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Divisible hypermodules

Hashem Bordbar and Irina Cristea

Abstract

The article is motivated by the recently published studies on injective and projective hypermodules. We present here a new characterization of the normal injective hypermodules. First we define the concept of zero-divisors over a hypermodule and based on it we introduce a new class of hypermodules, the one of divisible hypermodules. After presenting some of their fundamental properties, we will show that the class of normal injective R-hypermodules M and the class of divisible R-hypermodules M coincide whenever R is a hyperring with no zero-divisors over M. Finally, we answer to an open problem related to canonical hypergroups. In particular, we show that any canonical hypergroup can be endoweded with a $\mathbb{Z}$-hypermodule structure and it is a normal injective $\mathbb{Z}$-hypermodule if and only if it is a divisible $\mathbb{Z}$-hypermodule.

1 Introduction

In abstract algebra, an element r of a commutative unitary ring R is called a *zero-divisor* if there exists a nonzero element s in R such that $rs = 0$. If we consider now an R-module M, we can extend this definition and obtain the concept of *zero-divisors of* R *over* M. Thus, an element r of R is called *zero-divisor over* M (or *on* M) if there exists $m \in M \setminus \{0\}$ such that $rm = 0$. In [13] a *divisible module* is defined as a module A over an integral domain R having the property $rA = A$ for every $r \neq 0, r \in R$. Since any integral domain has no nonzero zero-divisors, this definition was then extended to an arbitrary

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unitary ring R. Therefore a module A over a unitary ring R is *divisible* if, for all non zero-divisors of R and for all $m \in M$ there exists $m' \in M$ such that $m = rm'$. Moreover, the injective objects have a fundamental role in the category theory. For instance, Dedekind-MacNeille completions are the injective objects in the category of posets [2] or a complete Boolean algebra is the injective object in the category of Boolean algebras [5]. In the category of modules, any every injective R-module is divisible, while viceversa holds if R is a Dedekind-domain [6].

Inspired by this nice characterization of injective modules, in this paper we aim to obtain a similar result but in hypercompositional algebra. This is algebra of hypercompositional structures, where the classical operation is substituted by hyperoperations, called also hyperproducts. A complete overview of the foundations of this theory has been recently published with a very well prepared list of references [12]. In 1956 Krasner [7] introduced the notion of hyperfield and later on the one of hyperring and hypermodule over a hyperring, known nowadays as Krasner hyperrings and Krasner hypermodules. Besides them there are also other types of hyperrings and hypermodules, as multiplicative hyperrings defined by Rota [16], or generalized hyperrings defined by Vougiouklis [22]. The additive structure of the Krasner hyperrings is a canonical hypergroup introduced by Mittas [14], that have a similar property of the abelian groups, as it is proven in the last part of this article. The first properties of the Krasner hypermodules were studied by Massouros [10] in connection with cyclicity and torsion free elements, argument that has been recently investigated from a categorical approach [15]. The study of the category of Krasner hypermodules has been initiated by Madanshekaf [9] and continued by Shojaei et al. [17, 18, 19]. The last group has recently started the investigation of the injectivity and projectivity properties of Krasner hypermodules [1]. This study was then continued by the authors of this manuscript [3], where they provided an alternative definition for the normal projective and injective R-hypermodules over Krasner hyperrings R, based on hyperideals of R and then exact chains of R-hypermodules. This manuscript goes in the same direction, aiming to provide a new characterization of normal injective R-hypermodules by using divisible R-hypermodules.

The remainder of this article is structured as follows. In the preliminary section we gather the basic notions related to Krasner hypermodules and their several types of homomorphisms, concluding it with the definition and characterization of normal injective hypermodules proved in [3]. Section 3 is dedicated to the study of the divisible hypermodules based on the definition of zero-divisors over a hypermodule. It is important to notice the difference between the similar concept of divisible module from classical algebra and our definition in hypercompositional algebra. In the definition of a divisible element of a module, the concept of non zero-divisor of a ring is used, while in the definition of a divisible element of a hypermodule a non zero-divisor over the hypermodule is involved. This section contains several fundamental properties of the divisible hypermodules. Section 4 starts with the construction of a new structure of R-hypermodule, using a finite or infinite nonempty family of R-hypermodules. This will help us to state and prove one of the main results of this paper. In particular, we show that every normal injective R-hypermodule M is a divisible R-hypermodule, whenever R is a Krasner hyperring with no zero-divisors over M. In the last part of this section, we answer to one open problem addressed in [20]. We show that every canonical hypergroup can be endowed with a $\mathbb{Z}$-hypermodule structure, as in classical algebra every abelian group has a $\mathbb{Z}$-module structure. Notice that this construction is different by the trivial one suggested in [21]. The conclusions of this study are covered in the last section of the manuscript.

2 Preliminaries

Throughout this paper, unless otherwise stated, R denotes a *Krasner hyperring*, that we will call, by short, *hyperring*, and $\mathcal{P}^*(R)$ denotes the family of all nonempty subsets of R.

Definition 2.1. [7] A *(Krasner) hyperring* is a hyperstructure $(R, +, \cdot)$ where

1. $(R, +)$ is a canonical hypergroup, i.e.,
   (a) $a, b \in R \Rightarrow a + b \subseteq R$,
   (b) $\forall a, b, c \in R, a + (b + c) = (a + b) + c$,
   (c) $\forall a, b \in R, a + b = b + a$,
   (d) $\exists 0 \in R, \forall a \in R, a + 0 = \{a\}$,
   (e) $\forall a \in R, \exists -a \in R$ such that $0 \in a + x \Leftrightarrow x = -a$,
   (f) $\forall a, b, c \in R, c \in a + b \Rightarrow a \in c + (-b)$.
2. $(R, \cdot)$ is a semigroup with a bilaterally absorbing element 0, i.e.,
   (a) $a, b \in R \Rightarrow a \cdot b \in R$,
   (b) $\forall a, b, c \in R, a \cdot (b \cdot c) = (a \cdot b) \cdot c$,
   (c) $\forall a \in R, 0 \cdot a = a \cdot 0 = 0$.
3. The product distributes from both sides over the hyperaddition, i.e.,
   (a) $\forall a, b, c \in R, a \cdot (b + c) = a \cdot b + a \cdot c$ and $(b + c) \cdot a = b \cdot a + c \cdot a$.

Moreover, a hyperring is called *commutative*, if $(R, \cdot)$ is commutative, i.e., $\forall a, b \in R, a \cdot b = b \cdot a$. Finally, if $(R, \cdot)$ is a monoid, i.e., $\exists 1 \in R$ such that $\forall a \in R, a \cdot 1 = a = 1 \cdot a$, then we say that R is a hyperring *with a unit element*, or a *unitary hyperring*.

Definition 2.2. A *hyperring homomorphism* is a mapping f from a hyperring $(R_1, +_{R_1}, \cdot_{R_1})$ to a hyperring $(R_2, +_{R_2}, \cdot_{R_2})$ with the unit elements 1_{R_1} and 1_{R_2} such that

1. $\forall a, b \in R_1, f(a +_{R_1} b) = f(a) +_{R_2} f(b)$.
2. $\forall a, b \in R_1, f(a \cdot_{R_1} b) = f(a) \cdot_{R_2} f(b)$.
3. $f(1_{R_1}) = 1_{R_2}$.

The concept of hypermodule over a Krasner hyperring was introduced by Krasner himself and studied later on more in detail for its algebraic properties in [10].

Definition 2.3. Let R be a hyperring with the unit element 1. A canonical hypergroup $(M, +)$ together with a left external map $R \times M \longrightarrow M$ defined by

$$(a, m) \mapsto a \cdot m = am \in M \tag{2.1}$$

such that for all $a, b \in R$ and $m_1, m_2 \in M$ we have

1. $(a + b)m_1 = am_1 + bm_1$,
2. $a(m_1 + m_2) = am_1 + am_2$,
3. $(ab)m_1 = a(bm_1)$,
4. $a0_M = 0_R m_1 = 0_M$,
5. $1m_1 = m_1$

is called a *left Krasner hypermodule over* R, or by short, a *left* R*-hypermodule*. Similarly, one may define a *right* R*-hypermodule*. For simplicity, in this paper we consider only left R-hypermodules, that we call R-hypermodules.

Proposition 2.4. *[4] Let R be a hyperring with unit element. Then R is an R-hypermodule.*

Definition 2.5. A *subhypermodule* N *of* M is a subhypergroup of M which is also closed under multiplication by elements of R.

In the following lemma, an example of subhypermodule is constructed.

Lemma 2.6. *[3] Let R be a hyperring, M be an R-hypermodule and $\{M_i\}_{i\in I}$ be a family of subhypermodules of M. Then the* sum *of this family is denoted by $\sum_{i\in I} M_i$ and it is the family of the sets $\sum_{i\in I} m_i$, where for every $i \in I$, $m_i \in M_i$. More specifically,*

$$M_1 + M_2 = \{m_1 + m_2 \mid m_1 \in M_1, m_2 \in M_2\}, \tag{2.2}$$

where $m_1 + m_2$ is a set (in particular a subset of M) and not only an element, since $+$ is a hyperoperation on M, while

$$M_1 + M_2 + M_3 = \{m_1 + m_2 + m_3 \mid m_1 \in M_1, m_2 \in M_2, m_3 \in M_3\},$$

where the set $m_1 + m_2 + m_3$ can be written as the union $\bigcup_{m\in m_1+m_2} m + m_3$.

Clearly, the structure $\sum_{i\in I} M_i$ is a subhypermodule of M and it is the smallest subhypermodule of M containing every M_i.

As already mentioned by Krasner and then very clear explained by Massouros [10], we may define more types of homomorphisms between R-hypermodules.

Definition 2.7. [10] Let M and N be two R-hypermodules. A multivalued function $f : M \longrightarrow \mathcal{P}^*(N)$ is called an *R-homomorphism* if:

(i) $\forall m_1, m_2 \in M, f(m_1 +_M m_2) \subseteq f(m_1) +_N f(m_2)$,

(ii) $\forall m \in M, \forall r \in R, f(r \cdot_M m) = r \cdot_N f(m)$,

while f is called *strong homomorphism* if instead of (i) we have

(i') $\forall m_1, m_2 \in M, f(m_1 +_M m_2) = f(m_1) +_N f(m_2)$.

A single-valued function $f : M \longrightarrow N$ is called a *strict R-homomorphism* if axioms (i) and (ii) are valid and it is called a *normal R-homomorphism* if (i') and (ii) are valid.

The family of all normal R-homomorphisms from M to N is denoted by $Hom^n_R(M,N)$, while the family of all strict homomorphisms from M to N is denoted by $Hom_R(M,N)$.

In the following we will recall some types of R-homomorphisms.

Definition 2.8. [3] Let $f \in Hom_R(M,N)$ (respectively $f \in Hom^n_R(M,N)$). Then f is called

(i) a *surjective (normal) R-homomorphism* if $Im(f) = N$.

(ii) an *injective (normal) R-homomorphism* if for all $m_1, m_2 \in M$, $f(m_1) = f(m_2)$ implies $m_1 = m_2$.

(iii) *(normal) R-isomorphism* if it is a bijective (normal) R-homomorphism.

We conclude this section, by recalling the characterizations of a normal injective R-hypermodule using hyperideals and then using exact chains of R-hypermodules and normal R-homomorphisms.

Theorem 2.9. *[3] Let R be a hyperring and N be an R-hypermodule. Then the following statements are equivalent:*

(1) *N is a normal injective R-hypermodule.*

(2) *For any hyperideal I of R, an inclusion hyperring homomorphism $i : I \longrightarrow R$ and a normal R-homomorphism $k : I \longrightarrow N$, there exists a normal R-homomorphism $h : R \longrightarrow N$ such that the diagram in Figure 1 has the composition structure, i.e., $hi = k$.*

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r, "i"] \arrow[d, "k"] & R \arrow[dl, dashed, "\exists h"] \\ & N & \end{tikzcd}$$

Figure 1: Composition structure of a diagram for a normal injective R-hypermodule, using hyperideals

Theorem 2.10. *[3] An R-hypermodule N is normal injective if it satisfies the following equivalent conditions.*

(i) *For any exact chain*

$$0 \longrightarrow M_1 \xrightarrow{\gamma} M_2 \xrightarrow{\delta} M_3 \longrightarrow 0 \tag{2.3}$$

of R-hypermodules and normal R-homomorphisms, the chain

$$0 \longrightarrow Hom_R^n(M_3, N) \xrightarrow{\Delta} Hom_R^n(M_2, N) \xrightarrow{\Gamma} Hom_R^n(M_1, N) \longrightarrow 0 \tag{2.4}$$

is exact, too.

(ii) *For any R-hypermodules M_1, M_2, N and normal R-homomorphisms $\gamma : M_1 \longrightarrow M_2$ and $k : M_1 \longrightarrow N$ such that the chain $0 \longrightarrow M_1 \xrightarrow{\gamma} M_2$ is exact, there exists a normal R-homomorphism $h : M_2 \longrightarrow N$ such that $h\gamma = k$.*

(iii) *For any hyperideal I of R, any inclusion hyperring homomorphism $i : I \longrightarrow R$, and normal R-homomorphism $k : I \longrightarrow N$, there exists a normal R-homomorphism $h : R \longrightarrow N$ such that $hi = k$.*

3 Divisible R-hypermodules

Based on the definition of zero-divisor element in a hyperring, and zero-divisor element over an R-hypermodule, we will define the concept of divisible R-hypermodule.

Definition 3.1. Let R be a hyperring. An element r of R is said to be a *right zero-divisor* if there exists a nonzero element $r' \in R$ such that $r'r = 0$. Similarly, a *left zero-divisor* element is defined as an element of R such that $rr' = 0$ for an element $r' \in R \setminus \{0\}$. If R is a commutative hyperring, then the right and the left zero-divisors coincide and we refer to them as *zero-divisors of R.*

Definition 3.2. Let R be a hyperring and M be an R-hypermodule. A nonzero element $r \in R$ is said to be a *zero-divisor over M* if there exists a nonzero element $m \in M$ such that $r \cdot_M m = 0_M$.

We denote by $Z(R)$ the set of all zero-divisors of the hyperring R, while by $Z_R(M)$ the set of all zero-divisors elements over M, i.e., $Z_R(M) = \{r \in R \setminus \{0\} \mid \exists m \in M, m \neq 0, r \cdot_M m = 0_M\}$.

Remark 3.3. If we consider the hyperring R as an R-hypermodule based on Proposition 2.4, then every zero-divisor element over R is a zero-divisor element of R, i.e., $Z_R(R) \subset Z(R)$.

Example 3.4. Let M be an R-hypermodule. It is obvious that any element of the annihilator $Ann_R(M) = \{r \in R \mid \forall m \in M, r \cdot_M m = 0\}$ is a zero-divisor element over M, but the converse is not true in general. For example suppose that $R = \{0, 1, 2\}$ and define the hyperaddition "+" and the multiplication "·" by the following tables:

$+$	0	1	2
0	0	1	2
1	1	R	1
2	2	1	$\{0,2\}$

$\cdot$	0	1	2
0	0	0	0
1	0	1	2
2	0	2	0

Then $(R, +, \cdot)$ is a hyperring [4]. Based on Proposition 2.4, it is an R-hypermodule. Now we can check that the element 2 is a zero-divisor element over the R-hypermodule R, which does not belong to $Ann_R(R) = \{0\}$.

Lemma 3.5. *Let R be a hyperring and M be an arbitrary R-hypermodule. If R is a hyperring such that $Z_R(M) = \emptyset$, then R has no zero-divisor elements.*

Proof. Suppose that M is an R-hypermodule and $Z_R(M) = \emptyset$. By reductio ad absurdum, suppose that R has a zero-divisor element r. Then there exists

$s \in R$, $s \neq 0$ such that $s \cdot_R r = 0_R$. Since $Z_R(M) = \emptyset$, it follows that, for every $m \in M$ such that $m \neq 0$ we have, $r \cdot_M m \neq 0_M$. But

$$s \cdot_M (r \cdot_M m) = (s \cdot_R r) \cdot_M m = 0_R \cdot_M m = 0_M,$$

which is a contradiction, since $s \in R$ and $Z_R(M) = \emptyset$. □

Definition 3.6. Let M be an R-hypermodule. A nonzero element m of M is said to be *divisible*, if for every non zero-divisor $r \in R$ over M, there exists $m' \in M$ such that $m = rm'$. Moreover, if each element of M is a divisible element, then M is said to be a *divisible R-hypermodule*.

Example 3.7. On the set $R = \{0, 1, a, b\}$ define the hyperaddition "+" and the multiplication "·" by the following tables:

$+$	0	1	a	b
0	0	1	a	b
1	1	R	$\{1, a, b\}$	$\{1, a, b\}$
a	a	$\{1, a, b\}$	R	$\{1, a, b\}$
b	b	$\{1, a, b\}$	$\{1, a, b\}$	R

$\cdot$	0	1	a	b
0	0	0	0	0
1	0	1	a	b
a	0	a	b	1
b	0	b	1	a

Then $(R, +, \cdot)$ is a hyperring. Based on Proposition 2.4, R is an R-hypermodule. The divisible elements of R are 1, a and b. Thus, R is a divisible R-hypermodule.

Lemma 3.8. *Let M be an R-hypermodule. If M is a divisible R-hypermodule, then $M = rM$, where $r \in R$ is an arbitrary non zero-divisor element over M. Moreover, if $Z_R(M) = \emptyset$ and M is a divisible R-hypermodule, then for each $r \in R$, we have $rM = M$.*

Proof. The proof is straightforward based on Definition 3.6 and Lemma 3.5. □

In the following we will investigate some properties of divisible R-hypermodules.

Proposition 3.9. *Let M_1 and M_2 be two divisible R-hypermodules. Then the sum $M_1 + M_2$ is a divisible R-hypermodule, too.*

Proof. Using Lemma 2.6, $M_1 + M_2$ is an R-hypermodule in which each element is a set of the form $m_1 + m_2$ with $m_1 \in M_1$ and $m_2 \in M_2$. Suppose that the set $m_1 + m_2$ is an arbitrary element of $M_1 + M_2$ and r is not a zero-divisor over M. Since M_1 and M_2 are divisible R-hypermodules, there exist $m_1' \in M_1$ and $m_2' \in M_2$ such that $m_1 = rm_1'$ and $m_2 = rm_2'$. Thus,

$$m_1 + m_2 = rm_1' + rm_2' = r(m_1' + m_2'),$$

where $m_1' + m_2'$ is another element of $M_1 + M_2$. Therefore, $M_1 + M_2$ is clearly a divisible R-hypermodule. □

Proposition 3.10. *Let M be a divisible R-hypermodule and N be an R-subhypermodule of M. Then the R-hypermodule quotient $\frac{M}{N}$ is a divisible hypermodule, too.*

Proof. Suppose that the set $m+N$ is an element of the R-hypermodule $\frac{M}{N}$ and $r \in R$ is not a zero-divisor over M. Then there exists $m' \in M$ such that $m = rm'$. Thus,

$$m + N = rm' + N = r(m' + N),$$

which shows that $\frac{M}{N}$ is a divisible R-hypermodule. □

Proposition 3.11. *Let M and N be R-hypermodules and $f \in Hom_R^n(M,N)$ be a surjective normal R-homomorphism. If M is a divisible R-hypermodule, then N is divisible, too.*

Proof. The proof is straightforward. □

4 Characterization of normal injective hypermodules

The aim of this section is to give a new charazterization of normal injective R-hypermodules as divisible R-hypermodules, for a particular class of hyperrings R.

We start this section by defining a new R-hypermodule structure, starting with a family of R-hypermodules.

Definition 4.1. Let $\{M_i\}_{i\in I}$ be a nonempty family of R-hypermodules, where I is an arbitrary finite or infinite index set, and $M = \prod M_i$ be the set of all families $\{m_i\}$, where $i \in I$ and $m_i \in M_i$. For any $r \in R$ and $\{m_i\}_{i\in I}, \{m_i'\}_{i\in I} \in \prod M_i$, define

$$\{m_i\}_{i\in I} +_M \{m_i'\}_{i\in I} = \{m_i +_{M_i} m_i'\}_{i\in I}, \tag{4.1}$$

$$r \cdot_M \{m_i\}_{i\in I} = \{r \cdot_{M_i} m_i\}_{i\in I}. \tag{4.2}$$

Since for each $i \in I$, the structure $(M_i, +_{M_i})$ is a canonical hypergroup, we easily conclude that $(M, +_M)$ is a canonical hypergroup, too. Moreover, for arbitrary elements $r, s \in R$ and $\{m_i\}_{i\in I}, \{m_i'\}_{i\in I} \in M$ we have:

1. $(r+s) \cdot_M \{m_i\}_{i\in I} = r \cdot_M \{m_i\}_{i\in I} +_M s \cdot_M \{m_i\}_{i\in I}$,
2. $r \cdot_M (\{m_i\}_{i\in I} +_M \{m_i'\}_{i\in I}) = r \cdot_M \{m_i\}_{i\in I} +_M r \cdot_M \{m_i'\}_{i\in I}$,
3. $(rs) \cdot_M \{m_i\}_{i\in I} = r \cdot_M (s \cdot_M \{m_i\}_{i\in I})$,
4. $r \cdot_M 0_M = 0_R \cdot_M \{m_i\}_{i\in I} = 0_M$,
5. $1 \cdot_M \{m_i\}_{i\in I} = \{m_i\}_{i\in I}$.

Therefore, the set $M = \prod M_i$ is endowed with an R-hypermodule structure.

The next result shows a correspondence between the normal injectivity of the R-hypermodule $\prod M_i$ and the same property of the R-hypermodules M_i.

Theorem 4.2. *Let $\{M_i\}_{i\in I}$ be a nonempty family of R-hypermodules. Then the R-hypermodule $\prod M_i$ in Definition 4.1 is normal injective if and only if any R-hypermodule M_i is normal injective, for $i \in I$.*

Proof. For each $j \in I$ let denote the injection normal R-homomorphism by ϕ_j and define it as follows, for $m_j \in M_j$:

$$\phi_j : M_j \longrightarrow \prod M_i, \quad \phi_j(m_j) = \{m_k\}_{k\in I}, \tag{4.3}$$

such that $m_k = m_j$ for $k = j$ and $m_k = 0_{M_k}$ otherwise. Besides, for each $j \in I$, denote the projection normal R-homomorphism by ψ_j and define it by

$$\psi_j : \prod M_i \longrightarrow M_j, \quad \psi_j(\{m_k\}_{k\in I}) = m_j. \tag{4.4}$$

First, suppose that the R-hypermodule $\prod M_i$ is normal injective. In order to show that for each $i \in I$, M_i is a normal injective R-hypermodule, we use the characterizations in Theorem 2.10. Hence, let N_1 and N_2 be two arbitrary R-hypermodules, $\gamma : N_1 \longrightarrow N_2$ and $\delta : N_1 \longrightarrow M_i$ be normal R-homomorphisms such that the chain $0 \longrightarrow N_1 \xrightarrow{\gamma} N_2$ is exact. Now, consider the following diagram

$$\begin{tikzcd} 0 \arrow[r] & N_1 \arrow[r, "\gamma"] \arrow[d, "\delta"] & N_2 \\ & M_i & \end{tikzcd}$$

that can be extended to a new one, by using the injection normal R-hypermorphism $\phi_i : M_i \to \prod M_i$. Since $\prod M_i$ is a normal injective R-hypermodule and the composition of two injection normal R-homomorphisms is still an injection normal R-homomorphism, by Theorem 2.9, it follows that there exists $f \in Hom^n_R(N_2, \prod M_i)$ such that the big diagram in Figure 2 has the composition structure, i.e., $f\gamma = \phi_i\delta$.

Using the projection normal R-hypermorphism ψ_i, we can define now $g : N_2 \longrightarrow M_i$ by $g(n_2) = \psi_i f(n_2)$ for each $n_2 \in N_2$. It is clear that g is a well-defined normal R-homomorphism. Moreover, for $n_1 \in N_1$, we have

$$g\gamma(n_1) = \psi_i f\gamma(n_1) = \psi_i\phi_i\delta(n_1) = \delta(n_1).$$

$$\begin{tikzcd}
0 \arrow[r] & N_1 \arrow[r, "\gamma"] \arrow[d, "\delta"] & N_2 \arrow[dl, dashed, "\exists g"'] \arrow[ddl, dashed, "\exists f"] \\
& M_i \arrow[d, shift left, "\phi_i"] & \\
& \prod M_i \arrow[u, shift left, "\psi_i"] &
\end{tikzcd}$$

Figure 2: Composition structure of a diagram for the R-hypermodule $\prod M_i$

Thus, there exists $g \in Hom_R^n(N_2, M_i)$ such that the small diagram in Figure 2 has the composition structure, i.e., $g\gamma = \delta$. Therefore, M_i is a normal injective R-hypermodule for each $i \in I$.

Conversely, suppose that for each $i \in I$, the R-hypermodule M_i is normal injective. In order to show that $\prod M_i$ is a normal injective R-hypermodule, consider the following diagram where the row of normal homomorphisms is exact and N_1 and N_2 are two arbitrary R-hypermodules:

$$\begin{tikzcd}
0 \arrow[r] & N_1 \arrow[r, "\gamma"] \arrow[d, "\delta"] & N_2 \\
& \prod M_i &
\end{tikzcd}$$

This diagram can be extended to the following one, by using the projection normal R-homomorphism $\psi_i : \prod M_i \to M_i$, where the composition $\psi_i\delta$ remains a normal R-homomorphism. Since M_i is a normal injective R-hypermodule, there exists $f_i \in Hom_R^n(N_2, M_i)$ such that the diagram in Figure 3 has the composition structure, i.e., $f_i\gamma = \psi_i\delta$.

$$\begin{tikzcd}
0 \arrow[r] & N_1 \arrow[r, "\gamma"] \arrow[d, "\delta"] & N_2 \arrow[ddl, dashed, "\exists f_i"'] \\
& \prod M_i \arrow[d, "\psi_i"] & \\
& M_i &
\end{tikzcd}$$

Figure 3: Composition structure of a diagram for the R-hypermodule M_i

Now define the normal R-homomorphism $g : N_2 \longrightarrow \prod M_i$ by $g(n_2) = \phi_i f_i(n_2)$ for each $n_2 \in N_2$, where ϕ_i is an injection normal R-homomorphism. For $n_1 \in N_1$, we get

$$g\gamma(n_1) = \phi_i f_i \gamma(n_1) = \phi_i \psi_i \delta(n_1) = \delta(n_1),$$

which shows that $\prod M_i$ is a normal injective R-hypermodule. $\square$

Proposition 4.3. *Let $\{M_i\}_{i\in I}$ be a nonempty family of divisible R-hypermodules, where I is a finite or infinite index set. Then the R-hypermodule $M = \prod M_i$ in Definition 4.1 is a divisible R-hypermodule.*

Proof. Let $r \in R$ be a non zero-divisor element over $\prod M_i$ and $\{m_i\}_{i\in I}$ be an element of the R-hypermodule $\prod M_i$. Since, for each $i \in I$, M_i is a divisible R-hypermodule, it follows that there exists $m_i' \in M_i$ such that $m_i = r \cdot_{M_i} m_i'$. Thus,

$$\{m_i\}_{i\in I} = \{r \cdot_{M_i} m_i'\}_{i\in I} = r \cdot_M \{m_i'\}_{i\in I},$$

where $\{m_i'\}_{i\in I}$ is an element of $\prod M_i$. Therefore, $\prod M_i$ is a divisible R-hypermodule. $\square$

The next theorem shows that every normal injective R-hypermodule is a divisible R-hypermodule, whenever R is a hyperring such that $Z_R(M) = \emptyset$.

Theorem 4.4. *Let M be a normal injective R-hypermodule, where R is a hyperring such that $Z_R(M) = \emptyset$. Then M is a divisible R-hypermodule.*

Proof. Suppose that M is a normal injective R-hypermodule, where R is a hyperring with $Z_R(M) = \emptyset$. In order to show that M is divisible, let $m \in M$ and $r \in R$ be such that $m \neq 0_M$ and $r \neq 0_R$. Then r is not a zero-divisor element over M. Suppose that $< r >= R \cdot_R r$ is the hyperideal generated by r, i.e., $R \cdot_R r = \{r' \cdot_R r \mid r' \in R\}$ and $\phi :< r >\longrightarrow M$ is defined by $\phi(r' \cdot_R r) = r' \cdot_M m$. Besides, using Proposition 2.4, we know that $< r >$ is an R-hypermodule. Hence, ϕ is a multivalued function between R-hypermodules $< r >$ and M. Moreover, ϕ is a well defined normal R-homomorphism since for $a_1,\ a_2 \in< r >$ and $r' \in R$ we have $a_1 = r_1 \cdot_R r$ and $a_2 = r_2 \cdot_R r$ where $r_1,\ r_2 \in R$, and then we get

(i)

$$\phi(a_1 +_{<r>} a_2) = \phi(r_1 \cdot_R r +_{<r>} r_2 \cdot_R r) = \phi((r_1 + r_2) \cdot_R r) = (r_1 + r_2) \cdot_M m$$

$$= r_1 \cdot_M m +_M r_2 \cdot_M m = \phi(r_1 \cdot_R r) +_M \phi(r_2 \cdot_R r) = \phi(a_1) +_M \phi(a_2).$$

(*ii*)

$$\phi(r' \cdot_{<r>} a_1) = \phi(r' \cdot_R (r_1 \cdot_R r)) = \phi((r' \cdot_R r_1) \cdot_R r)$$
$$= (r' \cdot_R r_1) \cdot_M m = r' \cdot_M (r_1 \cdot_M m) = r' \cdot_M \phi(r_1 \cdot_R r) = r' \cdot_M \phi(a_1).$$

(*iii*) If $r' \cdot_{<r>} r = 0_R$, then $(r' \cdot_R r) \cdot_M m = r' \cdot_M (r \cdot_M m) = 0_M$. Since r is not a zero-divisor over M, we conclude that $r \cdot_M m \neq 0$. Thus, $r' = 0_R$. Therefore, $\phi(r' \cdot_{<r>} r) = r' \cdot_M m = 0_M$ and ϕ is a well defined normal R-homomorphism.

Now, consider the following diagram, where $\phi : < r > \longrightarrow M$ is defined by $\phi(r' \cdot_{<r>} r) = r' \cdot_M m$ and i is the inclusion function.

$$\begin{tikzcd}
0 \arrow[r] & < r > \arrow[r, "i"] \arrow[d, "\phi"] & R \arrow[dl, dashed, "\psi"'] \\
 & M &
\end{tikzcd}$$

Figure 4: Diagram for the normal injective R-hypermodule M

Since M is a normal injective R-hypermodule, there exists an R-homomorphism $\psi : R \longrightarrow M$ such that the diagram in Figure 4 has the composition structure, i.e., $\psi i = \phi$. Thus, for $1_R \in R$ we have

$$\phi(1_R \cdot_{<r>} r) = 1_R \cdot_M m = m,$$

and

$$\phi(1_R \cdot_{<r>} r) = \psi i(1_R \cdot_{<r>} r) = \psi(i(1_R \cdot_{<r>} r)) = \psi(1_R \cdot_R r) = r \cdot_M \psi(1_R).$$

Therefore,

$$m = r \cdot_M \psi(1_R),$$

which means that M is a divisible R-hypermodule. $\square$

We now discuss about the converse of Theorem 4.4.

Theorem 4.5. *Let R be a hyperring and M be a divisible R-hypermodule such that $Z_R(M) = \emptyset$. Then M is a normal injective R-hypermodule.*

Proof. Suppose that R is a hyperring and M is a divisible R-hypermodule, with $Z_R(M) = \emptyset$. In order to show that M is a normal injective R-hypermodule, consider the following diagram, where I is a hyperideal of R, $i : I \longrightarrow R$ is the inclusion hyperring homomorphism and $k : I \longrightarrow M$ is a normal R-homomorphism.

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r, "i"] \arrow[d, "k"] & R \arrow[dl, dashed, "h"'] \\ & M & \end{tikzcd}$$

Figure 5: Diagram for the divisible R-hypermodule M

Let assume that $I \neq 0$ and consider a nonzero element $a \in I$. Since M is a divisible R-hypermodule, we conclude that there exists $m \in M$ such that $k(a) = a \cdot_M m$. Besides, let t be an arbitrary element of I. Then we have

$$a \cdot_M k(t) = k(a \cdot_R t) = k(t \cdot_R a) = t \cdot_M k(a) = t \cdot_M (a \cdot_M m) = a \cdot_M (t \cdot_M m).$$

Thus,

$$0_M \in a \cdot_M (k(t) - t \cdot_M m),$$

then there exists $b \in k(t) - t \cdot_M m$ such that $0_M = a \cdot_M b$. Since $Z_R(M) = \emptyset$ and $a \neq 0$, we should have $b = 0_M$. Hence, $0_M \in k(t) - t \cdot_M m$ and so $k(t) = t \cdot_M m$, for every element $t \in I$, because M has a canonical hypergroup structure. Now if we define the R-homomorphism $h : R \longrightarrow M$ for each $r \in R$ by $h(r) = r \cdot_M m$, then h is a normal R-homomorphism and the diagram in Figure 5 has the composition structure, i.e., $hi = k$. Therefore, using Theorem 2.9, M is a normal injective R-hypermodule. □

Combining Theorem 4.4 and Theorem 4.5 we obtain the next characterization of a normal injective R-hypermodule, for R being a hyperring with no zero-divisors over M.

Corollary 4.6. *Let R be a hyperring and M be an R-hypermodule such that $Z_R(M) = \emptyset$. Then M is divisible if and only if M is normal injective R-hypermodule.*

We conclude this section with one fundamental result concerning the canonical hypergroups.

Proposition 4.7. *Every canonical hypergroup G has a structure of a $\mathbb{Z}$-hypermodule.*

Proof. Let $\mathbb{Z}$ be the set of integers. First we endow $\mathbb{Z}$ with a structure of commutative Krasner hyperring with unit. For doing this, define the hyperoperation $\oplus$ as follows: for any $a_1, a_2 \in \mathbb{Z}$,

$$a_1 \oplus a_2 = \{0, a_1 +_{\mathbb{Z}} a_2\} \setminus \{a_1, a_2\}.$$

Then we can check that $(\mathbb{Z}, \oplus)$ is a canonical hypergroup. Moreover if we consider the operation $\cdot_{\mathbb{Z}}$ as the normal multiplication in $\mathbb{Z}$, then $(\mathbb{Z}, \oplus, \cdot_{\mathbb{Z}})$ is a commutative Krasner hyperring with the unit element 1.

Suppose now that $(G, +_G)$ is an arbitrary canonical hypergroup. For all $g \in G$ and $n \in \mathbb{Z}$, define the external multiplication $\cdot$ as follows:

$$n \cdot g = \begin{cases} g +_G g +_G \ldots +_G g \quad (n \ times) & \text{if } n > 0, \\ 0_G & \text{if } n = 0, \\ (-g) +_G (-g) +_G \ldots +_G (-g) \quad (-n \ times) & \text{if } n < 0. \end{cases} \tag{4.5}$$

For all $n_1, n_2 \in \mathbb{Z}$ and $g_1, g_2 \in G$, we have the following properties:

(i) $(n_1 \oplus n_2) \cdot g_1 = n_1 \cdot g +_G n_2 \cdot g$,

(ii) $n_1 \cdot (g_1 +_G g_2) = n_1 \cdot g_1 +_G n_1 \cdot g_2$,

(iii) $(n_1 \cdot_{\mathbb{Z}} n_2) \cdot g_1 = n_1 \cdot (n_2 \cdot g_1)$,

(iv) $n_1 \cdot 0_G = 0_{\mathbb{Z}} \cdot g_1 = 0_G$,

(v) $1_{\mathbb{Z}} \cdot g = g$.

Thus, this multiplication turns G into a $\mathbb{Z}$-hypermodule. □

Since $\mathbb{Z}$ is a hyperring with no zero-divisors over the $\mathbb{Z}$-hypermodule $\mathbb{Z}$ for any canonical hypergroup G as a $\mathbb{Z}$-hypermodule, we have the following result.

Theorem 4.8. *Every canonical hypergroup G is a normal injective $\mathbb{Z}$-hypermodule if and only if G is a divisible $\mathbb{Z}$-hypermodule.*

Proof. Let G be a canonical hypergroup. Using Proposition 4.7, we know that G is a $\mathbb{Z}$-hypermodule. Moreover, $Z_{\mathbb{Z}}(G) = \emptyset$, therefore, by using Corollary 4.6, we conclude that G is a divisible $\mathbb{Z}$-hypermodule if and only if G is a normal injective $\mathbb{Z}$-hypermodule. □

Example 4.9. Let R be a commutative Krasner hyperring and let A be a hyperideal of R. Clearly, $\frac{R}{A} = \{r + A \mid r \in R\}$ has a canonical hypergroup structure. We need to provide $\frac{R}{A}$ with a left external map $R \times \frac{R}{A} \longrightarrow \frac{R}{A}$. For this goal, define the multiplication

$$s \cdot (r + A) = (s \cdot_R r) + A,$$

for each $s \in R$ and $r + A \in \frac{R}{A}$. Let $r_1, r_2 \in R$ be such that $r_1 + A = r_2 + A$ in $\frac{R}{A}$, and let $s \in R$. Thus, $r_1 - r_2 \subseteq A$ and we conclude that

$$s \cdot_R (r_1 - r_2) = s \cdot_R r_1 - s \cdot_R r_2 \subseteq A.$$

Therefore, $(s \cdot_R r_1) + A = (s \cdot_R r_2) + A$, which means that the left external map is well-defined. It is routine to check that $\frac{R}{A}$ becomes an R-hypermodule with respect to the left external map.

Proposition 4.7 shows that $\mathbb{Z}$ is a commutative Krasner hyperring with the unit element 1. Therefore, $\frac{\mathbb{Z}}{6\mathbb{Z}}$ is a $\mathbb{Z}$-hypermodule. Moreover, clearly $2 \neq 0$ in $\mathbb{Z}$ and $3 + 6\mathbb{Z} \neq 0_{\frac{\mathbb{Z}}{6\mathbb{Z}}}$, however, $2 \cdot (3 + 6\mathbb{Z}) = 0_{\frac{\mathbb{Z}}{6\mathbb{Z}}}$. Thus, $2 \in \mathbb{Z}$ is a zero-divisor element over $\frac{\mathbb{Z}}{6\mathbb{Z}}$. Similarly, the elements 3 and 4 are zero-divisor elements over $\frac{\mathbb{Z}}{6\mathbb{Z}}$ since $3 \cdot (2 + 6\mathbb{Z}) = 0_{\frac{\mathbb{Z}}{6\mathbb{Z}}}$ and $4 \cdot (3 + 6\mathbb{Z}) = 0_{\frac{\mathbb{Z}}{6\mathbb{Z}}}$. Therefore, $Z_{\mathbb{Z}}(\frac{\mathbb{Z}}{6\mathbb{Z}}) = \{2, 3, 4\}$. Besides, for all non zero-divisor elements over $\frac{\mathbb{Z}}{6\mathbb{Z}}$, it is routine to check that each element of $\frac{\mathbb{Z}}{6\mathbb{Z}}$ is a divisible element. Thus, $\frac{\mathbb{Z}}{6\mathbb{Z}}$ is a divisible $\mathbb{Z}$-hypermodule.

5 Conclusions and future work

In hypercompositional algebra, a homomorphism can be seen as a multivalued function, or as a single-valued one. In the category of Krasner hypermodules, a normal R-homomorphism between two hypermodules M and N over a Krasner hyperring R is a single-valued function preserving both, the hyperaddition and the external multiplication, and it is called normal injective if it is an injective function between M and N. The study of normal injective R-hypermodules have been started in [3], when the authors presented characterizations of this kind of hypermodules using hyperideals and exact chains. The goal of this paper has been to state and prove a new equivalent presentation of normal injective R-hypermodules as a new type of R-hypermodules. For achieving this, we have first defined the notion of zero-divisor element of a Krasner hyperring R over an R-hypermodule and then the one of divisible R-hypermodules. We proved that any normal injective R-hypermodule M is a divisible R-hypermodule whenever R is a hyperring with no zero-divisors over M. The paper ends with a non-trivial construction of a $\mathbb{Z}$-hypermodule structure on a canonical hypergroup.

The research on the injectivity of R-hypermodules helps us to investigate another property that appears also in classical algebra. Mainly, our future goal is to prove that every R-hypermodule can be embedded in a normal injective R-hypermodule. Of course, these properties could be extended also to other classes of hypermodules, as for example the weak hypermodules defined in [11].

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Hashem Bordbar,
Centre for Information Technologies and Applied Mathematics,
University of Nova Gorica,
Vipavska 13, SI-5000 Nova Gorica, Slovenia.
Email: hashem.bordbar@ung.si

Irina Cristea,
Centre for Information Technologies and Applied Mathematics,
University of Nova Gorica,
Vipavska 13, SI-5000 Nova Gorica, Slovenia.
Email: irina.cristea@ung.si