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DNA codes over finite local Frobenius non-chain rings of length 5 and nilpotency index 4

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Abstract

A one to one correspondence between the elements of a finite local Frobenius non-chain ring of length 5 and nilpotency index 4, and k-tuples of DNA codewords is established. Using this map the structure of DNA codes over these rings is determined, the length of the code is relatively prime to the characteristic of the residue field of the ring.

1 Introduction

In [2], Adleman gave studies on DNA computing by solving an instance of NP-complete problem over DNA molecules. A single DNA strand is a sequence of four possible nucleotides: adenine (A), guanine (G), cytosine (C) and thymine (T). DNA has two strands that are governed by the rule called Watson Crick complement (WCC), that is, A pairs with T and G pairs with C. We denote the WCC as $\overline{A} = T$, $\overline{T} = A$, $\overline{C} = G$, $\overline{G} = C$.

The structure of DNA is used as a model for constructing good error correcting codes and conversely error correcting codes that enjoy similar properties with DNA structure are also used to understand DNA itself. Several papers have proposed different techniques to construct a set of DNA codewords. In [18], authors used stochastic search algorithms to design codewords

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that are suitable for DNA computing. Linear and cyclic codes have also extensively been used to construct DNA codes by several authors in [1], [3], [5], [10], [11], [12], [13], [16], [17], [19].
In [1], [11] and [17], nucleotides were identified with the elements of the rings $\mathrm{GF}(4)$, $\mathbb{Z}_4$ and $\mathrm{GF}(2)[u]/\langle u^2-1\rangle$, respectively, and DNA codes over these rings were studied. Nucleotide pairs $\{A,C,G,T\}^2$ were identified with elements of the chain ring $\mathrm{GF}(2)[\mathrm{T}]/\langle \mathrm{T}^4-1\rangle$ and DNA codes over this ring were studied in [19]. k-tuples of nucleotides were identified with elements of finite local Frobenius non-chain rings of length 4 and DNA codes over those rings were studied in [3]. The idea in these works was to find a bijection $\Phi : \mathrm{R} \to \{A,C,G,T\}^k$ with the properties $\Phi(a+\kappa) = \overline{\Phi(a)}$ and $\Phi(\rho a) = rev(\Phi(a))$, for some $\rho, \kappa \in \mathrm{R}$, and for all $a \in \mathrm{R}$.

On the other hand, the finite local Frobenius non-chain rings of length 5 and nilpotency index 4 were determined in [9] and the structure of reversible cyclic codes over those rings was established in [4], when the length of the code is relatively prime to the characteristic of the residue field of the ring. Now it would be interesting to identify these rings with k-tuples of nucleotides and study DNA codes over these rings.

The purpose of this paper is to identify Finite local Frobenius non-chain rings of length 5 whose maximal ideal has nilpotency index 4 and residue field $\mathrm{GF}(2^d)$ with k-tuples of nucleotides, for some k, and determine the structure of DNA codes over these rings, of length relatively prime to the characteristic of the residue field of the ring. The paper is organized as follows: in Section 2, basic facts on commutative finite local rings, properties of the reciprocal matrix and properties finite local Frobenius non-chain ring of length 5 whose maximal ideal has nilpotency index 4 are given. In Section 3, we give conditions for the existence of a correspondence between finite local rings and k-tuples of nucleotides. In Section 4 we use the correspondence between $\{A,C,G,T\}^5$ and a finite local Frobenius non-chain ring of length 5, nilpotency index 4 and residue field $\mathrm{GF}(2^2)$ for describing the structure of DNA codes over this ring.

2 Preliminaries

Throughout this work $\mathrm{GF}(q)$ denotes the finite field with $q = p^d$ elements, p a prime, and all rings are assumed to be finite, commutative with unit element.

Let Ω be a non empty set and $\mathfrak{O}$ be an equivalence relation in Ω. The family of equivalence classes modulo $\mathfrak{O}$ is a partition of Ω, conversely if $\{\mathfrak{O}_i\}_{i\in\mathrm{I}}$ is a partition of Ω, the set $\mathfrak{O}$ of all pairs (ω_1, ω_2) such that ω_1, ω_2 are in the same member of the partition is an equivalence relation in Ω. That is, every equivalence relation in Ω corresponds uniquely to a partition of Ω and conversely.

Let R be a ring and M an R-module. The annihilator ideal of M in R is defined as $\mathrm{ann}_{\mathrm{R}}(\mathrm{M}) := \{a \in \mathrm{R} : am = 0, \forall\, m \in \mathrm{M}\}$. The length of M, denoted by $\ell_{\mathrm{R}}(\mathrm{M})$, is the length of a composition series for M, see [15]. If R has the unique maximal ideal $\mathfrak{m}$, it is called local, $\mathrm{k} = \mathrm{R}/\mathfrak{m}$ its residue field, there is an integer $t \geq 1$ such that $\mathfrak{m}^t = \langle 0 \rangle$, called the nilpotency index of $\mathfrak{m}$, and $|\mathrm{M}| = |\mathrm{GF}(q)|^{\ell_{\mathrm{R}}(\mathrm{M})}$, see [6]. The local ring R will be denoted by the triple $(\mathrm{R}, \mathfrak{m}, \mathrm{GF}(q))$.

Let $(\mathrm{R}, \mathfrak{m}, \mathrm{GF}(q))$ be a finite local ring, $\mathrm{f} \in \mathrm{R}[\mathrm{T}]$ and $n \in \mathbb{N}$ with $(p, n) = 1$. $^{-} : \mathrm{R}[\mathrm{T}] \to \mathrm{GF}(q)[\mathrm{T}]$ is the ring homomorphism that maps $a \mapsto a + \mathfrak{m}$ and the variable T to T. f is called basic irreducible if $\overline{\mathrm{f}}$ is irreducible in $\mathrm{GF}(q)[\mathrm{T}]$. If f is monic basic irreducible and $\deg(\mathrm{f}) = s$, the ring $\mathrm{B} = \mathrm{R}[\mathrm{T}]/\langle \mathrm{f} \rangle = \{a_0 + a_1\mathrm{T} + \cdots + a_{s-1}\mathrm{T}^{s-1} : a_i \in \mathrm{R}\}$ is a local separable extension of R with maximal ideal $\mathfrak{m}\mathrm{B}$ and residue field $\mathrm{GF}(q^s)$. Furthermore if $\mathbb{T} \subset \mathrm{R}$ is a set of representatives of $\mathrm{GF}(q)$ and I is an ideal of R, the set $\mathbb{T}_s := \{a_0 + a_1\mathrm{T} + \cdots + a_{s-1}\mathrm{T}^{s-1} : a_i \in \mathbb{T}\} \subset \mathrm{B}$ is a set of representatives of $\mathrm{GF}(q^s)$ and $\ell_{\mathrm{R}}(\mathrm{I}) = \ell_{\mathrm{B}}(\mathrm{IB})$, see [6]. Hensel's Lemma guarantees that $\mathrm{T}^n - 1$ is the product of a unique family of monic basic irreducible pairwise coprime polynomials in $\mathrm{R}[\mathrm{T}]$, see [7] and [15].

The reciprocal of the polynomial f is defined as $\mathrm{f}^* = \mathrm{T}^{\deg(\mathrm{f})}\mathrm{f}(\frac{1}{\mathrm{T}})$, the polynomial f is called self-reciprocal if f^* is associate of f. Let $\mathrm{f}_1, \ldots, \mathrm{f}_r$ the unique family of monic basic irreducible pairwise coprime polynomials such that $\mathrm{T}^n - 1 = \mathrm{f}_1 \cdots \mathrm{f}_r$. From [7], [8] and [14] we have the following: (1) f_i^* is associate of f_j if and only if $\overline{\mathrm{f}_i^*}$ is associate of $\overline{\mathrm{f}_j}$, in particular f_i is self-reciprocal if and only if $\overline{\mathrm{f}_i}$ is self-reciprocal; (2) Over fields, self reciprocal polynomials have even degree with the only exception of $\mathrm{T} + 1$. In particular, if f_i is self reciprocal and $\mathrm{f}_i \neq \mathrm{T} - 1$, then f_i has even degree; (3) After renumbering there are non negative integers r_1, r_2 such that $r = r_1 + 2r_2$, f_{2i-1}^* is associate of f_{2i}, $1 \leq i \leq r_1$, and f_{2r_1+i} is self-reciprocal polynomial, $1 \leq i \leq r_2$.
For the remainder of the manuscript the following notation will be used:
(1) The order of the sequence $\mathrm{f}_1, \ldots, \mathrm{f}_r$ is fixed according to the last assertion; (2) For $u \in \{1, \ldots, r\}$, u^* denotes the index of the polynomial which is associate of f_u, observe that $(u^*)^* = u$; (3) For $\mathrm{U} \subseteq \{1, \ldots, r\}$ let $\mathrm{U}^* = \{u^* : u \in \mathrm{U}\}$.
U is called self-reciprocal if $\mathrm{U}^* = \mathrm{U}$, observe that U is a self-reciprocal set if and only if $\{1, \ldots, r\} \backslash \mathrm{U}$ is a self-reciprocal set. $\emptyset$ is considered a self-reciprocal set.

The reciprocal matrix is helpful to determine reversible constacyclic codes, its properties was treated in [4] and [8]. Observe that if $\mathrm{f} \in \mathrm{GF}(q)[\mathrm{T}]$ is an irreducible polynomial, then $\psi : \mathrm{GF}(q)[\mathrm{T}]/\langle \mathrm{f}^* \rangle \to \mathrm{GF}(q)[\mathrm{T}]/\langle \mathrm{f} \rangle$ given by $\frac{\mathrm{h}^* + \langle \mathrm{f} \rangle}{\mathrm{T}^{\deg(\mathrm{h})} + \langle \mathrm{f} \rangle}$ is an isomorphism over $\mathrm{GF}(q)$.

Definition 1. *Let* $(\mathrm{R}, \mathfrak{m}, \mathrm{GF}(q))$ *be a finite local ring,* $\mathrm{f} \in \mathrm{R}[\mathrm{T}]$ *a monic basic irreducible polynomial,* $n \geq \deg(\mathrm{f})$*,* ψ *as above and* $\mathrm{H} = (a_{ij})$ *a matrix over* $\mathrm{GF}(q^{\deg(\mathrm{f})}) = \mathrm{GF}(q)[\mathrm{T}]/\langle \overline{\mathrm{f}^*} \rangle$*, the matrix* $(\overline{\mathrm{T}}^n \psi(a_{ij}))$ *in* $\mathrm{GF}(q)[\mathrm{T}]/\langle \bar{\mathrm{f}} \rangle$ *is called the n-th reciprocal matrix of* H *with respect* f *and is denoted by* $\mathrm{H}^*_{(n,\mathrm{f})}$.

Let $(\mathrm{R}, \mathfrak{m}, \mathrm{GF}(q))$, f, n and $\mathrm{H} = (a_{ij})$ as in Definition 1. Suppose f is self reciprocal, $\deg(\mathrm{f}) = 2s$ and $\mathrm{f} | \mathrm{T}^n - 1$. Let $\mathrm{g} \in \mathrm{GF}(q)[\mathrm{T}]$ be an irreducible polynomial of degree s and $\alpha \in \mathrm{GF}(q^{2s})$ be a root of g. From [3], we have

$$\mathrm{H}^*_{(n,\mathrm{f})} = \mathrm{H} \Leftrightarrow \mathrm{H} \textit{ is a matrix over } \mathrm{GF}(q^s)$$
$$\Leftrightarrow \mathrm{H} = (\mathrm{h}_{ij}(\alpha)), \textit{ for some } h_{ij} \in \mathrm{GF}(q)[\mathrm{T}] \textit{ with } \deg(\mathrm{h}_{ij}) < s.$$

The ring R is a chain ring if the lattice of its ideals is a chain under set-theoretic inclusion . The ring R is a chain ring if and only if R is local and its maximal ideal is principal. The ideals of the finite chain ring R are $\langle \pi^{\ell_{\mathrm{R}}(\mathrm{R})} \rangle = \langle 0 \rangle \subset \langle \pi^{\ell_{\mathrm{R}}(\mathrm{R})-1} \rangle \subset \ldots \subset \langle \pi \rangle \subset \mathrm{R}$. A finite local ring $(\mathrm{R}, \mathfrak{m}, \mathrm{GF}(q))$ is Frobenius if $\mathrm{ann}_{\mathrm{R}}(\mathfrak{m})$ is the unique minimal ideal of R. If $(\mathrm{R}, \mathfrak{m}, \mathrm{GF}(q))$ is a finite local Frobenius ring and I is an ideal of R, then $\ell_{\mathrm{R}}(\mathrm{I}) + \ell_{\mathrm{R}}(\mathrm{ann}_{\mathrm{R}}(\mathrm{I})) = \ell_{\mathrm{R}}(\mathrm{R})$. $\mathfrak{F}_5^4$ denotes the family of finite local Frobenius non-chain rings of length 5 with nilpotency index 4. Recently, the rings in the family $\mathfrak{F}_5^4$ with residue field $\mathrm{GF}(2^d)$ were described in [9] and for completeness we recall this result here:

Proposition 1. *Let* $(\mathrm{R}, \mathfrak{m}, \mathrm{GF}(2^d)) \in \mathfrak{F}_5^4$*, then* R *is isomorphic to one of the following rings:*

(*a*) *If* $3 | 2^d - 1$
(*1*) $\mathrm{GF}(2^d)[\mathrm{X}, \mathrm{Y}]/\langle \mathrm{X}^2 - \mathrm{Y}^3, \mathrm{XY} \rangle$,
(*2*) $\mathrm{GR}(2^2, d)[\mathrm{X}, \mathrm{Y}]/\langle \mathrm{Y}^2 - 2, \mathrm{X}^2 - \mathrm{Y}^3, \mathrm{XY} \rangle$,
(*3*) $\mathrm{GR}(2^2, d)[\mathrm{X}, \mathrm{Y}]/\langle \mathrm{Y}^3 - 2, \mathrm{X}^2 - \mathrm{Y}^3, \mathrm{XY} \rangle$,
(*4*) $\mathrm{GR}(2^2, d)[\mathrm{X}, \mathrm{Y}]/\langle \zeta\mathrm{Y}^3 - 2, \mathrm{X}^2 - \mathrm{Y}^3, \mathrm{XY} \rangle$,
(*5*) $\mathrm{GR}(2^2, d)[\mathrm{X}, \mathrm{Y}]/\langle \zeta^2\mathrm{Y}^3 - 2, \mathrm{X}^2 - \mathrm{Y}^3, \mathrm{XY} \rangle$,
(*6*) $\mathrm{GR}(2^3, d)[\mathrm{X}]/\langle 2^2 - \mathrm{X}^3, 2\mathrm{X} \rangle$,
(*7*) $\mathrm{GR}(2^3, d)[\mathrm{X}]/\langle 2^2 - \zeta\mathrm{X}^3, 2\mathrm{X} \rangle$,
(*8*) $\mathrm{GR}(2^3, d)[\mathrm{X}]/\langle 2^2 - \zeta^2\mathrm{X}^3, 2\mathrm{X} \rangle$,
(*9*) $\mathrm{GR}(2^4, d)[\mathrm{X}]/\langle \mathrm{X}^2 - 2^3, 2\mathrm{X} \rangle$.

In cases (4) *and* (5)*,* $\{0, 1, \ldots, \zeta^{2^d-2}\}$ *is the Teichmüller set of the Galois ring* $\mathrm{GR}(2^2, d)$*. In cases* (7) *and* (8)*,* $\{0, 1, \ldots, \zeta^{2^d-2}\}$ *is the Teichmüller set of the Galois ring* $\mathrm{GR}(2^3, d)$*.*

(*b*) *If* $3 \not| 2^d - 1$
(*1*) $\mathrm{GF}(2^d)[\mathrm{X},\mathrm{Y}]/\langle \mathrm{X}^2 - \mathrm{Y}^3, \mathrm{XY}\rangle$,
(*2*) $\mathrm{GR}(2^2, d)[\mathrm{X},\mathrm{Y}]/\langle \mathrm{Y}^2 - 2, \mathrm{X}^2 - \mathrm{Y}^3, \mathrm{XY}\rangle$,
(*3*) $\mathrm{GR}(2^2, d)[\mathrm{X},\mathrm{Y}]/\langle \mathrm{Y}^3 - 2, \mathrm{X}^2 - \mathrm{Y}^3, \mathrm{XY}\rangle$,
(*6*) $\mathrm{GR}(2^3, d)[\mathrm{X}]/\langle 2^2 - \mathrm{X}^3, 2\mathrm{X}\rangle$,
(*9*) $\mathrm{GR}(2^4, d)[\mathrm{X}]/\langle \mathrm{X}^2 - 2^3, 2\mathrm{X}\rangle$.

Let $(\mathrm{R}, \mathfrak{m}, \mathrm{GF}(q)) \in \mathfrak{F}_5^4$, $\mathrm{f} \in \mathrm{R}[\mathrm{T}]$ a monic basic irreducible polynomial of degree s, and $\vec{\mathrm{v}} = (a, b) \in \mathrm{GF}(q^s)^2$. For the reminder of the manuscript $(\mathrm{B} = \mathrm{R}[\mathrm{T}]/\langle \mathrm{f}\rangle, \mathfrak{m}\mathrm{B}, \mathrm{GF}(q^s))$ will be the separable extension of R determined by f, $\mathbb{T} \subset \mathrm{R}$ will be a set of representatives of $\mathrm{GF}(q)$, $\tilde{\theta} = \{\theta_1, \theta_2\}$ will be a sequence of elements of B and the following notation will be used.

(1) $\mathbb{T}_s = \{a_0 + a_1\mathrm{T} + \cdots + a_{s-1}\mathrm{T}^{s-1} : a_i \in \mathbb{T}\} \subset \mathrm{B}$ will be the set of representatives of $\mathrm{B}/\mathfrak{m}\mathrm{B} = \mathrm{GF}(q^s)$;

(2) For $a \in \mathrm{GF}(q^s)$, the only representative of a in $\mathbb{T}_s$ will be denoted by $a^{\mathbb{T}_s}$;

(3) The ideal of B, $\langle a^{\mathbb{T}_s}\theta_1 + b^{\mathbb{T}_s}\theta_2\rangle$, will be denoted by $\vec{\mathrm{v}}_{\tilde{\theta}}^{\mathbb{T}_s}$.

(4) A fixed minimal R-generating set $\{\alpha_1, \alpha_2\}$ of $\mathfrak{m}$ will be considered.
If the ring R is one of the rings $(1) - (5)$ mentioned in Proposition 1, $\alpha_1 = \mathrm{x}$ and $\alpha_2 = \mathrm{y}$.
If the ring R is one of the rings in cases $(6) - (8)$ mentioned in Proposition 1, $\alpha_1 = 2$ and $\alpha_2 = \mathrm{x}$.
If the ring R is the ring (9) mentioned in Proposition 1, $\alpha_1 = \mathrm{x}$ and $\alpha_2 = 2$.
When we take a minimal R-generating set for $\mathfrak{m}$ we understand that $\{\alpha_1, \alpha_2\}$ is the ordered minimal R-generating set for $\mathfrak{m}$.

(5) The fixed minimal R-generating set for $\mathrm{ann}_{\mathrm{R}}(\mathfrak{m}^2)$ will be $\{\alpha_1, \alpha_2^2\}$.
When we take a minimal R-generating set for $\mathrm{ann}_{\mathrm{R}}(\mathfrak{m}^2)$ we understand this is the ordered minimal R-generating set for $\mathrm{ann}_{\mathrm{R}}(\mathfrak{m}^2)$.

(6) We write $\tilde{\alpha}$ for $\{\alpha_1, \alpha_2\}$ and $\tilde{\beta}$ for $\{\alpha_1, \alpha_2^2\}$.

The lattice of ideals of a separable extension of a ring in the family $\mathfrak{F}_5^4$ and the annihilator ideal of the ideals of a ring in the family $\mathfrak{F}_5^4$ were determined in [9] and [4], respectively. For completeness we state these results here.

Lemma 1. *Let* $(\mathrm{R}, \mathfrak{m}, \mathrm{GF}(q)) \in \mathfrak{F}_5^4$, $\mathbb{T}$, $\mathbb{T}_s$, $\tilde{\alpha} = \{\alpha_1, \alpha_2\}$, $\mathrm{f} \in \mathrm{R}[\mathrm{T}]$, $\deg(\mathrm{f}) = s$, $(\mathrm{B} = \mathrm{R}[\mathrm{T}]/\langle \mathrm{f}\rangle, \mathfrak{m}\mathrm{B}, \mathrm{GF}(q^s))$ *be as above, then:*

(*1*) *The ideals of length* $0, 1, 4, 5$ *of* B *are* $\langle 0\rangle$, $\mathfrak{m}^3\mathrm{B}$, $\mathfrak{m}\mathrm{B}$, B, *respectively.*

(*2*) *Ideals of length* 2 *of* B *are between* $\mathrm{ann}_{\mathrm{R}}(\mathfrak{m}^2)\mathrm{B}$ *and* $\mathfrak{m}^3\mathrm{B}$. *These ideals are:*

$$\mathfrak{m}^2\mathrm{B} = (0,1)^{\mathbb{T}_s}_{\tilde{\beta}}, (1,\lambda_1)^{\mathbb{T}_s}_{\tilde{\beta}}, \ldots, (1,\lambda_{q^s})^{\mathbb{T}_s}_{\tilde{\beta}} \quad \lambda_i \in \mathrm{GF}(q^s).$$

(*3*) *Ideals of length* 3 *of* B *are between* $\mathfrak{m}\mathrm{B}$ *and* $\mathfrak{m}^2\mathrm{B}$. *These ideals are:*

$$(0,1)^{\mathbb{T}_s}_{\tilde{\alpha}}, (1,0)^{\mathbb{T}_s}_{\tilde{\alpha}} + \mathfrak{m}^2\mathrm{B} = \mathrm{ann}_{\mathrm{B}}(\mathfrak{m}^2\mathrm{B}), (1,\lambda_1)^{\mathbb{T}_s}_{\tilde{\alpha}}, \ldots, (1,\lambda_{q^s-1})^{\mathbb{T}_s}_{\tilde{\alpha}},$$

$$\lambda_i \in \mathrm{GF}(q^s) \setminus \{0\}.$$

The ideal $(1,0)^{\mathbb{T}_s}_{\tilde{\alpha}} + \mathfrak{m}^2\mathrm{B} = \mathrm{ann}_{\mathrm{B}}(\mathfrak{m}^2\mathrm{B}) = \langle \alpha_1, \alpha_2^2\rangle\mathrm{B}$ is simply denoted by $(1,0)^{\mathbb{T}_s}_{\tilde{\alpha}}$.

Lemma 2. *Let* $(\mathrm{R},\mathfrak{m},\mathrm{GF}(q)) \in \mathfrak{F}^4_5$, $\mathbb{T}$ *and* $\tilde{\alpha} = \{\alpha_1,\alpha_2\}$ *be as above, then:* $\mathrm{ann}_{\mathrm{R}}(\alpha_2^2) = \langle \alpha_1, \alpha_2^2\rangle$, $\mathrm{ann}_{\mathrm{R}}(\alpha_1) = \langle \alpha_2\rangle$ *and*

(*1*) $\mathrm{ann}_{\mathrm{R}}(\alpha_1 + \lambda_1\alpha_2^2) = \langle \alpha_1 + \lambda_2\alpha_2\rangle$, $\lambda_1, \lambda_2 \in \mathbb{T}$, $\bar{\lambda}_1\bar{\lambda}_2 = -1$, *for the rings* $(1)-(6)$, (9) *in Proposition 1.*

(*2*) $\mathrm{ann}_{\mathrm{R}}(\alpha_1 + \lambda_1\alpha_2^2) = \langle \alpha_1 + \lambda_2\alpha_2\rangle$, $\lambda_1, \lambda_2 \in \mathbb{T}$, $\bar{\lambda}_1\bar{\lambda}_2 = -\bar{\zeta}$, *for the ring* (7) *in Proposition 1.*

(*3*) $\mathrm{ann}_{\mathrm{R}}(\alpha_1 + \lambda_1\alpha_2^2) = \langle \alpha_1 + \lambda_2\alpha_2\rangle$, $\lambda_1, \lambda_2 \in \mathbb{T}$, $\bar{\lambda}_1\bar{\lambda}_2 = -\overline{\zeta^2}$, *for the ring* (8) *in Proposition 1.*

Observation 1. *Under the notation as in Lemma 1. Since* $\mathfrak{m}(\alpha_2\mathrm{B}) = \alpha_2^2\mathrm{B} = \mathfrak{m}^2\mathrm{B}$ *and* $(\mathrm{ann}_{\mathrm{B}}(\mathfrak{m}^2)\mathrm{B})(\alpha_2\mathrm{B}) = (\langle \alpha_1,\alpha_2^2\rangle\mathrm{B})(\alpha_2\mathrm{B}) = \alpha_2^3\mathrm{B} \subseteq \alpha_1\mathrm{B}$, *then:*

$$(\alpha_1\mathrm{B} : \alpha_2\mathrm{B}) = \{\mathrm{a} \in \mathrm{R} : a\alpha_2 \in \alpha_1\mathrm{B}\} = \mathrm{ann}_{\mathrm{B}}(\mathfrak{m}^2)\mathrm{B}.$$

The next results will be useful for the existence of compatible functions defined on a rings $(\mathrm{R},\mathfrak{m},\mathrm{GF}(2^d))$ in the family $\mathfrak{F}^4_5$. Observe the following:

$$\begin{gathered}\ell_{\mathrm{R}}(\mathrm{ann}_{\mathrm{R}}(\rho-1)) = 3 \Leftrightarrow \ell_{\mathrm{R}}(\rho-1) = 2\\ \Leftrightarrow \langle \rho - 1\rangle = (0,1)^{\mathbb{T}_s}_{\tilde{\beta}} \text{ or } \langle \rho-1\rangle = (1,\lambda)^{\mathbb{T}_s}_{\tilde{\beta}}, \lambda \in \mathrm{GF}(2^d),\\ \Leftrightarrow \rho = 1 + a_0\alpha_2^2, a_0 \in \mathrm{R}^*, \text{ or } \rho = 1 + a_0(\alpha_1 + \lambda\alpha_2^2), a_0 \in \mathrm{R}^*, \lambda \in \mathbb{T}.\end{gathered}$$

Lemma 3. *Let* R *be any of the following rings:*

(*1*) $\mathrm{GF}(2^d)[\mathrm{X},\mathrm{Y}]/\langle \mathrm{X}^2 - \mathrm{Y}^3, \mathrm{XY}\rangle$,

(*2*) $\mathrm{GR}(2^2,d)[\mathrm{X},\mathrm{Y}]/\langle \mathrm{Y}^2 - 2, \mathrm{X}^2 - \mathrm{Y}^3, \mathrm{XY}\rangle$,

(*3*) $\mathrm{GR}(2^2,d)[\mathrm{X},\mathrm{Y}]/\langle \mathrm{Y}^3 - 2, \mathrm{X}^2 - \mathrm{Y}^3, \mathrm{XY}\rangle$, (*4*) $\mathrm{GR}(2^2, d)[\mathrm{X}, \mathrm{Y}]/\langle \zeta \mathrm{Y}^3 - 2, \mathrm{X}^2 - \mathrm{Y}^3, \mathrm{XY}\rangle$,

(*5*) $\mathrm{GR}(2^2, d)[\mathrm{X}, \mathrm{Y}]/\langle \zeta^2 \mathrm{Y}^3 - 2, \mathrm{X}^2 - \mathrm{Y}^3, \mathrm{XY}\rangle$.

The only elements $\rho, \kappa \in \mathrm{R}$ *which satisfy* $2\kappa = 0$, $\rho^2 = 1$, $\kappa = \rho\kappa$, $\kappa \notin \langle \rho - 1\rangle$ *and* $\ell_{\mathrm{R}}(\mathrm{ann}_{\mathrm{R}}(\rho - 1)) = 3$ *are:*

$$\rho = 1 + a_0\mathrm{y}^2, \quad \kappa = a_1\mathrm{x} + a_2\mathrm{y}^2, \quad a_0, a_1 \in \mathrm{R}^*, a_2 \in \mathrm{R}.$$

In this case $\ell_{\mathrm{R}}(\kappa) = \ell_{\mathrm{R}}(\mathrm{x} + a_1^{-1}a_2\mathrm{y}^2) = 2$.

Proof: Case I: If $\rho = 1 + a_0\mathrm{y}^2$, $a_0 \in \mathrm{R}^*$, relation $\kappa = \rho\kappa$ implies $\kappa \in \mathrm{ann}_{\mathrm{R}}(\mathrm{y}^2) = \langle \mathrm{x}, \mathrm{y}^2\rangle$ and $\kappa = a_1\mathrm{x} + a_2\mathrm{y}^2$, $a_1, a_2 \in \mathrm{R}$. Now since $\kappa \notin \langle \rho - 1\rangle = \langle \mathrm{y}^2\rangle$ then $a_1 \in \mathrm{R}^*$.
Case II: If $\langle \rho - 1\rangle = \langle \mathrm{x} + \lambda\mathrm{y}^2\rangle$, then $\langle \rho - 1\rangle^2 = \langle 2(\rho - 1)\rangle = \langle 2\rangle\langle \mathrm{x} + \lambda\mathrm{y}^2\rangle = \langle 0\rangle = \langle \mathrm{x} + \lambda\mathrm{y}^2\rangle^2 = \langle \mathrm{x}^2\rangle$, a contradiction.

Lemma 4. *Let* R *be any of the rings:*

(*1*) $\mathrm{GR}(2^3, d)[\mathrm{X}]/\langle 2^2 - \mathrm{X}^3, 2\mathrm{X}\rangle$,

(*2*) $\mathrm{GR}(2^3, d)[\mathrm{X}]/\langle 2^2 - \zeta\mathrm{X}^3, 2\mathrm{X}\rangle$,

(*3*) $\mathrm{GR}(2^3, d)[\mathrm{X}]/\langle 2^2 - \zeta^2\mathrm{X}^3, 2\mathrm{X}\rangle$.

The only elements $\rho, \kappa \in \mathrm{R}$ *which satisfy* $2\kappa = 0$, $\rho^2 = 1$, $\kappa = \rho\kappa$, $\kappa \notin \langle \rho - 1\rangle$ *and* $\ell_{\mathrm{R}}(\mathrm{ann}_{\mathrm{R}}(\rho - 1)) = 3$ *are:*

(*1*) $\rho = 1 + a_0(2 + \lambda\mathrm{x}^2)$, $\kappa = a_1\mathrm{x}^2$, *where* $\lambda \in \mathbb{T}$, $a_0 \in 1 + \mathfrak{m}$, $a_1 \in \mathrm{R}^*$.
In this case $\ell_{\mathrm{R}}(\kappa) = \ell_{\mathrm{R}}(\mathrm{x}^2) = 2$.

(*2*) $\rho = 1 + 2a_0$, $\kappa = a_1\mathrm{x}$, *where* $a_0 \in \mathrm{R}^*$, $a_1 \notin \langle 2, \mathrm{x}^2\rangle$.
In this case $\ell_{\mathrm{R}}(\kappa) = \ell_{\mathrm{R}}(a_1\mathrm{x}) = \begin{cases} \ell_{\mathrm{R}}(\mathrm{x}) = 3 & a_1 \in \mathrm{R}^* \\ \ell_{\mathrm{R}}(\mathrm{x}^2) = 2 & a_1 \in \mathfrak{m} \setminus \langle 2, \mathrm{x}^2\rangle \end{cases}$.

Proof: Case I: If $\langle \rho - 1\rangle = \langle \mathrm{x}^2\rangle$, relations $2\kappa = 0$ and $\kappa = \rho\kappa$ imply $\kappa \in \mathrm{ann}_{\mathrm{R}}(2) \cap \mathrm{ann}_{\mathrm{R}}(\mathrm{x}^2) = \mathrm{ann}_{\mathrm{R}}(2, \mathrm{x}^2) = \langle \mathrm{x}^2\rangle = \langle \rho - 1\rangle$, which is not possible.
Case II: If $\rho = 1 + a_0(2 + \lambda\mathrm{x}^2)$, $\lambda \neq 0$ and $a_0 \in \mathrm{R}^*$, then $\rho^2 = 1 = 1 + 4a_0 + 4a_0^2$, $4a_0(1 + a_0) = 0$, $1 + a_0 \in \mathrm{ann}_{\mathrm{R}}(\mathrm{x}^3) = \mathfrak{m}$ and $a_0 \in 1 + \mathfrak{m}$. Relations $\kappa = \rho\kappa$ and $2\kappa = 0$ imply $\kappa \in \mathrm{ann}_{\mathrm{R}}(2 + \lambda\mathrm{x}^2) \cap \mathrm{ann}_{\mathrm{R}}(2) = \mathrm{ann}_{\mathrm{R}}(2, \mathrm{x}^2) = \langle \mathrm{x}^2\rangle$ and $\kappa = a_1\mathrm{x}^2$, $a_1 \in \mathrm{R}$. Now if $a_1 \in \mathrm{m}$, then $\kappa = a_1\mathrm{x}^2 \in \mathfrak{m}^3 \subseteq \langle \rho - 1\rangle$, hence $a_1 \in \mathrm{R}^*$.
Case III: If $\rho = 1 + 2a_0$, $a_0 \in \mathrm{R}^*$. Relation $2\kappa = 0$ and Lemma 2 imply $\kappa \in \mathrm{ann}_{\mathrm{R}}(2) = \langle \mathrm{x}\rangle$ and $\kappa = a_1\mathrm{x}$, $a \in \mathrm{R}$. Now by Remark 1 and because $\kappa \notin \langle \rho - 1\rangle = \langle 2\rangle$, then $a_1 \notin (\langle 2\rangle : \langle \mathrm{x}\rangle) = \{\mathrm{a} \in \mathrm{R} : a\mathrm{x} \in \langle 2\rangle\} = \langle 2, \mathrm{x}^2\rangle$.

Lemma 5. *Let* $\mathrm{R} = \mathrm{GR}(2^4, d)[\mathrm{X}]/\langle \mathrm{X}^2 - 2^3, 2\mathrm{X}\rangle$. *There are not elements* $\rho, \kappa \in \mathrm{R}$ *which satisfy* $2\kappa = 0$, $\rho^2 = 1$, $\kappa = \rho\kappa$, $\kappa \notin \langle \rho - 1\rangle$ *and* $\ell_{\mathrm{R}}(\mathrm{ann}_{\mathrm{R}}(\rho - 1)) = 3$.

Proof: Case I: If $\langle \rho - 1\rangle = \langle 4\rangle$, then $\langle \rho - 1\rangle^2 = \langle 2(\rho - 1)\rangle = \langle 8\rangle = \langle 4\rangle^2 = \langle 16\rangle$, a contradiction.
Case II: If $\langle \rho - 1\rangle = \langle \mathrm{x} + 4\lambda\rangle$, relations $\kappa = \rho\kappa$ and $2\kappa = 0$ imply $\kappa \in \mathrm{ann}_{\mathrm{R}}(\rho - 1) \cap \mathrm{ann}_{\mathrm{R}}(2) = \mathrm{ann}_{\mathrm{R}}(\mathrm{x} + 4\lambda) \cap \mathrm{ann}_{\mathrm{R}}(2) = \mathrm{ann}_{\mathrm{R}}(\mathrm{x}, 2) = \mathfrak{m}^3 \subseteq \langle \rho - 1\rangle$, which is not possible.

3 The identification between a local ring and the DNA alphabet

In this section we give conditions for the existence of a compatible function between a finite local ring of odd length and k-tuples of nucleotides. These compatible functions exist for some rings in the family $\mathfrak{F}_5^4$.

Definition 2. *Let k be an integer,* R *be a ring,* $\rho \in \mathrm{R}^*$ *and* $\kappa \in \mathrm{R}$.
(1) $\tilde{\rho}$ *denotes the permutation* $\mathrm{R} \mapsto \mathrm{R}$ *given by* $\tilde{\rho}(a) = \rho a$*;*
(2) $\tilde{\kappa}$ *denotes the permutation* $\mathrm{R} \mapsto \mathrm{R}$ *given by* $\tilde{\kappa}(a) = a + \kappa$*;*
(3) $r : \{A, C, G, T\}^k \to \{A, C, G, T\}^k$ *denotes the reverse operation,*

$$(c_1, \ldots, c_k) \mapsto (c_k, \ldots, c_1);$$

(4) $^- : \{A, C, G, T\}^k \to \{A, C, G, T\}^k$ *denotes the complement operation,*

$$(c_1, \ldots, c_k) \mapsto \overline{(c_1, \ldots, c_k)} = (\bar{c}_1, \ldots, \bar{c}_k), \overline{A} = T, \overline{T} = A, \overline{C} = G, \overline{G} = C.$$

(5) A bijection $\Phi : \mathrm{R} \to \{A, C, G, T\}^k$ *is called compatible if there are* $\rho, \kappa \in \mathrm{R}$ *such that the following diagrams are commutative.*

$$\begin{array}{ccc} \mathrm{R} & \xrightarrow{\Phi} & \{A,C,G,T\}^k \\ \downarrow{\scriptstyle \tilde{\kappa}} & & \downarrow{\scriptstyle -} \\ \mathrm{R} & \xrightarrow{\Phi} & \{A,C,G,T\}^k \end{array} \qquad \begin{array}{ccc} \mathrm{R} & \xrightarrow{\Phi} & \{A,C,G,T\}^k \\ \downarrow{\scriptstyle \tilde{\rho}} & & \downarrow{\scriptstyle r} \\ \mathrm{R} & \xrightarrow{\Phi} & \{A,C,G,T\}^k \end{array}$$

Let $(\mathrm{R}, \mathfrak{m}, \mathrm{GF}(2^d))$ is a finite local ring and $\Phi : \mathrm{R} \to \{A, T, C, G\}^k$ a compatible bijection, for $\rho, \kappa \in \mathrm{R}$. Relation $|\mathrm{M}| = |\mathrm{GF}(2^d)|^{\ell_{\mathrm{R}}(\mathrm{M})}$, where M is a module over R implies $|\mathrm{R}| = 2^{d\ell_{\mathrm{R}}(\mathrm{R})} = |\{A, C, G, T\}^k| = 4^k$ and $k = \frac{d\ell_{\mathrm{R}}(\mathrm{R})}{2}$.

The following two Lemmas and the remark are from [3].

Lemma 6. *Let* Ω *be a non-empty set,* $\omega \in \Omega$ *and* ψ, ϕ *two permutations of* Ω *such that* $\psi^2 = \phi^2 = id_\Omega$ *and* $\psi \circ \phi = \phi \circ \psi$. *Let* $[\omega]_\phi^\psi = \{\omega, \psi(\omega), \phi(\omega), \psi\phi(\omega)\}$, *then* $\{[\omega]_\phi^\psi : \omega \in \Omega\}$ *is a partition of* Ω.

Let Ω, ψ, ϕ and $[\omega]_\phi^\psi$ be as in Lemma 6. The equivalence relation corresponding to the classes $[\omega]_\phi^\psi$ is denoted by $\mathfrak{O}_\phi^\psi$. The following subsets of Ω are useful to describe $\mathfrak{O}_\phi^\psi$:
(1) $\Lambda_\phi = \{\omega \in \Omega : \phi(\omega) = \omega\}$, (2) $\Lambda_\phi^\psi = \{\omega \in \Omega : \phi(\omega) = \psi(\omega)\}$,
(3) $\Lambda_\Omega = \Omega \setminus (\Lambda_\phi \cup \Lambda_\phi^\psi)$.

Observation 2. *Let $k \geq 2$ an integer, R a ring, $\rho \in \mathrm{R}^* \setminus \{1\}$ and $\kappa \in \mathrm{R} \setminus \{0\}$. Suppose $2\kappa = 0$, $\rho^2 = 1$ and $\kappa = \rho\kappa$.*
(1) *If in Lemma 6, we take $\Omega = \mathrm{R}$, $\psi = \tilde{\kappa}$ and $\phi = \tilde{\rho}$, then:*

(a) *$\tilde{\kappa}(a) \neq a$, for all $a \in \mathrm{R}$, $\tilde{\rho} \neq id_{\mathrm{R}}$, $\tilde{\kappa}$ and $\tilde{\rho}$ have order two and $\tilde{\kappa} \circ \tilde{\rho} = \tilde{\rho} \circ \tilde{\kappa}$;*

(b) $$\Lambda_{\tilde{\rho}} = \mathrm{ann}_{\mathrm{R}}(\rho - 1) \quad \textit{and} \quad \Lambda_{\tilde{\rho}}^{\tilde{\kappa}} = \begin{cases} a_1 + \mathrm{ann}_{\mathrm{R}}(\rho - 1) & \textit{if } (\rho - 1)a_1 = \kappa \\ \emptyset & \textit{if } \kappa \notin \langle \rho - 1 \rangle. \end{cases}$$

(2) *If in Lemma 6, we take $\Omega = \{A, C, G, T\}^k$, $\psi = {}^-$ and $\phi = r$, then:*

(a) *$\bar{\boldsymbol{c}} \neq \boldsymbol{c}$, for all $\boldsymbol{c} \in \{A, C, G, T\}^k$, $r \neq id_{\{A,C,G,T\}^k}$, r and ${}^-$ have order two and $r \circ {}^- = {}^- \circ r$;*

(b) $$\Lambda_r = \begin{cases} (c_1, \ldots, c_{\frac{k}{2}}, c_{\frac{k}{2}}, \ldots, c_1) & \textit{if } k \textit{ is even} \\ (c_1, \ldots, c_{\frac{k-1}{2}}, c_{\frac{k+1}{2}}, c_{\frac{k-1}{2}}, \ldots, c_1) & \textit{if } k \textit{ is odd}, \end{cases}$$
$$\Lambda_r^- = \begin{cases} (c_1, \ldots, c_{\frac{k}{2}}, \bar{c}_{\frac{k}{2}}, \ldots, \bar{c}_1) & \textit{if } k \textit{ is even} \\ \emptyset & \textit{if } k \textit{ is odd}, \end{cases}$$
$$|\Lambda_r| = \begin{cases} 2^k & \textit{if } k \textit{ is even} \\ 2^{k+1} & \textit{if } k \textit{ is odd} \end{cases} \quad \textit{and} \quad |\Lambda_r^-| = \begin{cases} 2^k & \textit{if } k \textit{ is even} \\ 0 & \textit{if } k \textit{ is odd}. \end{cases}$$

Lemma 7. *Let Ω, ψ, ϕ and $[\omega]_\phi^\psi$ be as in Lemma 6. Suppose $\psi(\omega) \neq \omega$, $\forall \omega \in \Omega$, and let $\Lambda_\phi = \{\omega \in \Omega : \phi(\omega) = \omega\}$, $\Lambda_\phi^\psi = \{\omega \in \Omega : \phi(\omega) = \psi(\omega)\}$ and $\Lambda_\Omega = \Omega \setminus (\Lambda_\phi \cup \Lambda_\phi^\psi)$, then:*

(1) *$|[\omega]_\phi^\psi| = 2 \Leftrightarrow [\omega]_\phi^\psi = \{\omega, \psi(\omega)\} \Leftrightarrow \omega \in \Lambda_\phi$ or $\omega \in \Lambda_\phi^\psi$.*

(2) *$|[\omega]_\phi^\psi| = 4 \Leftrightarrow \omega \in \Lambda_\Omega$.*

(3) *If $\omega \in \Lambda_\phi$ ($\omega \in \Lambda_\phi^\psi$), then $[\omega]_\phi^\psi \subseteq \Lambda_\phi$ ($[\omega]_\phi^\psi \subseteq \Lambda_\phi^\psi$).*
In particular $|\{[\omega]_\phi^\psi : \omega \in \Lambda_\phi\}| = \frac{|\Lambda_\phi|}{2}$, $|\{[\omega]_\phi^\psi : \omega \in \Lambda_\phi^\psi\}| = \frac{|\Lambda_\phi^\psi|}{2}$ and $|\{[\omega]_\phi^\psi : \omega \in \Lambda_\Omega\}| = \frac{|\Omega| - |\Lambda_\phi| - |\Lambda_\phi^\psi|}{4}$.

(4) *The number of equivalence classes with* 2 *elements is* $\frac{|\Lambda_\phi|+|\Lambda_\phi^\psi|}{2}$.
The number of equivalence classes with 4 *elements is* $\frac{|\Omega|-|\Lambda_\phi|-|\Lambda_\phi^\psi|}{4}$.
The number of equivalence classes is $\frac{|\Omega|+|\Lambda_\phi|+|\Lambda_\phi^\psi|}{4}$.

Let $\mathrm{R} = \mathrm{GF}(2^d)[\mathrm{X}, \mathrm{Y}]/\langle \mathrm{X}^2 - \mathrm{Y}^3, \mathrm{XY}\rangle$ the ring (1) in Proposition 1, $\rho = 1 + \mathrm{y}^2$ and $\kappa = \mathrm{x} \in \mathrm{R}$. R has residue field $\mathrm{GF}(2^d)$, every element of R can be written uniquely in the form $a_0 + a_1\mathrm{x} + a_2\mathrm{y} + a_3\mathrm{y}^2 + a_4\mathrm{y}^3$, where $a_i \in \mathrm{GF}(2^d)$. By Lemma 3, $2\kappa = 0$, $\rho^2 = 1$, $\kappa = \rho\kappa$, $\kappa \notin \langle \rho - 1\rangle$ and $\ell_{\mathrm{R}}(\mathrm{ann}_{\mathrm{R}}(\rho - 1)) = 3$.

Example 1. *Let* $\mathrm{R} = \mathrm{GF}(2^2)[\mathrm{X}, \mathrm{Y}]/\langle \mathrm{X}^2-\mathrm{Y}^3, \mathrm{XY}\rangle$, $\rho = 1+\mathrm{y}^2$ *and* $\kappa = \mathrm{x} \in \mathrm{R}$.

(1) $\Lambda_{\tilde\rho} = \mathrm{ann}_{\mathrm{R}}(\rho - 1) = \langle \mathrm{x}, \mathrm{y}^2\rangle = \{a_1\mathrm{x} + a_2\mathrm{y}^2 + a_3\mathrm{y}^3 : a_i \in \mathrm{GF}(2^2)\}$,
$|\{[a]_{\tilde\rho}^{\tilde\kappa} : a \in \Lambda_{\tilde\rho}\}| = \frac{|\Lambda_{\tilde\rho}|}{2} = 32$ *and*
$[a_1\mathrm{x} + a_2\mathrm{y}^2 + a_3\mathrm{y}^3]_{\tilde\rho}^{\tilde\kappa} = \{a_1\mathrm{x} + a_2\mathrm{y}^2 + a_3\mathrm{y}^3, (a_1 + 1)\mathrm{x} + a_2\mathrm{y}^2 + a_3\mathrm{y}^3\}$.

(2) $\Lambda_{\mathrm{R}} = \{a_0 + a_1\mathrm{x} + a_2\mathrm{y} + a_3\mathrm{y}^2 + a_4\mathrm{y}^3 : a_i \in \mathrm{GF}(2^2), a_0 \neq 0$ *or* $a_2 \neq 0\}$,
$|\{[a]_{\tilde\rho}^{\tilde\kappa} : a \in \Lambda_R\}| = \frac{|\mathrm{R}|-|\Lambda_{\tilde\rho}|}{4} = 4^4 - 4^2 = 240$ *and*
$[a_0+a_1\mathrm{x}+a_2\mathrm{y}+a_3\mathrm{y}^2+a_4\mathrm{y}^3]_{\tilde\rho}^{\tilde\kappa} = \{a_0+a_1\mathrm{x}+a_2\mathrm{y}+a_3\mathrm{y}^2+a_4\mathrm{y}^3, a_0+(a_1+1)\mathrm{x} + a_2\mathrm{y} + a_3\mathrm{y}^2 + a_4\mathrm{y}^3, a_0 + a_1\mathrm{x} + a_2\mathrm{y} + (a_0 + a_3)\mathrm{y}^2 + (a_2 + a_4)\mathrm{y}^3, a_0 + (a_1 + 1)\mathrm{x} + a_2\mathrm{y} + (a_0 + a_3)\mathrm{y}^2 + (a_2 + a_4)\mathrm{y}^3\}$.

Example 2. *Let* $\{A, C, G, T\}^5$, *we have*

(1) $\Lambda_r = \{(c_1, c_2, c_3, c_4, c_5) : c_1 = c_5, c_2 = c_4\}$,
$|\{[\boldsymbol{c}]_r^- : \boldsymbol{c} \in \Lambda_r\}| = \frac{|\Lambda_r|}{2} = 32$ *and*
$[(c_1, c_2, c_3, c_2, c_1)]_{\tilde\rho}^{\tilde\kappa} = \{(c_1, c_2, c_3, c_2, c_1), (\overline{c_1}, \overline{c_2}, \overline{c_3}, \overline{c_2}, \overline{c_1})\}$.

(2) $\Lambda_{\{A,C,G,T\}^5} = \{(c_1, c_2, c_3, c_4, c_5) : c_1 \neq c_5$ *or* $c_2 \neq c_4\}$,
$|\{[\boldsymbol{c}]_r^- : \boldsymbol{c} \in \Lambda_{\{A,C,G,T\}^5}\}| = \frac{|\{A,C,G,T\}^5|-|\Lambda_{\{A,C,G,T\}^5}|}{4} = 240$ *and*
$[(c_1, c_2, c_3, c_4, c_5)]_{\tilde\rho}^{\tilde\kappa} = \{(c_1, c_2, c_3, c_4, c_5), (\overline{c_1}, \overline{c_2}, \overline{c_3}, \overline{c_4}, \overline{c_5}), (c_5, c_4, c_3, c_2, c_1), (\overline{c_5}, \overline{c_4}, \overline{c_3}, \overline{c_2}, \overline{c_1})\}$.

The following Proposition is the main result of this section.

Proposition 2. *Let* $(\mathrm{R}, \mathfrak{m}, \mathrm{GF}(2^d))$ *be a finite local ring. There is a compatible function* $\Phi : \mathrm{R} \to \{A, C, G, T\}^{\frac{d\ell_{\mathrm{R}}(\mathrm{R})}{2}}$, *for some* $\rho, \kappa \in \mathrm{R}$, *if and only if there exist* $\rho, \kappa \in \mathrm{R}$, *with* $\kappa \neq 0$, $2\kappa = 0$, $\rho^2 = 1$, $\kappa = \rho\kappa$ *and one of the following relations is satisfied:*

(1) R *is one of the following rings:* $\mathrm{GF}(4)$, $\mathbb{Z}_4$, $\mathrm{GF}(2)[\mathrm{T}]/\langle \mathrm{T}^2\rangle$;

(2) $\ell_{\mathrm{R}}(\mathrm{R})$ *is odd,* $\ell_{\mathrm{R}}(\mathrm{R}) \geq 3$, $d = 2$, $\kappa \notin \langle \rho - 1\rangle$ *and* $\ell_{\mathrm{R}}(\mathrm{ann}_{\mathrm{R}}(\rho - 1)) = \frac{\ell_{\mathrm{R}}(\mathrm{R})+1}{2}$;

(*3*) $\ell_{\mathrm{R}}(\mathrm{R})$ *is even,* $4 \not| \ell_{\mathrm{R}}(\mathrm{R})$, $d = 1$, $\kappa \notin \langle \rho - 1\rangle$ *and* $\ell_{\mathrm{R}}(\mathrm{ann}_{\mathrm{R}}(\rho - 1)) = \frac{\ell_{\mathrm{R}}(\mathrm{R})}{2} + 1$*;*

(*4*) $\ell_{\mathrm{R}}(\mathrm{R})$ *is even,* $4|d\ell_{\mathrm{R}}(\mathrm{R})$, $\kappa \in \langle \rho - 1\rangle$ *and* $\ell_{\mathrm{R}}(\mathrm{ann}_{\mathrm{R}}(\rho - 1)) = \frac{\ell_{\mathrm{R}}(\mathrm{R})}{2}$.

Proof: We consider the classes $[a]_{\tilde{\rho}}^{\tilde{\kappa}} = \{a, a+\kappa, \rho a, \rho a + \kappa\}$ and $[\boldsymbol{c}]_r^- = \{\boldsymbol{c}$, $\bar{\boldsymbol{c}}$, $r(\boldsymbol{c})$, $\overline{r(\boldsymbol{c})}\}$, in R and $\{A, C, G, T\}^k$, respectively.
$\Rightarrow$) For the first relations we have:
(*a*) $\Phi(a+\kappa) = \overline{\Phi(a)} \neq \Phi(a)$, then $\kappa \neq 0$;
(*b*) $\Phi(2\kappa) = \overline{\Phi(\kappa)} = \overline{\overline{\Phi(0)}} = \Phi(0)$, then $2\kappa = 0$;
(*c*) $\Phi(\rho^2) = r(\Phi(\rho)) = r[r(\Phi(1))] = \Phi(1)$, then $\rho^2 = 1$;
(*d*) $\Phi(\rho\kappa) = r(\Phi(\kappa)) = r(\overline{\Phi(0)}) = \overline{r\Phi(0)} = \overline{\Phi(\rho 0)} = \overline{\Phi(0)} = \Phi(\kappa)$, then $\rho\kappa = \kappa$.
Concerning the number $\frac{d\ell_{\mathrm{R}}(\mathrm{R})}{2}$, the possibilities are $\frac{d\ell_{\mathrm{R}}(\mathrm{R})}{2} = 1$, $\frac{d\ell_{\mathrm{R}}(\mathrm{R})}{2} \geq 3$ is odd and $\frac{d\ell_{\mathrm{R}}(\mathrm{R})}{2}$ is even. Each of one of these cases is treated in the following lines. Observe that Φ maps bijectively $\Lambda_{\tilde{\rho}}$ into Λ_r.

(1) $\frac{d\ell_{\mathrm{R}}(\mathrm{R})}{2} = 1 \Leftrightarrow \Phi(a) = r(\Phi(a)) = \Phi(\rho a)$, $\forall\, a \in \mathrm{R} \Leftrightarrow \rho = 1 \Leftrightarrow$
$d = 2$ and $\ell_{\mathrm{R}}(\mathrm{R}) = 1$ or $d = 1$ and $\ell_{\mathrm{R}}(\mathrm{R}) = 2 \Leftrightarrow$
$\mathrm{R} \cong \mathrm{GF}(4)$ or $\mathrm{R} \cong \mathbb{Z}_4$ or $\mathrm{R} \cong \mathrm{GF}(2)[\mathrm{T}]/\langle \mathrm{T}^2\rangle$.

(2) $\frac{d\ell_{\mathrm{R}}(\mathrm{R})}{2}$ is odd and $\frac{d\ell_{\mathrm{R}}(\mathrm{R})}{2} \geq 3 \Leftrightarrow r(\boldsymbol{c}) \neq \bar{\boldsymbol{c}}$, $\forall \boldsymbol{c} \in \{A, T, C, G\}^{\frac{d\ell_{\mathrm{R}}(\mathrm{R})}{2}} \Leftrightarrow$
$\Phi(\rho a) \neq \Phi(a+\kappa)$, $\forall a \in \mathrm{R} \Leftrightarrow \kappa \notin \langle \rho - 1\rangle$.
Then $|\Lambda_{\tilde{\rho}}| = |\Lambda_r| = |\mathrm{ann}_{\mathrm{R}}(\rho - 1)| = 2^{d\ell_{\mathrm{R}}(\mathrm{ann}_{\mathrm{R}}(\rho-1))} = 2^{\frac{d\ell_{\mathrm{R}}(\mathrm{R})}{2}+1}$ and

$$\ell_{\mathrm{R}}(\mathrm{ann}_{\mathrm{R}}(\rho - 1)) = \frac{\ell_{\mathrm{R}}(\mathrm{R})}{2} + \frac{1}{d}.$$

If $\ell_{\mathrm{R}}(\mathrm{R})$ is odd, then $d = 2$ and $\ell_{\mathrm{R}}(\mathrm{ann}_{\mathrm{R}}(\rho - 1)) = \frac{\ell_{\mathrm{R}}(\mathrm{R})+1}{2}$.
If $\ell_{\mathrm{R}}(\mathrm{R})$ is even, then $d = 1$, $4 \not| \ell_{\mathrm{R}}(\mathrm{R})$ and $\ell_{\mathrm{R}}(\mathrm{ann}_{\mathrm{R}}(\rho - 1)) = \frac{\ell_{\mathrm{R}}(\mathrm{R})}{2} + 1$.

(3) $\frac{d\ell_{\mathrm{R}}(\mathrm{R})}{2}$ is even $\Leftrightarrow$ there exists $\boldsymbol{c} \in \{A, T, C, G\}^{\frac{d\ell_{\mathrm{R}}(\mathrm{R})}{2}}$ with $r(\boldsymbol{c}) = \bar{\boldsymbol{c}} \Leftrightarrow$
there exists $a \in \mathrm{R}$ with $\Phi(\rho a) = \Phi(a+\kappa) \Leftrightarrow \kappa \in \langle \rho - 1\rangle$. Then $|\Lambda_{\tilde{\rho}}| = |\Lambda_r| = |\mathrm{ann}_{\mathrm{R}}(\rho - 1)| = 2^{d\ell_{\mathrm{R}}(\mathrm{ann}_{\mathrm{R}}(\rho-1))} = 2^{\frac{d\ell_{\mathrm{R}}(\mathrm{R})}{2}}$, $\ell_{\mathrm{R}}(\mathrm{ann}_{\mathrm{R}}(\rho - 1)) = \frac{\ell_{\mathrm{R}}(\mathrm{R})}{2}$ and $\ell_{\mathrm{R}}(\mathrm{R})$ is even.

$\Leftarrow$) Case (1) was treated in [1], [11] and [17].
Case (2). Suppose $\ell_{\mathrm{R}}(\mathrm{R})$ is odd, $\ell_{\mathrm{R}}(\mathrm{R}) \geq 3$, $d = 2$, $\rho, \kappa \in \mathrm{R}$, with $2\kappa = 0$, $\rho^2 = 1$, $\kappa = \rho\kappa$, $\kappa \notin \langle \rho - 1\rangle$ and $\ell_{\mathrm{R}}(\mathrm{ann}_{\mathrm{R}}(\rho - 1)) = \frac{\ell_{\mathrm{R}}(\mathrm{R})+1}{2}$. By Remark 2,

$|\Lambda_{\tilde{\rho}}^{\tilde{\kappa}}| = |\Lambda_r^{-}| = 0$ and let

$$l_1 = |\{[a]_{\tilde{\rho}}^{\tilde{\kappa}} : a \in \Lambda_{\tilde{\rho}}\}| = |\{[\boldsymbol{c}]_r^{-} : \boldsymbol{c} \in \Lambda_r\}| = 2^{\ell_{\mathrm{R}}(\mathrm{R})},$$
$$l_2 = |\{[a]_{\tilde{\rho}}^{\tilde{\kappa}} : a \in \Lambda_R\}| = |\{[\boldsymbol{c}]_r^{-} : \boldsymbol{c} \in \Lambda_{\{A,C,G,T\}^{\ell_{\mathrm{R}}(\mathrm{R})}}\}| = \frac{4^{\ell_{\mathrm{R}}(\mathrm{R})} - 2^{\ell_{\mathrm{R}}(\mathrm{R})+1}}{4},$$

$M = \{f_1, \ldots, f_{l_1} | f_i \in \Lambda_{\tilde{\rho}}\} \cup \{h_1, \ldots, h_{l_2} | h_i \in \Lambda_R\}$ be a set of fixed representatives of the classes $[a]_{\tilde{\rho}}^{\tilde{\kappa}}$ and $N = \{\boldsymbol{f}_1, \ldots, \boldsymbol{f}_{l_1} | \boldsymbol{f}_i \in \Lambda_r\} \cup \{\boldsymbol{h}_1, \ldots, \boldsymbol{h}_{l_2} | \boldsymbol{h}_i \in \Lambda_{\{A,C,G,T\}^{\ell_{\mathrm{R}}(\mathrm{R})}}\}$ be a set of fixed representatives of the classes $[\boldsymbol{c}]_r^{-}$. Using Lemma 7, we define the bijection $\Phi : \mathrm{R} \to \{A, C, G, T\}^{\ell_{\mathrm{R}}(\mathrm{R})}$ in the following way:

(1) For $i \in \{1, \ldots, l_1\}$, $[f_i]_{\tilde{\rho}}^{\tilde{\kappa}} = \{f_i, f_i+\kappa\}$ and we put $\Phi(f_i) = \boldsymbol{f}_i$, $\Phi(f_i+\kappa) = \overline{\boldsymbol{f}}_i$;

(2) For $i \in \{1, \ldots, l_2\}$, $[h_i]_{\tilde{\rho}}^{\tilde{\kappa}} = \{h_i, h_i + \kappa, \rho h_i, \rho h_i + \kappa\}$ and we put $\Phi(h_i) = \boldsymbol{h}_i$,$\Phi(h_i + \kappa) = \overline{\boldsymbol{h}_i}$, $\Phi(\rho h_i) = r(\boldsymbol{h}_i)$, $\Phi(\rho h_i + \kappa) = \overline{r(\boldsymbol{h}_i)}$.

Φ satisfies the desired properties. We have the following cases

(1) For $i \in \{1, \ldots, l_1\}$, $[f_i]_{\tilde{\rho}}^{\tilde{\kappa}} = \{f_i, f_i + \kappa\}$, $\rho f_i = f_i$, $\rho(f_i + \kappa) = f_i + \kappa$, $\boldsymbol{f}_i \in \Lambda_r$, $r(\boldsymbol{f}_i) = \boldsymbol{f}_i$, $\overline{r(\boldsymbol{f}_i)} = \overline{\boldsymbol{f}}_i$ and:
   (a) $\overline{\Phi(f_i)} = \overline{\boldsymbol{f}}_i = \Phi(f_i + \kappa)$,
   (b) $r(\Phi(f_i)) = r(\boldsymbol{f}_i) = \boldsymbol{f}_i = \Phi(f_i) = \Phi(\rho f_i)$,
   (c) $\overline{\Phi(f_i + \kappa)} = \overline{\overline{\boldsymbol{f}}}_i = \boldsymbol{f}_i = \Phi(f_i) = \Phi(f_i + 2\kappa)$,
   (d) $r(\Phi(f_i + \kappa)) = r(\overline{\boldsymbol{f}}_i) = \overline{\boldsymbol{f}}_i = \Phi(f_i + \kappa) = \Phi(\rho(f_i + \kappa))$.

(2) For $i \in \{1, \ldots, l_2\}$, $[h_i]_{\tilde{\rho}}^{\tilde{\kappa}} = \{h_i, h_i + \kappa, \rho h_i, \rho h_i + \kappa\}$, $[\boldsymbol{h}_i]_r^{-} = \{\boldsymbol{h}_i, \overline{\boldsymbol{h}_i}, r(\boldsymbol{h}_i), \overline{r(\boldsymbol{h}_i)}\}$ and:
   (a) $\overline{\Phi(h_i + \kappa)} = \overline{\overline{\boldsymbol{h}_i}} = \boldsymbol{h}_i = \Phi(h_i) = \Phi(h_i + 2\kappa)$,
   (b) $r(\Phi(h_i + \kappa)) = r(\overline{\boldsymbol{h}_i}) = \overline{r(\boldsymbol{h}_i)} = \Phi(\rho(h_i + \kappa))$,
   (c) $\overline{\Phi(\rho h_i)} = \overline{r(\boldsymbol{h}_i)} = \Phi(\rho h_i + \kappa)$,
   (d) $r(\Phi(\rho h_i)) = r(r(\boldsymbol{h}_i)) = \boldsymbol{h}_i = \Phi(h_i) = \Phi(\rho^2 h_i)$,
   (e) $\overline{\Phi(\rho h_i + \kappa)} = \overline{\overline{r(\boldsymbol{h}_i)}} = r(\boldsymbol{h}_i) = \Phi(\rho h_i) = \Phi(\rho h_i + 2\kappa)$,
   (f) $r(\Phi(\rho h_i + \kappa)) = r(\overline{r(\boldsymbol{h}_i)}) = \overline{\boldsymbol{h}_i} = \Phi(h_i + \kappa) = \Phi(\rho(\rho h_i + \kappa))$.

Case (3) was treated in [3]. Case (4) is treated in a similar way as case (2) and details are omitted.

The following is a particular case of Proposition 2, when the ring R is a chain ring. Case (1) follows from Proposition 2 and case (2) was treated in [3].

Corollary 1. *Let* $(\mathrm{R}, \langle\pi\rangle, \mathrm{GF}(2^d))$ *be a finite chain ring. There is a compatible function* $\Phi : \mathrm{R} \to \{A, C, G, T\}^{\frac{d\ell_{\mathrm{R}}(\mathrm{R})}{2}}$*, for some* $\rho, \kappa \in \mathrm{R}$*, if and only if* $\mathrm{char}(\mathrm{R}) \in \{2, 4\}$ *and one of the following relations is satisfied:*

(*1*) R *is one of the following rings:* $\mathrm{GF}(4)$, $\mathbb{Z}_4$, $\mathrm{GF}(2)[\mathrm{T}]/\langle \mathrm{T}^2\rangle$*;*

(*2*) $\ell_{\mathrm{R}}(\mathrm{R})$ *is even and* $4 | d\ell_{\mathrm{R}}(\mathrm{R})$*;*

(*3*) $\ell_{\mathrm{R}}(\mathrm{R})$ *is even,* $4 \not| \ell_{\mathrm{R}}(\mathrm{R})$*,* $d = 1$*, and if* $\mathrm{char}(\mathrm{R}) = 4$ *and* $\kappa = v_1\pi^{\frac{\ell_{\mathrm{R}}(\mathrm{R})}{2}-1}$*,* $v_1 \in \mathrm{R}^*$*, then* $2 \in \langle \pi^{\frac{\ell_{\mathrm{R}}(\mathrm{R})}{2}+1}\rangle$*;*

(*4*) $\ell_{\mathrm{R}}(\mathrm{R})$ *is odd,* $\ell_{\mathrm{R}}(\mathrm{R}) \geq 3$ *and* $d = 2$*.*

Proof: The assertion follows from Proposition 2 and the following relations: Case (3)

(i) $\ell_{\mathrm{R}}(\mathrm{ann}_{\mathrm{R}}(\rho - 1)) = \frac{\ell_{\mathrm{R}}(\mathrm{R})}{2} + 1 \Leftrightarrow \ell_{\mathrm{R}}(\rho - 1) = \frac{\ell_{\mathrm{R}}(\mathrm{R})}{2} - 1 \Leftrightarrow$ $\langle \rho - 1\rangle = \langle \pi^{\frac{\ell_{\mathrm{R}}(\mathrm{R})}{2}+1}\rangle \Leftrightarrow \rho = 1 + u\pi^{\frac{\ell_{\mathrm{R}}(\mathrm{R})}{2}+1}$, $u \in \mathrm{R}^*$.

(ii) $\kappa \notin \langle \rho - 1\rangle$ and $\rho\kappa = \kappa \Leftrightarrow \kappa \notin \langle \pi^{\frac{\ell_{\mathrm{R}}(\mathrm{R})}{2}+1}\rangle$ and $\kappa \in \mathrm{ann}_{\mathrm{R}}(\rho - 1) =$ $\langle \pi^{\frac{\ell_{\mathrm{R}}(\mathrm{R})}{2}-1}\rangle \Leftrightarrow \kappa = v_1\pi^{\frac{\ell_{\mathrm{R}}(\mathrm{R})}{2}-1}$ or $\kappa = v_2\pi^{\frac{\ell_{\mathrm{R}}(\mathrm{R})}{2}}$, $v_1, v_2 \in \mathrm{R}^*$.

(iii) $2\kappa = 0 \Leftrightarrow 2 \in \mathrm{ann}_{\mathrm{R}}(\kappa) = \begin{cases} \langle \pi^{\frac{\ell_{\mathrm{R}}(\mathrm{R})}{2}+1}\rangle & if\ \kappa = v_1\pi^{\frac{\ell_{\mathrm{R}}(\mathrm{R})}{2}-1} \\ \langle \pi^{\frac{\ell_{\mathrm{R}}(\mathrm{R})}{2}}\rangle & if\ \kappa = v_2\pi^{\frac{\ell_{\mathrm{R}}(\mathrm{R})}{2}} \end{cases} \Leftrightarrow$

$\mathrm{char}(\mathrm{R}) \in \{2, 4\}$, and if $\mathrm{char}(\mathrm{R}) = 4$ and $\kappa = v_1\pi^{\frac{\ell_{\mathrm{R}}(\mathrm{R})}{2}-1}$, then $2 \in \langle \pi^{\frac{\ell_{\mathrm{R}}(\mathrm{R})}{2}+1}\rangle$.

Case (4)

(i) $\ell_{\mathrm{R}}(\mathrm{ann}_{\mathrm{R}}(\rho - 1)) = \frac{\ell_{\mathrm{R}}(\mathrm{R})+1}{2} \Leftrightarrow \ell_{\mathrm{R}}(\rho - 1) = \frac{\ell_{\mathrm{R}}(\mathrm{R})-1}{2} \Leftrightarrow \langle \rho - 1\rangle = \langle \pi^{\frac{\ell_{\mathrm{R}}(\mathrm{R})+1}{2}}\rangle$ $\Leftrightarrow \rho = 1 + u\pi^{\frac{\ell_{\mathrm{R}}(\mathrm{R})+1}{2}}$, $u \in \mathrm{R}^*$.

(ii) $\kappa \notin \langle \rho - 1\rangle$ and $\rho\kappa = \kappa \Leftrightarrow \kappa \notin \langle \pi^{\frac{\ell_{\mathrm{R}}(\mathrm{R})+1}{2}}\rangle$ and $\kappa \in \mathrm{ann}_{\mathrm{R}}(\rho - 1) =$ $\langle \pi^{\frac{\ell_{\mathrm{R}}(\mathrm{R})-1}{2}}\rangle \Leftrightarrow \kappa = v\pi^{\frac{\ell_{\mathrm{R}}(\mathrm{R})-1}{2}}$.

(iii) $2\kappa = 0 \Leftrightarrow 2 \in \mathrm{ann}_{\mathrm{R}}(\pi^{\frac{\ell_{\mathrm{R}}(\mathrm{R})-1}{2}}) = \langle \pi^{\frac{\ell_{\mathrm{R}}(\mathrm{R})+1}{2}}\rangle \Leftrightarrow \mathrm{char}(\mathrm{R}) \in \{2, 4\}$.

The following is a particular case of Proposition 2, when the ring is in the family $\mathfrak{F}_5^4$.

Corollary 2. *Let* $(\mathrm{R}, \mathfrak{m}, \mathrm{GF}(2^d)) \in \mathfrak{F}_5^4$. *There is a compatible function* $\Phi : \mathrm{R} \to \{A, C, G, T\}^{\frac{d\ell_{\mathrm{R}}(\mathrm{R})}{2}}$, *for some* $\rho, \kappa \in \mathrm{R}$, *if and only if* R *is one of the following rings*

(*1*) $\mathrm{GF}(2^2)[\mathrm{X}, \mathrm{Y}]/\langle \mathrm{X}^2 - \mathrm{Y}^3, \mathrm{XY}\rangle$,

(*2*) $\mathrm{GR}(2^2, 2)[\mathrm{X}, \mathrm{Y}]/\langle \mathrm{Y}^2 - 2, \mathrm{X}^2 - \mathrm{Y}^3, \mathrm{XY}\rangle$,

(*3*) $\mathrm{GR}(2^2, 2)[\mathrm{X}, \mathrm{Y}]/\langle \mathrm{Y}^3 - 2, \mathrm{X}^2 - \mathrm{Y}^3, \mathrm{XY}\rangle$,

(*4*) $\mathrm{GR}(2^2, 2)[\mathrm{X}, \mathrm{Y}]/\langle \zeta\mathrm{Y}^3 - 2, \mathrm{X}^2 - \mathrm{Y}^3, \mathrm{XY}\rangle$,

(*5*) $\mathrm{GR}(2^2, 2)[\mathrm{X}, \mathrm{Y}]/\langle \zeta^2\mathrm{Y}^3 - 2, \mathrm{X}^2 - \mathrm{Y}^3, \mathrm{XY}\rangle$,

(*6*) $\mathrm{GR}(2^3, 2)[\mathrm{X}]/\langle 2^2 - \mathrm{X}^3, 2\mathrm{X}\rangle$,

(*7*) $\mathrm{GR}(2^3, 2)[\mathrm{X}]/\langle 2^2 - \zeta\mathrm{X}^3, 2\mathrm{X}\rangle$,

(*8*) $\mathrm{GR}(2^3, 2)[\mathrm{X}]/\langle 2^2 - \zeta^2\mathrm{X}^3, 2\mathrm{X}\rangle$.

Proof: The assertion follows from Proposition 2 and Lemmas 3, 4 and 5.

Example 3. *Let* $\mathrm{R} = \mathrm{GF}(2^2)[\mathrm{X}, \mathrm{Y}]/\langle \mathrm{X}^2 - \mathrm{Y}^3, \mathrm{XY}\rangle$, $\rho = 1 + \mathrm{y}^2$, $\kappa = \mathrm{x}$. *Using notation as in proof of Proposition 2 and by Examples 1, 2, let* $M = \{f_1, \ldots, f_{32}, h_1, \ldots, h_{240} | f_i \in \Lambda_{\tilde{\rho}}, h_i \in \Lambda_R\}$ *the set of fixed representatives of the classes* $[a]_{\tilde{\rho}}^{\tilde{\kappa}}$ *and* $N = \{\boldsymbol{f}_1, \ldots, \boldsymbol{f}_{32}, \boldsymbol{h}_1, \ldots, \boldsymbol{h}_{240} | \boldsymbol{f}_i \in \Lambda_r, \boldsymbol{h}_i \in \Lambda_{\{A,C,G,T\}^5}\}$ *the set of fixed representatives of the classes* $[\boldsymbol{c}]_r^{-}$.
Let $i \in \{1, \ldots, 32\}$, $f_i = a_1^i\mathrm{x} + a_2^i\mathrm{y}^2 + a_3^i\mathrm{y}^3$, $\boldsymbol{f}_i = (c_1^i, c_2^i, c_3^i, c_2^i, c_1^i)$, *the map* $\Phi : \mathrm{GF}(2^2)[\mathrm{X}, \mathrm{Y}]/\langle \mathrm{X}^2 - \mathrm{Y}^3, \mathrm{XY}\rangle \to \{A, C, G, T\}^5$ *is given by:*

$f \in [a_1^i\mathrm{x} + a_2^i\mathrm{y}^2 + a_3^i\mathrm{y}^3]_{\tilde{\rho}}^{\tilde{\kappa}}$	$\Phi(f)$
$a_1^i\mathrm{x} + a_2^i\mathrm{y}^2 + a_3^i\mathrm{y}^3$	$(c_1^i, c_2^i, c_3^i, c_2^i, c_1^i)$
$(a_1^i + 1)\mathrm{x} + a_2^i\mathrm{y}^2 + a_3^i\mathrm{y}^3$	$(\overline{c_1^i}, \overline{c_2^i}, \overline{c_3^i}, \overline{c_2^i}, \overline{c_1^i})$

Table 1: The map Φ

Let $j \in \{1, \ldots, 240\}$, $h_j = a_0^j + a_1^j\mathrm{x} + a_2^j\mathrm{y} + a_3^j\mathrm{y}^2 + a_4^j\mathrm{y}^3$, $a_0^j \neq 0$ *or* $a_2^j \neq 0$, *and* $\boldsymbol{h}_j = (c_1^j, c_2^j, c_3^j, c_4^j, c_5^j)$, $c_1 \neq c_5$ *or* $c_2 \neq c_4$, *the map* Φ *is given by*

$h \in [a_0^j + a_1^j\mathrm{x} + a_2^j\mathrm{y} + a_3^j\mathrm{y}^2 + a_4^j\mathrm{y}^3]_{\tilde{\rho}}^{\tilde{\kappa}}$	$\Phi(h)$
$a_0^j + a_1^j\mathrm{x} + a_2^j\mathrm{y} + a_3^j\mathrm{y}^2 + a_4^j\mathrm{y}^3$	$(c_1^j, c_2^j, c_3^j, c_4^j, c_5^j)$
$a_0^j + (a_1^j+1)\mathrm{x} + a_2^j\mathrm{y} + a_3^j\mathrm{y}^2 + a_4^j\mathrm{y}^3$	$(\overline{c_1^j}, \overline{c_2^j}, \overline{c_3^j}, \overline{c_4^j}, \overline{c_5^j})$
$a_0^j + a_1^j\mathrm{x} + a_2^j\mathrm{y} + (a_0^j + a_3^j)\mathrm{y}^2 + (a_2^j + a_4^j)\mathrm{y}^3$	$(c_5^j, c_4^j, c_3^j, c_2^j, c_1^j)$
$a_0^j + (a_1^j+1)\mathrm{x} + a_2^j\mathrm{y} + (a_0^j + a_3^j)\mathrm{y}^2 + (a_2^j + a_4^j)\mathrm{y}^3$	$(\overline{c_5^j}, \overline{c_4^j}, \overline{c_3^j}, \overline{c_2^j}, \overline{c_1^j})$

Table 2: The map Φ

4 DNA codes over rings in $\mathfrak{F}_5^4$

In the rest of this manuscript we only consider the following local rings in the family $\mathfrak{F}_5^4$, see Corollary 2:

(1) $\mathrm{GF}(2^2)[\mathrm{X},\mathrm{Y}]/\langle \mathrm{X}^2 - \mathrm{Y}^3, \mathrm{XY}\rangle$,

(2) $\mathrm{GR}(2^2,2)[\mathrm{X},\mathrm{Y}]/\langle \mathrm{Y}^2 - 2, \mathrm{X}^2 - \mathrm{Y}^3, \mathrm{XY}\rangle$,

(3) $\mathrm{GR}(2^2,2)[\mathrm{X},\mathrm{Y}]/\langle \mathrm{Y}^3 - 2, \mathrm{X}^2 - \mathrm{Y}^3, \mathrm{XY}\rangle$,

(4) $\mathrm{GR}(2^2,2)[\mathrm{X},\mathrm{Y}]/\langle \zeta\mathrm{Y}^3 - 2, \mathrm{X}^2 - \mathrm{Y}^3, \mathrm{XY}\rangle$,

(5) $\mathrm{GR}(2^2,2)[\mathrm{X},\mathrm{Y}]/\langle \zeta^2\mathrm{Y}^3 - 2, \mathrm{X}^2 - \mathrm{Y}^3, \mathrm{XY}\rangle$,

(6) $\mathrm{GR}(2^3,2)[\mathrm{X}]/\langle 2^2 - \mathrm{X}^3, 2\mathrm{X}\rangle$,

(7) $\mathrm{GR}(2^3,2)[\mathrm{X}]/\langle 2^2 - \zeta\mathrm{X}^3, 2\mathrm{X}\rangle$,

(8) $\mathrm{GR}(2^3,2)[\mathrm{X}]/\langle 2^2 - \zeta^2\mathrm{X}^3, 2\mathrm{X}\rangle$.

Let $(\mathrm{R}, \mathfrak{m}, \mathrm{GF}(2^2)) \in \mathfrak{F}_5^4$ and $\Phi : \mathrm{R} \longrightarrow \{A, C, G, T\}^5$ be a compatible function, for $\rho, \kappa \in \mathrm{R}$, Φ can be extended to R^n in the obvious way

$$\Phi(a_0, \ldots, a_{n-1}) = (\Phi(a_0), \ldots, \Phi(a_{n-1})).$$

A code over the nucleotides of length m is a subset of $\{A, C, T, G\}^m$. A code over the nucleotides of length m is complementary if it is invariant under the mapping $^- : \{A, C, T, G\}^m \to \{A, C, T, G\}^m$ given by $(a_1, \ldots, a_m) \mapsto (\bar{a}_1, \ldots, \bar{a}_m)$, where $\bar{A} = T$, $\bar{T} = A$, $\bar{C} = G$, $\bar{G} = C$. A linear cyclic code of length n over R is a submodule of R^n invariant under the permutation $\sigma : \mathrm{R}^n \to \mathrm{R}^n$ given by $(a_0, \ldots, a_{n-1}) \mapsto (a_{n-1}, a_0, \ldots, a_{n-2})$. These codes can be thought of as ideals in the quotient ring $\mathrm{R}[\mathrm{T}]/\langle \mathrm{T}^n - 1\rangle$ via the isomorphism from R^n to $\mathrm{R}[\mathrm{T}]/\langle \mathrm{T}^n - 1\rangle$ defined by $(a_0, \ldots, a_{n-1}) \mapsto a_0 + \ldots + a_{n-1}\mathrm{T}^{n-1} +$

$\langle \mathrm{T}^n - 1\rangle$, see [14]. Reversible codes are codes invariant under the mapping $r : \mathrm{R}^n \longrightarrow \mathrm{R}^n$ given by $(a_0, \ldots, a_{n-1}) \mapsto (a_{n-1}, \ldots, a_0)$. The linear code C of length n over R is called complementary if $\Phi(\mathrm{C})$ is complementary and is DNA if it is cyclic, reversible and complementary.

In this work all codes over R are considered linear and their lengths are relatively prime to the characteristic of the residue field of R and the following notation will be used:
(1) If $\mathrm{F}|\mathrm{T}^n - 1$, let $\widehat{\mathrm{F}} = \frac{\mathrm{T}^n-1}{\mathrm{F}}$,
(2) We will just write $a_0 + a_1\mathrm{T} + \ldots + a_{n-1}\mathrm{T}^{n-1}$ for the corresponding coset $a_0 + a_1\mathrm{T} + \ldots + a_{n-1}\mathrm{T}^{n-1} + \langle \mathrm{T}^n - 1\rangle$ in $\mathrm{R[T]}/\langle \mathrm{T}^n - 1\rangle$.
(3) $\mathrm{T}^n - 1 = \mathrm{f}_1 \cdots \mathrm{f}_r$ will be the unique representation of $\mathrm{T}^n - 1$ as a product of monic basic irreducible pairwise coprime polynomials in R[T].

From [6], the decomposition of a cyclic code over $(\mathrm{R}, \mathfrak{m}, \mathrm{GF}(q)) \in \mathfrak{F}_5^4$ can be found in the following way: Let $\mathrm{T}^n - 1 = \mathrm{f}_1 \cdots \mathrm{f}_r$ be as above, $s_i = \deg(\mathrm{f}_i)$ and $\mathrm{B}_i = \mathrm{R[T]}/\langle \mathrm{f}_i\rangle$. By the Chinese Remainder Theorem, $\mathrm{R[T]}/\langle \mathrm{T}^n-1\rangle \cong \oplus_{i=1}^r \mathrm{B}_i$, then any ideal I of $\mathrm{R[T]}/\langle \mathrm{T}^n-1\rangle$ is a direct sum of ideals of B_i. Since $\ell_{\mathrm{R}}(\mathrm{R}) = \ell_{\mathrm{B}_i}(\mathrm{B}_i) = 5$, then there is a partition of $[1, \ldots, r]$, $\mathrm{U}_0, \mathrm{U}_1, \mathrm{U}_2, \mathrm{U}_3, \mathrm{U}_4, \mathrm{U}_l$, such that $\mathrm{U}_i = \{u : \ell_{\mathrm{B}_u}(\mathrm{I}_u) = i\}$ and

$$\mathrm{I} = \bigoplus_{u\in \mathrm{U}_0} \mathrm{I}_u \oplus \bigoplus_{u\in \mathrm{U}_1} \mathrm{I}_u \oplus \bigoplus_{u\in \mathrm{U}_2} \mathrm{I}_u \oplus \bigoplus_{u\in \mathrm{U}_3} \mathrm{I}_u \oplus \bigoplus_{u\in \mathrm{U}_4} \mathrm{I}_u \oplus \bigoplus_{u\in \mathrm{U}_5} \mathrm{I}_u$$

The following results are on the structure of cyclic reversible codes over a ring in the family $\mathfrak{F}_5^4$, see [4] and [9].

Theorem 1. *Let* $(\mathrm{R}, \mathfrak{m}, \mathrm{GF}(q)) \in \mathfrak{F}_5^4$, $\tilde{\alpha} = \{\alpha_1, \alpha_2\}$, $\tilde{\beta} = \{\alpha_1, \alpha_2^2\}$, $\mathbb{T}$ *and* $\mathbb{T}_s$ *as above and* $\mathrm{f}_1, \ldots, \mathrm{f}_r$ *the unique monic basic irreducible pairwise coprime polynomials such that* $\mathrm{T}^n - 1 = \mathrm{f}_1 \cdots \mathrm{f}_r$, $s_i = \deg(\mathrm{f}_i)$. *Let* C *be a cyclic code of length* n *over* R. *Then*

(*1*) *There exists a unique partition of* $[1, r]$, $\mathrm{U}_0, \mathrm{U}_1, \mathrm{U}_2, \mathrm{U}_3, \mathrm{U}_4, \mathrm{U}_5$.

(*2*) *For each* $i \in \{2, 3\}$ *and each* $u \in \mathrm{U}_i$, *there is a unique* $\vec{\mathrm{v}}_u \in \{(0, 1), (1, \lambda) : \lambda \in \mathrm{GF}(q^s)\}$ *such that the corresponding ideal, in* $\mathrm{R[T]}/\langle \mathrm{T}^n - 1\rangle$, *of* C *is*

$$\langle \mathfrak{m}^3 \widehat{\prod_{u\in \mathrm{U}_1} \mathrm{f}_u}, \mathfrak{m} \widehat{\prod_{u\in \mathrm{U}_4} \mathrm{f}_u}, \widehat{\prod_{u\in \mathrm{U}_5} \mathrm{f}_u}, (\vec{\mathrm{v}}_u)_{\tilde{\beta}}^{\mathbb{T}_{s_u}} \widehat{\mathrm{f}}_u, (\vec{\mathrm{v}}_w)_{\tilde{\alpha}}^{\mathbb{T}_{s_w}} \widehat{\mathrm{f}}_w : u \in \mathrm{U}_2, w \in \mathrm{U}_3\rangle$$

and

$$|\mathrm{C}| = q^{5\sum_{u\in \mathrm{U}_5} s_u + 4\sum_{u\in \mathrm{U}_4} s_u + 3\sum_{\nu\in \mathrm{U}_3} s_u + 2\sum_{u\in \mathrm{U}_2} s_u + \sum_{u\in \mathrm{U}_1} s_u}.$$

Theorem 2. *Let* $(\mathrm{R}, \mathfrak{m}, \mathrm{GF}(q)) \in \mathfrak{F}_5^4$, $\tilde{\alpha} = \{\alpha_1, \alpha_2\}$, $\tilde{\beta} = \{\alpha_1, \alpha_2^2\}$, $\mathbb{T}$, $\mathbb{T}_s$, $\mathrm{f}_1, \ldots, \mathrm{f}_r$ *and* $s_i = \deg(\mathrm{f}_i)$, C *as in Theorem 1 and*

(*1*) $\mathrm{U}_0, \mathrm{U}_1, \mathrm{U}_2, \mathrm{U}_3, \mathrm{U}_4, \mathrm{U}_5$ *the unique partition of* $\{1, \ldots, r\}$*, associated to* C,

(*2*) $\{\vec{\mathrm{v}}_u : u \in \mathrm{U}_2 \cup \mathrm{U}_3\}$ *the vectors such that the corresponding ideal, in* $\mathrm{R}[\mathrm{T}]/\langle \mathrm{T}^n - 1\rangle$*, of* C *is*

$$\langle \mathfrak{m}^3 \widehat{\prod_{u\in \mathrm{U}_1} \mathrm{f}_u}, \mathfrak{m} \widehat{\prod_{u\in \mathrm{U}_4} \mathrm{f}_u}, \widehat{\prod_{u\in \mathrm{U}_5} \mathrm{f}_u}, (\vec{\mathrm{v}}_u)_{\tilde{\beta}}^{\mathbb{T}_{s_u}} \widehat{\mathrm{f}}_u, (\vec{\mathrm{v}}_w)_{\tilde{\alpha}}^{\mathbb{T}_{s_w}} \widehat{\mathrm{f}}_w : u \in \mathrm{U}_2, w \in \mathrm{U}_3\rangle.$$

The following conditions are equivalent:

(*1*) C *is a reversible code;*

(*2*) U_i *is self-reciprocal, for* $i \in \{1, \ldots, 5\}$*, and* $\vec{\mathrm{v}}_u = (\vec{\mathrm{v}}_{u^*})^*_{(n,\mathrm{f}_u)}$*, for* $u \in \mathrm{U}_2 \cup \mathrm{U}_3$*;*

(*3*) U_i *is self-reciprocal, for* $i \in \{0, \ldots, 5\}$*, and* $\vec{\mathrm{v}}_u = (\vec{\mathrm{v}}_{u^*})^*_{(n,\mathrm{f}_u)}$*, for* $u \in \mathrm{U}_2 \cup \mathrm{U}_3$.

The following is the main result of this section. Recall that C is complementary, if and only if $(\kappa, \ldots, \kappa) \in \mathrm{C}$, see [3].

Theorem 3. *Let* $(\mathrm{R}, \mathfrak{m}, \mathrm{GF}(q)) \in \mathfrak{F}_5^4$, $\tilde{\alpha} = \{\alpha_1, \alpha_2\}$, $\tilde{\beta} = \{\alpha_1, \alpha_2^2\}$, $\mathbb{T}$, $\mathbb{T}_s$, $\mathrm{f}_1, \ldots, \mathrm{f}_r$ *and* $s_i = \deg(\mathrm{f}_i)$, C *as in Theorem 1 and*

(*1*) $\mathrm{U}_0, \mathrm{U}_1, \mathrm{U}_2, \mathrm{U}_3, \mathrm{U}_4, \mathrm{U}_5$ *the unique partition of* $\{1, \ldots, r\}$*, associated to* C,

(*2*) $\{\vec{\mathrm{v}}_u : u \in \mathrm{U}_2 \cup \mathrm{U}_3\}$ *the vectors such that the corresponding ideal, in* $\mathrm{R}[\mathrm{T}]/\langle \mathrm{T}^n - 1\rangle$*, of* C *is*

$$\langle \mathfrak{m}^3 \widehat{\prod_{u\in \mathrm{U}_1} \mathrm{f}_u}, \mathfrak{m} \widehat{\prod_{u\in \mathrm{U}_4} \mathrm{f}_u}, \widehat{\prod_{u\in \mathrm{U}_5} \mathrm{f}_u}, (\vec{\mathrm{v}}_u)_{\tilde{\beta}}^{\mathbb{T}_{s_u}} \widehat{\mathrm{f}}_u, (\vec{\mathrm{v}}_w)_{\tilde{\alpha}}^{\mathbb{T}_{s_w}} \widehat{\mathrm{f}}_w : u \in \mathrm{U}_2, w \in \mathrm{U}_3\rangle.$$

The following conditions are equivalent:

(*1*) C *is DNA code;*

(*2*) U_i *is self-reciprocal, for* $i \in \{1, \ldots, 5\}$, $\vec{\mathrm{v}}_u = (\vec{\mathrm{v}}_{u^*})^*_{(n,\mathrm{f}_u)}$*, for* $u \in \mathrm{U}_2 \cup \mathrm{U}_3$*, and one of the following relations is satisfied:*
(*a*) *if* $\ell_{\mathrm{R}}(\kappa) = 2$ *and* $\langle \kappa\rangle = \mathfrak{m}^2$*, then* $\mathrm{I}_r = \langle \kappa\rangle$ *or* $3 \leq \ell_{\mathrm{R}}(\mathrm{I}_r)$*;*
(*b*) *if* $\ell_{\mathrm{R}}(\kappa) = 2$ *and* $\langle \kappa\rangle \neq \mathfrak{m}^2$*, then* $\mathrm{I}_r \in \{\langle \kappa\rangle, \mathrm{ann}_{\mathrm{R}}(\mathfrak{m}^2), \mathfrak{m}, \mathrm{R}\}$*;*
(*c*) *if* $\ell_{\mathrm{R}}(\kappa) = 3$*, then* $\mathrm{I}_r \in \{\langle \kappa\rangle, \mathfrak{m}, \mathrm{R}\}$.

(*3*) U_i *is self-reciprocal, for* $0 \in \{1, \ldots, 5\}$, $\vec{\mathrm{v}}_u = (\vec{\mathrm{v}}_{u^*})^*_{(n,\mathrm{f}_u)}$*, for* $u \in \mathrm{U}_2 \cup \mathrm{U}_3$, $r \in \cup_{i=\ell_{\mathrm{R}}(\kappa)}^5 \mathrm{U}_i$ *and one of the following relations is satisfied:*
(*a*) *if* $\ell_{\mathrm{R}}(\kappa) = 2$, $\langle \kappa\rangle = \mathfrak{m}^2$ *and* $r \in \mathrm{U}_2$*, then* $(\vec{\mathrm{v}}_r)_{\tilde{\beta}} = \langle \kappa\rangle$*;*
(*b*) $\ell_{\mathrm{R}}(\kappa) = 2$, $\langle \kappa\rangle = \mathfrak{m}^2$ *and* $r \in \mathrm{U}_3 \cup \mathrm{U}_4 \cup \mathrm{U}_5$*;*

(*c*) *if* $\ell_{\mathrm{R}}(\kappa) = 2$, $\langle\kappa\rangle \neq \mathfrak{m}^2$ *and* $r \in \mathrm{U}_2$, *then* $(\vec{\mathrm{v}}_r)_{\tilde{\beta}} = \langle\kappa\rangle$*;*
(*d*) $\ell_{\mathrm{R}}(\kappa) = 2$, $\langle\kappa\rangle \neq \mathfrak{m}^2$, $r \notin \mathrm{U}_2$ *and* $\mathrm{I}_r \in \{\mathrm{ann}_{\mathrm{R}}(\mathfrak{m}^2), \mathfrak{m}, \mathrm{R}\}$*;*
(*e*) *if* $\ell_{\mathrm{R}}(\kappa) = 3$ *and* $r \in \mathrm{U}_3$, *then* $(\vec{\mathrm{v}}_r)_{\tilde{\beta}} = \langle\kappa\rangle$*;*
(*f*) $\ell_{\mathrm{R}}(\kappa) = 3$, $r \notin \mathrm{U}_3$ *and* $\mathrm{I}_r \in \{\mathfrak{m}, \mathrm{R}\}$.

Proof: Recall the following:
(1) Under the polynomial representation, $(\kappa, \ldots, \kappa)$ in R^n corresponds with $\kappa\frac{\mathrm{T}^n-1}{\mathrm{T}-1}$ in $\mathrm{R}[\mathrm{T}]/\langle \mathrm{T}^n - 1\rangle$,
(2) The ideal generated by $\kappa\frac{\mathrm{T}^n-1}{\mathrm{T}-1}$ in $\mathrm{R}[\mathrm{T}]/\langle \mathrm{T}^n - 1\rangle$ is identified with $\langle\vec{0}\rangle \oplus \kappa\mathrm{R}[\mathrm{T}]/\langle \mathrm{f}_r\rangle$, in $\oplus_{i=1}^{r}\mathrm{R}[\mathrm{T}]/\langle \mathrm{f}_i\rangle$.
(3) $\mathrm{U}_0, \mathrm{U}_1, \mathrm{U}_2, \mathrm{U}_3, \mathrm{U}_4, \mathrm{U}_5$, are given by $\mathrm{U}_i = \{u : \ell_{\mathrm{R}[\mathrm{T}]/\langle \mathrm{f}_u\rangle}(\mathrm{I}_u) = i\}$.
(4) We may assume $\mathrm{R}[\mathrm{T}]/\langle \mathrm{f}_r\rangle$ is the ring R.
Let $\mathrm{C} \cong \mathrm{I}_1 \oplus \ldots \oplus \mathrm{I}_r$, I_i is an ideal of $\mathrm{R}[\mathrm{T}]/\langle \mathrm{f}_i\rangle$, then:

$$(\kappa, \ldots, \kappa) \in \mathrm{C} \Leftrightarrow \langle\vec{0}\rangle \oplus \langle\kappa\rangle \subseteq \mathrm{I}_1 \oplus \ldots \oplus \mathrm{I}_r \Leftrightarrow \langle\kappa\rangle \subseteq \mathrm{I}_r$$

if and only if, by Lemma 1, one of the following hold:
(*a*) if $\ell_{\mathrm{R}}(\kappa) = 2$ and $\langle\kappa\rangle = \mathfrak{m}^2$, then $\mathrm{I}_r = \langle\kappa\rangle$ or $3 \leq \ell_{\mathrm{R}}(\mathrm{I}_r)$;
(*b*) if $\ell_{\mathrm{R}}(\kappa) = 2$ and $\langle\kappa\rangle \neq \mathfrak{m}^2$, then $\mathrm{I}_r \in \{\langle\kappa\rangle, \mathrm{ann}_{\mathrm{R}}(\mathfrak{m}^2), \mathfrak{m}, \mathrm{R}\}$;
(*c*) if $\ell_{\mathrm{R}}(\kappa) = 3$, then $\mathrm{I}_r \in \{\langle\kappa\rangle, \mathfrak{m}, \mathrm{R}\}$.
$\Leftrightarrow r \in \cup_{i=\ell_{\mathrm{R}}(\kappa)}^{5}\mathrm{U}_i$ and one of the following relations is satisfied:
(*a*) if $\ell_{\mathrm{R}}(\kappa) = 2$, $\langle\kappa\rangle = \mathfrak{m}^2$ and $r \in \mathrm{U}_2$, then $(\vec{\mathrm{v}}_r)_{\tilde{\beta}} = \langle\kappa\rangle$;
(*b*) $\ell_{\mathrm{R}}(\kappa) = 2$, $\langle\kappa\rangle = \mathfrak{m}^2$ and $r \in \mathrm{U}_3 \cup \mathrm{U}_4 \cup \mathrm{U}_5$;
(*c*) if $\ell_{\mathrm{R}}(\kappa) = 2$, $\langle\kappa\rangle \neq \mathfrak{m}^2$ and $r \in \mathrm{U}_2$, then $(\vec{\mathrm{v}}_r)_{\tilde{\beta}} = \langle\kappa\rangle$;
(*d*) $\ell_{\mathrm{R}}(\kappa) = 2$, $\langle\kappa\rangle \neq \mathfrak{m}^2$, $r \notin \mathrm{U}_2$ and $\mathrm{I}_r \in \{\mathrm{ann}_{\mathrm{R}}(\mathfrak{m}^2), \mathfrak{m}, \mathrm{R}\}$;
(*e*) if $\ell_{\mathrm{R}}(\kappa) = 3$ and $r \in \mathrm{U}_3$, then $(\vec{\mathrm{v}}_r)_{\tilde{\beta}} = \langle\kappa\rangle$;
(*f*) $\ell_{\mathrm{R}}(\kappa) = 3$, $r \notin \mathrm{U}_3$ and $\mathrm{I}_r \in \{\mathfrak{m}, \mathrm{R}\}$.
The assertion follows from Theorem 2.

The following example is given illustrating the above results.

Example 4. *Let* $\mathrm{GF}(2^2) = \{0, 1, \zeta, \zeta^2\}$, $\zeta^2 = \zeta + 1$, $\mathrm{R} = \mathrm{GF}(2^2)[\mathrm{X}, \mathrm{Y}]/\langle \mathrm{X}^2 - \mathrm{Y}^3, \mathrm{XY}\rangle$, $\rho = 1 + \mathrm{y}^2$, $\kappa = \mathrm{x}$ *as in Example 3. By Hensel's Lemma,* $\mathrm{T}^3 - 1 = \mathrm{f}_1\mathrm{f}_2\mathrm{f}_3$*, where:* $\mathrm{f}_1 = \mathrm{T} - \zeta^2$, $\mathrm{f}_2 = \mathrm{T} - \zeta$, $\mathrm{f}_3 = \mathrm{T} - 1 \in \mathrm{GF}(2)[\mathrm{T}] \subset \mathrm{R}[\mathrm{T}]$.
We have $\mathrm{f}_1^* = \zeta^2\mathrm{f}_2$, $\mathrm{f}_2^* = \zeta\mathrm{f}_2$, $\mathrm{f}_3^* = \mathrm{f}_3$, $r_1 = 1$ *and* $r_2 = 1$.
If $\mathrm{U}_3 = \{3\}$, $\mathrm{U}_0 = \mathrm{U}_1 = \mathrm{U}_2 = \mathrm{U}_4 = \mathrm{U}_5 = \emptyset$, $\vec{\mathrm{v}}_3 = (1, 0)$*, the code*

$$\langle \mathfrak{m}^3 \widehat{\prod_{u\in \mathrm{U}_1} \mathrm{f}_u}, \mathfrak{m}\widehat{\prod_{u\in \mathrm{U}_4} \mathrm{f}_u}, \widehat{\prod_{u\in \mathrm{U}_5} \mathrm{f}_u}, (\vec{\mathrm{v}}_u)_{\tilde{\beta}}^{\mathbb{T}_{s_u}}\widehat{\mathrm{f}}_u, (\vec{\mathrm{v}}_w)_{\tilde{\alpha}}^{\mathbb{T}_{s_w}}\widehat{\mathrm{f}}_w : u \in \mathrm{U}_2, w \in \mathrm{U}_3\rangle =$$

$$\langle \mathrm{x}\widehat{\mathrm{f}_3}, \mathrm{y}^2\widehat{\mathrm{f}_3}\rangle = \langle \mathrm{x}\mathrm{f}_1\mathrm{f}_2, \mathrm{y}^2\mathrm{f}_1\mathrm{f}_2\rangle$$

is a DNA code with $|\mathrm{C}| = 4^{5(0)+4(0)+3(1)+2(0)+1(0)} = 4^3 = 64$ *elements*

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