LESN\,(5.7489, 1.3002, 0.2266)$. We have:
$\varepsilon_X = 0.6713, \varepsilon_Y = 0.2266$,
$\theta_X = -0.4169, \theta_Y = 5.7489$,
$\sigma_X\,(1-\varepsilon_X) = 0.9273 \cdot (1-0.6713) = 0.3048$,
$\sigma_Y\,(1-\varepsilon_Y) = 1.3002 \cdot (1-0.2266) = 1.0056$,
$\sigma_X\,(1+\varepsilon_X) = 0.9273 \cdot (1+0.6713) = 1.5498$,
$\sigma_Y\,(1+\varepsilon_Y) = 1.3002 \cdot (1+0.2266) = 1.5948$.
From theorem 4.1 it results $X \nprec_{st} Y$ because $\sigma_X\,(1-\varepsilon_X) < \sigma_Y\,(1-\varepsilon_Y)$.
From theorem 4.3 it results $X \nprec_{lr} Y$ because $\theta_X \neq \theta_Y$.
Remarks. (1) There exists $c_1 > 0$ such that $\overline{F}_X\,(c_1) > \overline{F}_Y\,(c_1)$;

(2) There exists $k_1 \in \mathbb{Z}, k_1 \geq 1$ such that $EX^{k_1} > EY^{k_1}$;
(3) There exists $c_2 > 0$ such that $r_X(c_2) < r_Y(c_2)$.
Proof. (1) and (2) result from $X \nprec_{st} Y$.
(3) If $X \prec_{hr} Y$ then $X \prec_{st} Y$, contradiction!
Then $X \nprec_{hr} Y$. $\square$

7 Conclusions

In this article we analyzed different stochastic orders between log-epsilon-skew-normal distributions. We gave necessary and/or sufficient conditions for these orders. These results improve the literature of stochastic ordering between log-normal distributions and provide ideas for generalizations in other research papers.

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