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Prime preideals on bounded EQ-algebras

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Abstract

EQ-algebras were introduced by Novák in [14] as an algebraic structure of truth values for fuzzy type theory (FFT). In [1], Borzooei et. al. introduced the notion of preideal in bounded EQ-algebras. In this paper, we introduce various kinds of preideals on bounded EQ-algebras such as $\wedge$-*prime*, $\otimes$-*prime*, $\cap$-*prime*, $\cap$-*irreducible*, *maximal* and then we investigate some properties and the relations among them. Specially, we prove that in a prelinear and involutive bounded EQ-algebra, any proper preideal is included in a $\wedge$-prime preideal. In the following, we show that the set of all $\wedge$-prime preideals in a bounded EQ-algebra is a T_0 space and under some conditions, it is compact, connected, and Hausdorff. Moreover, we show that the set of all maximal preideals of a prelinear involutive bounded EQ-algebra is an Uryshon (Hausdorff) space and for a finite EQ-algebra, it is T_3 and T_4 space. Finally, we introduce a contravariant functor from the categories of bounded EQ-algebras to the category of topological spaces.

1 Introduction

Fuzzy type theory was developed as a counterpart of the classical higher-order logic. Since the algebra of truth values is no longer a residuated lattice, a specific algebra called an EQ-algebra was proposed by Novák [14] and it continued in [6], [5]. It is proved EQ-algebras overlap with residuated lattices but are not identical with them. Novák and De Baets in [14] introduced

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various kinds of EQ-algebras. El-Zekey in [5] introduced prelinear good EQ-algebras and proved that a prelinear good EQ-algebra is a distributive lattice. The concepts of prefilter and filter on EQ-algebras were defined in [14] and in [6], prime (pre)filter was defined and proved the quotient of prelinear EQ-algebra induced by prime filter is a chain. The other types of (pre)filters of EQ-algebras were studied in [3], [11], [15], [8]. In [16], by using filter, Yang et. al. induced uniform topology on EQ-algebras and proved that topological space is disconnected. Also, in [17], they used filters to construct topological EQ-algebras and proved the binary operations of EQ-algebras are continuous. Ideals theory is a very effective tool for studying various algebraic and logical systems. From logical point of view, various ideals correspond to various of provable formula. The notions of preideals and ideals in EQ-algebras were defined in [1]. In [10], prime and maximal ideals of residuated lattices were introduced and proved the set of all prime (maximal) ideals of a residuated lattice is a compact T_0 (Hausdorff) topological space. With this inspirations, we define prime, $\otimes$-prime, $\cap$-prime, and maximal (pre)ideals of EQ-algebras. We investigate some properties and the relations between them and prove the quotient structures induced by $\wedge$-prime and maximal ideals of a prelinear EQ-algebra is chain or simple, respectively. We show for an EQ-algebra, the set of all $\wedge$-prime preideals of it, is a T_0-topological space and under some conditions, the set of all $\wedge$-prime preideals is a compact, connected and Hausdorff space. Also, we prove that the set of all maximal preideals of a prelinear IEQ-algebra is a Hausdorff and Urysohn topological space and for a finite IEQ-algebra, it is T_3 and T_4-space. Finally, we introduce a contravariant functor from the category of EQ-algebras to the category of topological spaces with continuous maps.

2 Preliminaries

In this section, we gather some basic notions relevant to EQ- algebras which will be needed in the next sections.

Definition 2.1. [6] An *EQ-algebra* is an algebraic structure $\mathcal{E} = (E, \wedge, \otimes, \sim, 1)$ of type $(2, 2, 2, 0)$, where for any $a, b, c, d \in E$, the following statements hold:
($E1$) $(E, \wedge, 1)$ is a $\wedge$-semilattice with top element 1. For any $a, b \in E$, we set $a \leqslant b$ if and only if $a \wedge b = a$.
($E2$) $(E, \otimes, 1)$ is a (commutative) monoid and $\otimes$ is isotone with respect to $\leqslant$.
($E3$) $a \sim a = 1$.
($E4$) $((a \wedge b) \sim c) \otimes (d \sim a) \leqslant (c \sim (d \wedge b))$.
($E5$) $(a \sim b) \otimes (c \sim d) \leqslant (a \sim c) \sim (b \sim d)$.

$(E6)$ $(a \wedge b \wedge c) \sim a \leqslant (a \wedge b) \sim a$.
$(E7)$ $a \otimes b \leqslant a \sim b$.

The operations " $\wedge$ ", " $\otimes$ ", and " $\sim$ " are called *meet*, *multiplication*, and *fuzzy equality*, respectively. For any $a, b \in E$, we defined the binary operation *implication* on E by, $a \to b = (a \wedge b) \sim a$. Also, in particular $1 \to a = 1 \sim a = \tilde{a}$. If E contains a bottom element 0, then we denote it by BEQ-algebra and an unary operation $\neg$ is defined on E by $\neg a = a \sim 0 = a \to 0$.

Let $\mathcal{E} = (E, \wedge, \otimes, \sim, 1)$ be an EQ-algebra. Then $\mathcal{E}$ is called *separated* if for any $a, b \in E$, $a \sim b = 1$ implies $a = b$, *good* if for any $a \in E$ we get $a \sim 1 = a$, *involutive (IEQ-algebra)*, if $\mathcal{E}$ is a BEQ-algebra and for any $a \in E$ we have $\neg\neg a = a$, *lattice-ordered EQ-algebra*, if it has a lattice reduct*, *prelinear EQ-algebra* if for any $a, b \in E$ the set $\{a \to b, b \to a\}$ has the unique upper bound 1, *lattice EQ-algebra (or ℓEQ-algebra)*, if it is a lattice-ordered EQ-algebra and for any $a, b, c, d \in E$,

$$((a \vee b) \sim c) \otimes (d \sim a) \leqslant (d \vee b) \sim c.$$

Definition 2.2. [12] Let $\mathcal{E}$ be a lattice-ordered BEQ-algebra. The set of all $a \in E$ such that $a \vee \neg a = 1$ and $a \wedge \neg a = 0$ is called *Boolean center* of $\mathcal{E}$ and denoted by $B(\mathcal{E})$.

Proposition 2.3. [6] *Let $\mathcal{E}$ be a good EQ-algebra, $\{a_i\}_{i \in I} \subseteq E$ and $c \in E$. Then*

$$(\bigvee_{i \in I} a_i) \to c = \bigwedge_{i \in I} (a_i \to c)$$

Proposition 2.4. [11] *Let $\mathcal{E}$ be an EQ-algebra. Then the following statements are equivalent:*
(i) $\mathcal{E}$ is good,
(ii) $\mathcal{E}$ is separated and $a \to (b \to c) = b \to (a \to c)$, for any $a, b, c \in E$,
(iii) $\mathcal{E}$ is separated and $a \leq (a \to b) \to b$, for any $a, b \in E$.

Theorem 2.5. [5] *Every prelinear and good EQ-algebra $\mathcal{E} = (E, \wedge, \otimes, \sim, 1)$ is an ℓEQ-algebra, where by for any $a, b \in E$, the join operation is given by $a \vee b = ((a \to b) \to b) \wedge ((b \to a) \to a)$.*

Proposition 2.6. [6], [5] *Let $\mathcal{E}$ be an EQ-algebra. Then, for all $a, b, c \in E$, the following properties hold:*
(i) $a \to b = a \to (a \wedge b)$.
(ii) $b \leqslant a \to b$.
(iii) $a \to b \leqslant (c \to a) \to (c \to b)$ and $a \to b \leqslant (b \to c) \to (a \to c)$.

*Given an algebra $< E, F >$, where F is a set of operations on E and $F' \subseteq F$, then the algebra $< E, F' >$ is called the F'-reduct of $< E, F >$

(iv) If $a \leqslant b$, then $c \to a \leqslant c \to b$ and $b \to c \leqslant a \to c$.
(v) If $\mathcal{E}$ is separated, then $a \to b = 1$ if and only if $a \leqslant b$.
(vi) If $\mathcal{E}$ is good, then $a \to (b \to c) \leqslant (a \otimes b) \to c$.
(vii) If $\mathcal{E}$ is good and prelinear, then $(a \wedge b) \to c = (a \to c) \vee (b \to c)$.
(viii) If $\mathcal{E}$ is good and prelinear, then $a \to (b \vee c) = (a \to b) \vee (a \to c)$.

Let $\mathcal{E}$ be an EQ-algebra, $a, b, c \in E$ and $\emptyset \neq F \subseteq E$. Then F is called a *prefilter* of $\mathcal{E}$, if $1 \in F$ and if $a \in F$ and $a \to b \in F$, then $b \in F$, for any $a, b \in \mathcal{E}$. The set of all prefilters of $\mathcal{E}$ is denoted by $\mathcal{PF}(\mathcal{E})$. A prefilter F of $\mathcal{E}$ is called a *filter* of $\mathcal{E}$, if $a \to b \in F$, then $(a \otimes c) \to (b \otimes c) \in F$, for any $a, b, c \in \mathcal{E}$. Proper (pre)filter F is called *prime*, if $a \to b \in F$ or $b \to a \in F$, for any $a, b \in \mathcal{E}$ (See [6], [11]).

Remark 2.7. [14], [6] (*i*) Let F be a (pre)filter of EQ-algebra $\mathcal{E}$. If $a \in F$ and $a \leqslant b$, then $b \in F$.
(*ii*) If $\mathcal{E}$ is a separated EQ-algebra, then $F = \{1\} \subseteq E$ is a filter of $\mathcal{E}$.

Let $\mathcal{E} = (E, \wedge, \otimes, \sim, 0, 1)$ be a BEQ-algebra. For any $a, b \in E$, operation $a \oplus b$ is defined on E by $a \oplus b = \neg a \to b$. Moreover, for any $n \in \mathbb{N}$, we defined $a \oplus (a \oplus \cdots (a \oplus a) \cdots) = na$ and $0a = 0$.

Proposition 2.8. [1] *Let $\mathcal{E}$ be an IEQ-algebra. Then for any $a, b, c \in E$ the following statements hold: (i) $a \oplus b = b \oplus a$.*
(ii) $a \oplus (b \oplus c) = (a \oplus b) \oplus c$.
(iii) If $\mathcal{E}$ is prelinear, then $a \wedge (b \oplus c) \leqslant (a \wedge b) \oplus (a \wedge c)$.
(iv) If $\mathcal{E}$ is prelinear, then for any $n, m \in \mathbb{N}$, $na \wedge mb \leqslant (n + m)(a \wedge b)$.

Let $\mathcal{E} = (E, \wedge, \otimes, \sim, 0, 1)$ be a BEQ-algebra and I be a non-empty subset of E. Then I is called a *preideal* of $\mathcal{E}$, if for any $a, b, c \in E$, (I_1): If $a \leqslant b$ and $b \in I$, then $a \in I$, (I_2): If $a, b \in I$, then $a \oplus b \in I$. A preideal I of $\mathcal{E}$ is called an *ideal* of $\mathcal{E}$, if for any $a, b, c \in E$, (I_3): If $\neg(a \to b) \in I$, then $\neg((a \otimes c) \to (b \otimes c)) \in I$. The set of all preideals of $\mathcal{E}$ is denoted by $\mathcal{PI}(\mathcal{E})$ and the set of all ideals of $\mathcal{E}$ is denoted by $\mathcal{I}(\mathcal{E})$. It is clear that $\mathcal{I}(\mathcal{E}) \subseteq \mathcal{PI}(\mathcal{E})$. (See [1])

Proposition 2.9. [1] *Let $\varphi : \mathcal{E} \to \mathcal{G}$ be an EQ-homomorphism. If $I \in \mathcal{PI}(\mathcal{G})$, then $\varphi^{-1}(I) \in \mathcal{PI}(\mathcal{E})$.*

Theorem 2.10. [1] *Let $\mathcal{E}$ be good and I be a non-empty subset of E. Then I is a preideal of E if and only if $0 \in I$ and for any $a, b \in E$, $\neg(\neg a \to \neg b) \in I$ and $a \in I$ imply $b \in I$.*

Definition 2.11. [1] Let S be a non-empty subset of E. The smallest preideal of $\mathcal{E}$ containing S is called the *generated preideal* by S and it is denoted by $(S]_P$. It is also the intersection of all preideals of $\mathcal{E}$ containing S.

Proposition 2.12. [1] *Let E be an EQ-algebra, $a, b, x \in E$, and $I \in \mathcal{PI}(\mathcal{E})$. Then the following statements hold:*
(*i*) $(x]_P = \{a \in E| \ \ \exists n \in \mathbb{N}$ *such that* $a \leqslant nx\}$.
(*ii*) *If* $a \leqslant b$, *then* $(a]_P \subseteq (b]_P$.
(*iii*) *If* $\mathcal{E}$ *is involutive, then* $(I \cup \{a\}]_P = \{x \in E|x \leqslant na \oplus i$ *for some* $i \in I$ *and* $n \in \mathbb{N}\}$.
(*iv*) *Let* $I_1, I_2 \in \mathcal{PI}(\mathcal{E})$. *If* $\mathcal{E}$ *is involutive, then*

$$I_1 \vee I_2 = (I_1 \cup I_2]_P = \{x \in E|x \leqslant i_1 \oplus i_2 \text{ for some } i_1 \in I_1 \text{ and } i_2 \in I_2\}.$$

(*v*) *If* $\mathcal{E}$ *is involutive, then* $(a]_P \vee (b]_P = (a \oplus b]_P$.
(*vi*) *If* $\mathcal{E}$ *is involutive and prelinear, then* $(a]_P \cap (b]_P = (a]_P \wedge (b]_P = (a \wedge b]_P$.

Let X be a subset of E. The set of all complement elements (with respect to X) is defined by $N(X) = \{x \in E|\neg x \in X\}$.

Proposition 2.13. [1] *Let* $\mathcal{E}$ *be good. Then the following statements hold:*
(*i*) *If* $I \in \mathcal{PI}(\mathcal{E})$, *then* $N(I) \in \mathcal{PF}(\mathcal{E})$.
(*ii*) *If* $F \in \mathcal{PF}(\mathcal{E})$, *then* $N(F) \in \mathcal{PI}(\mathcal{E})$.

Theorem 2.14. [1] *Let* $\mathcal{E}$ *be good,* $I \in \mathcal{PI}(\mathcal{E})$ *and for any* $a, b \in E$, *binary relation* $\approx_I$ *on* E *is defined by* $a \approx_I b$ *if and only if* $\neg(a \sim b) \in I$. *Then*
(*i*) $\approx_I$ *is an equivalence relation on* $\mathcal{E}$.
(*ii*) *If* I *is an ideal of* $\mathcal{E}$, *then* $\approx_I$ *is a congruence relation.*
(*iii*) *If* I *is an ideal of* $\mathcal{E}$, *then* $\mathcal{E}/I = (E/I, \wedge_I, \otimes_I, \sim_I)$ *is a good* BEQ*-algebra, where for any* $a, b \in E$,

$$[a] \wedge_I [b] = [a \wedge b] \quad , \quad [a] \otimes_I [b] = [a \otimes b] \quad , \quad [a] \sim_I [b] = [a \sim b] \quad ,$$
$$[a] \rightarrow_I [b] = [a \rightarrow b].$$

(*iv*) *Let* $\mathcal{E}$ *be good and* $I \in \mathcal{I}(\mathcal{E})$. *Then for any* $a, b \in E$, *the relation* $[a] \leqslant [b]$ *if and only if* $\neg(a \rightarrow b) \in I$. *is an order on* E/I.

Note. From now on, in this paper, $\mathcal{E} = (E, \wedge, \otimes, \sim, 1)$ or simply $\mathcal{E}$ is a BEQ-algebra, unless otherwise state.

3 Prime preideals

In this section, we introduce the notions of various kinds of preideals on BEQ-algebras such as $\wedge$-*prime*, $\otimes$-*prime*, $\cap$-*prime*, $\cap$-*irreducible*, and *maximal preideals*. Also, we investigate some properties and the relations among them.

First, we introduce the concept of $\wedge$-*prime preideals* on BEQ-algebras and we show that the induced quotient structure by a $\wedge$-prime ideal is a chain.

Definition 3.1. Let I be a proper preideal of $\mathcal{E}$. Then I is called a $\wedge$-*prime preideal* of $\mathcal{E}$ if for any $a, b \in E$, $a \wedge b \in I$, satisfies $a \in I$ or $b \in I$.

Note. The notion of $\wedge$- *prime ideal* on BEQ-algebras, can be defined similarly.

Example 3.2. Let $E = \{0, a, b, c, d, e, f, g, 1\}$ be a lattice as Figure 1, and the operations $\otimes$ and $\sim$ are defined on E as Tables 1 and 2.

$\otimes$	0	a	b	c	d	e	f	g	1
0	0	0	0	0	0	0	0	0	0
a	0	0	a	0	0	a	0	0	a
b	0	a	b	0	a	b	0	a	b
c	0	0	0	0	0	0	c	c	c
d	0	0	a	0	0	a	c	c	d
e	0	a	b	0	a	b	c	d	e
f	0	0	0	c	c	c	f	f	f
g	0	0	a	c	c	d	f	f	g
1	0	a	b	c	d	e	f	g	1

Table 1

$\sim$	0	a	b	c	d	e	f	g	1
0	1	g	f	e	d	c	b	a	0
a	g	1	g	d	e	d	a	b	a
b	f	g	1	c	d	e	0	a	b
c	e	d	c	1	g	f	e	d	c
d	d	e	d	g	1	g	d	e	d
e	c	d	e	f	g	1	c	d	e
f	b	a	0	e	d	c	1	g	f
g	a	b	a	d	e	d	g	1	g
1	0	a	b	c	d	e	f	g	1

Table 2

$\to$	0	a	b	c	d	e	f	g	1
0	1	1	1	1	1	1	1	1	1
a	g	1	1	g	1	1	g	1	1
b	f	g	1	f	g	1	f	g	1
c	e	e	e	1	1	1	1	1	1
d	d	e	e	g	1	1	g	1	1
e	c	d	e	f	g	1	f	g	1
f	b	b	b	e	e	e	1	1	1
g	a	b	b	d	e	e	g	1	1
1	0	a	b	c	d	e	f	g	1

Table 3

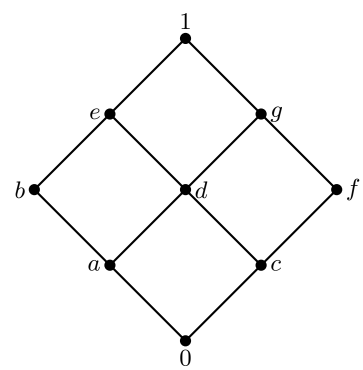


Figure 1

Then $\mathcal{E} = (E, \wedge, \otimes, \sim, 1)$ is a BEQ-algebra and the operation $\to$ is as Table 3. We can see that $I_1 = \{0, a, b\}$, and $I_2 = \{0, c, f\}$ are $\wedge$-prime preideals of $\mathcal{E}$. But $I_3 = \{0\}$ is a preideal of $\mathcal{E}$, which is not $\wedge$-prime. Because $a \wedge c = 0 \in I_3$, but $a, c \notin I_3$.

Proposition 3.3. *Let $\mathcal{E}$ be good and $I \in \mathcal{PI}(\mathcal{E})$ be proper. Then the following statements hold:*
(i) If for any $a, b \in E$, $\neg(a \to b) \in I$ or $\neg(b \to a) \in I$, then I is $\wedge$-prime.
(ii) If $\mathcal{E}$ is prelinear and I is $\wedge$-prime, then for any $a, b \in E$, $\neg(a \to b) \in I$ or $\neg(b \to a) \in I$.
(iii) Let $\mathcal{E}$ be prelinear. If $J \subseteq I$ and J is a $\wedge$-prime preideal of $\mathcal{E}$, then I is $\wedge$-prime, too.

Proof. (i) Let $a, b \in E$ such that $a \wedge b \in I$. Without loss of generality, we suppose $\neg(a \to b) \in I$. By Proposition 2.6(i), we have $a \to b = a \to (a \wedge b)$. Thus by Proposition 2.6(iii) and (iv), we get $a \to (a \wedge b) \leqslant \neg(a \wedge b) \to \neg a$ and so

$$\neg\big(\neg(a \wedge b) \to \neg a\big) \leqslant \neg(a \to (a \wedge b)) = \neg(a \to b) \in I.$$

By (I_1), $\neg\big(\neg(a \wedge b) \to \neg a\big) \in I$ and by Theorem 2.10, $a \in I$. Hence, I is $\wedge$-prime.
(ii) Since $\mathcal{E}$ is prelinear and good, by Theorem 2.5, $\mathcal{E}$ is an ℓEQ-algebra. Thus, for any $a, b \in E$, $(a \to b) \vee (b \to a) = 1$. From Proposition 2.3, we get $\neg(a \to b) \wedge \neg(b \to a) = 0 \in I$. Since I is $\wedge$-prime, we have $\neg(a \to b) \in I$ or $\neg(b \to a) \in I$.
(iii) Since J is $\wedge$-prime, by (ii) for any $a, b \in E$, $\neg(a \to b) \in J$ or $\neg(b \to a) \in J$. Moreover, since $J \subseteq I$, by (i), I a $\wedge$-prime, too. $\square$

Theorem 3.4. *Let $\mathcal{E}$ be good. Then the following statements hold:*
(i) If $\mathcal{E}$ is prelinear and I is a $\wedge$-prime preideals of $\mathcal{E}$, then $N(I)$ is a prime prefilter of $\mathcal{E}$.
(ii) If F is a prime prefilter of $\mathcal{E}$, then $N(F)$ is a $\wedge$-prime preideal of $\mathcal{E}$.

Proof. (i) Let $\mathcal{E}$ be good and I be a $\wedge$-prime preideal of $\mathcal{E}$. By Proposition 2.13(i), $N(I) \in \mathcal{PF}(\mathcal{E})$. By Proposition 3.3($ii$), for any $a, b \in E$, $a \to b \in N(I)$ or $b \to a \in N(I)$. Thus, $N(I)$ is a prime prefilter of $\mathcal{E}$.
(ii) Let F be a prime prefilter of $\mathcal{E}$. Then by Proposition 2.13(ii), we get $N(F) \in \mathcal{PI}(\mathcal{E})$. Since F is a prime prefilter of $\mathcal{E}$, for any $a, b \in E$, we have $a \to b \in F$ or $b \to a \in F$. By Proposition 2.4(iii), we have $a \to b \leqslant \neg\neg(a \to b)$ and $b \to a \leqslant \neg\neg(b \to a)$. By Remark 2.7($i$), for any $a, b \in E$, $\neg\neg(a \to b) \in F$ or $\neg\neg(b \to a) \in F$. Hence, for any $a, b \in E$, we have $\neg(a \to b) \in N(F)$ or $\neg(b \to a) \in N(F)$. Therefore, by Proposition 3.3(i), $N(F)$ is a $\wedge$-prime preideal of $\mathcal{E}$. □

Although, we proved in good EQ-algebras preideals and prefilters are dual of each others, but the most properties of preideals will be proved in a different ways.

Theorem 3.5. *Let $\mathcal{E}$ be good and prelinear and $I \in \mathcal{I}(\mathcal{E})$. Then I is a $\wedge$-prime preideal of $\mathcal{E}$ if and only if $\mathcal{E}/I$ is a chain.*

Proof. Let $a, b \in E$ and I be a $\wedge$-prime preideal of $\mathcal{E}$. Then by Proposition 3.3(ii), we have $\neg(a \to b) \in I$ or $\neg(b \to a) \in I$. By Theorem 2.14(ii), $[a]_I \leqslant [b]_I$ or $[b]_I \leqslant [a]_I$. Hence, $\mathcal{E}/I$ is a chain.
Conversely, suppose $\mathcal{E}/I$ is a chain. Thus, for any $a, b \in E$, we have $[a]_I \leqslant [b]_I$ or $[b]_I \leqslant [a]_I$ and so $\neg(a \to b) \in I$ or $\neg(b \to a) \in I$. Therefore, by Proposition 3.3(i), I is a $\wedge$-prime preideal of $\mathcal{E}$. □

Proposition 3.6. *Let $\mathcal{E}$ be prelinear and involutive. Then the following statements hold:*
(i) If P is a $\wedge$-prime preideal of $\mathcal{E}$, then

$$I_P = \{x \in E | x \wedge y = 0, \text{ for some } y \notin P\}$$

is a preideal of $\mathcal{E}$ and $I \subseteq P$.
(ii) If $I \in \mathcal{PI}(\mathcal{E})$ and $a, b \in E$ such that $a \wedge b \in I$, then $(I \cup \{a\}]_P \cap (I \cup \{b\}]_P = I$.

Proof. (i) Since $1 \notin P$ and $0 \wedge 1 = 0$, we have $0 \in I_P$ and I_P is non-empty. Let $a \leqslant b$ and $b \in I_P$. Then there exists $x \notin P$ such that $b \wedge x = 0$. Thus, $a \wedge x \leqslant b \wedge x = 0$ and $a \in I_P$. Suppose $a, b \in I_P$. Then there exist $x, y \notin P$ such that $a \wedge x = 0$ and $b \wedge y = 0$. Since P is a $\wedge$-prime preideal of $\mathcal{E}$, we have $x \wedge y \notin P$. Thus, by Proposition 2.8(iii), we have

$$(a \oplus b) \wedge (x \wedge y) \leqslant (a \wedge (x \wedge y)) \oplus (b \wedge (x \wedge y)) = 0 \oplus 0 = 0,$$

and so $a \oplus b \in I_P$. Therefore, $I_P \in \mathcal{PI}(\mathcal{E})$. Also, for any $a \in I_P$, there exists $y \in E \setminus P$ such that $a \wedge y = 0 \in P$. Since P is $\wedge$-prime, we obtain $a \in P$ and so $I_P \subseteq P$.
(ii) It is clear that $I \subseteq (I \cup \{a\}]_P \cap (I \cup \{b\}]_P$. Now, let $x \in (I \cup \{a\}]_P \cap (I \cup \{b\}]_P$. Then by Proposition 2.12(iii), there exist $n, m \in \mathbb{N}$ and $i, j \in I$ such that $x \leqslant na \oplus i$ and $x \leqslant mb \oplus j$. Thus, by Proposition 2.8(ii) and (iii), we have

$$\begin{aligned} x &\leqslant (na \oplus i) \wedge (mb \oplus j) \\ &\leqslant (na \wedge mb) \oplus (na \wedge j) \oplus (mb \wedge i) \oplus (i \wedge j) \\ &\leqslant ((n+m)(a \wedge b)) \oplus (na \wedge j) \oplus (mb \wedge i) \oplus (i \wedge j) \in I. \end{aligned}$$

Hence, $x \in I$, and so $(I \cup \{a\}]_P \cap (I \cup \{b\}]_P = I$. □

Definition 3.7. Let S be a non-empty subset of $\mathcal{E}$. Then S is called $\wedge$-*closed*, if $a \wedge b \in S$, for any $a, b \in S$.

Example 3.8. Let $\mathcal{E}$ be an BEQ-algebra as in Example 3.2. Simply $S = \{0, a, c\}$ is a $\wedge$-closed subset of $\mathcal{E}$. But $T = \{d, f, g\}$ is not a $\wedge$-closed subset of $\mathcal{E}$. Because $d \wedge f = c \notin T$.

Theorem 3.9. *Let $\mathcal{E}$ be prelinear and involutive and $I \in \mathcal{PI}(\mathcal{E})$. If S is a non-empty $\wedge$-closed subset of E such that $S \cap I = \emptyset$, then there exists a $\wedge$-prime preideal P such that $I \subseteq P$ and $P \cap S = \emptyset$.*

Proof. Let

$$\mathcal{I}_I = \{J \in \mathcal{PI}(\mathcal{E}) | I \subseteq J \text{ and } S \cap J = \emptyset\}$$

Since $I \in \mathcal{I}_I$, $\mathcal{I}_I \neq \emptyset$. By Zorn's Lemma, $\mathcal{I}_I$ has a maximal element such as P. Clear that $P \in \mathcal{PI}(\mathcal{E})$. Now, we show that P is $\wedge$-prime. By contrary, suppose that there exist $a, b \in E$ such that $a \wedge b \in P$, and $a, b \notin P$. By Proposition 3.6(ii), $(P \cup \{a\}]_P \cap (P \cup \{b\}]_P = P$. Since P is a maximal element of $\mathcal{I}_I$, $(P \cup \{a\}]_P \cap S \neq \emptyset$ and $(P \cup \{b\}]_P \cap S \neq \emptyset$. Now, suppose $s_1 \in (P \cup \{a\}]_P \cap S$ and $s_2 \in (P \cup \{b\}]_P \cap S$. Thus, there exist $n, m \in \mathbb{N}$ and $i, j \in P$ such that $s_1 \leqslant na \oplus i$ and $s_2 \leqslant mb \oplus j$. Since S is a $\wedge$-closed subset of E, we have $s_1 \wedge s_2 \in S$. On the other hand,

$$\begin{aligned} s_1 \wedge s_2 &\leqslant (na \oplus i) \wedge (mb \oplus j) \\ &\leqslant (na \wedge mb) \oplus (na \wedge j) \oplus (mb \wedge i) \oplus (i \wedge j) \\ &\leqslant ((n+m)(a \wedge b)) \oplus (na \wedge j) \oplus (mb \wedge i) \oplus (i \wedge j) \in P. \end{aligned}$$

Hence, $s_1 \wedge s_2 \in P$, and so $P \cap S \neq \emptyset$, which is a contradiction. Therefore, P is $\wedge$-prime. □

Corollary 3.10. *Let $\mathcal{E}$ be prelinear and involutive and $I \in \mathcal{PI}(\mathcal{E})$. Then the following statements hold:*
(i) For any $a \in E \setminus I$, there exists $\wedge$-prime preideal P such that $I \subseteq P$, and $a \notin P$.
(ii) I is the intersection of all $\wedge$-prime preideals of $\mathcal{E}$ which contain I.
(iii) The intersections of all $\wedge$-prime preideals of $\mathcal{E}$ is $\{0\}$

In the follows, we define the notion of $\otimes$-*prime*($\cap$-*prime*) preideals on BEQ-algebras.

Definition 3.11. Let I be a proper preideal of $\mathcal{E}$. Then I is called an
(i) $\otimes$-*prime,* if for any $a, b \in E$, $a \otimes b \in I$ satisfies $a \in I$ or $b \in I$.
(ii) $\cap$-*prime,* if for any $I_1, I_2 \in \mathcal{PI}(\mathcal{E})$, $I_1 \cap I_2 \subseteq I$ satisfies $I_1 \subseteq I$ or $I_2 \subseteq I$.

Example 3.12. (i) Let $\mathcal{E}$ be a BEQ-algebra as in Example 3.2 and $I_1 = \{0\}$, $I_2 = \{0, a, b\}$, and $I_3 = \{0, c, f\}$. Then we can see that I_2 and I_3 are $\cap$-prime preideals of $\mathcal{E}$. But I_3 is a preideals of $\mathcal{E}$ which is not $\otimes$-prime, because $a \otimes d = 0 \in I_3$ and $a, d \notin I_3$. Also, I_1 is a preideals of $\mathcal{E}$ which is not $\cap$-prime, because $I_2 \cap I_3 = I_1$ but $I_2 \nsubseteq I_1$ and $I_3 \nsubseteq I_1$.
(ii) Let $E = \{0, a, b, c, d, e, f, 1\}$ be a lattice as Figure 2, and the operations $\otimes$ and $\sim$ are defined on E as Tables 4 and 5.

$\otimes$	0	a	b	c	d	e	f	1
0	0	0	0	0	0	0	0	0
a	0	0	0	0	0	0	0	a
b	0	0	0	0	0	0	0	b
c	0	0	0	0	0	0	0	c
d	0	0	0	0	d	d	d	d
e	0	0	0	0	d	e	d	e
f	0	0	0	0	d	d	d	f
1	0	a	b	c	d	e	f	1

Table 4

$\sim$	0	a	b	c	d	e	f	1
0	1	e	f	d	c	a	b	0
a	e	1	d	f	c	a	c	a
b	f	d	1	e	c	c	b	b
c	d	f	e	1	c	c	c	c
d	c	c	c	c	1	f	e	d
e	a	a	c	c	f	1	d	e
f	b	c	b	c	e	d	1	f
1	0	a	b	c	d	e	f	1

Table 5

$\rightarrow$	0	a	b	c	d	e	f	1
0	1	1	1	1	1	1	1	1
a	e	1	e	1	1	1	1	1
b	f	f	1	1	1	1	1	1
c	d	f	e	1	1	1	1	1
d	c	c	c	c	1	1	1	1
e	a	a	c	c	f	1	f	1
f	b	c	b	c	e	e	1	1
1	0	a	b	c	d	e	f	1

Table 6

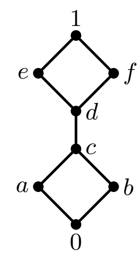


Figure 2

Then $\mathcal{E} = (E, \wedge, \otimes, \sim, 0, 1)$ is a BEQ-algebra and the operation $\rightarrow$ is as Table 6 [14]. Let $I = \{0, a, b, c\}$. It is easy to see that I is an $\otimes$-prime preideal of $\mathcal{E}$.

Proposition 3.13. *Any $\otimes$-prime preideal of $\mathcal{E}$ is $\wedge$-prime.*

Proof. Let I be an $\otimes$-prime preideal of $\mathcal{E}$ and for $a, b \in E$, $a \wedge b \in I$. Since $a \otimes b \leqslant a \wedge b$, we have $a \otimes b \in I$ and so $a \in I$ or $b \in I$. Thus I is $\wedge$-prime. □

In the following example, we show that the converse of Proposition 3.13 may not be true, in general.

Example 3.14. Let $\mathcal{E}$ be a BEQ-algebra as in Example 3.2. Then $I_1 = \{0, a, b\}$ is a $\wedge$-prime preideal of $\mathcal{E}$, while it is not $\otimes$-prime. Because $e \otimes d = a \in I_1$, but $e \notin I_1$ and $d \notin I_1$.

Definition 3.15. Let I be a proper preideal of $\mathcal{E}$. Then I is called an $\cap$-*irreducible*, if for any $I_1, I_2 \in \mathcal{PI}(\mathcal{E})$, $I_1 \cap I_2 = I$ satisfies $I_1 = I$ or $I_2 = I$

Example 3.16. Let $\mathcal{E}$ be a BEQ-algebra as in Example 3.2. Then $I_1 = \{0, a, b\}$ and $I_2 = \{0, c, f\}$ are $\cap$-irreducible preideals of $\mathcal{E}$.

Theorem 3.17. *Let $I \in \mathcal{PI}(\mathcal{E})$. Then the following statements hold:*
(i) If I is $\wedge$-prime, then I is $\cap$-prime.
(ii) If I is $\cap$-prime, then I is $\cap$-irreducible.

Proof. (i) Let $I_1, I_2 \in \mathcal{PI}(\mathcal{E})$ such that $I_1 \cap I_2 \subseteq I$. If $I_1 \nsubseteq I$ and $I_2 \nsubseteq I$, then there exist $a \in I_1 \setminus I$ and $b \in I_2 \setminus I$. Since $a \wedge b \leqslant a, b$, we have $a \wedge b \in I_1 \cap I_2$ and so $a \wedge b \in I$. Also, from I is $\wedge$-prime, we have $a \in I$ or $b \in I$, which is a contradiction. Hence, $I_1 \subseteq I$ or $I_2 \subseteq I$.
(ii) The proof is clear □

In the following example, we can see that the converse of Theorem 3.17 does not hold, in general.

Example 3.18. Let $E = \{0, a, b, c, d, , e, f, m, 1\}$ be a lattice as Figure 3 and the operations $\otimes$ and $\sim$ are defined on E as Tables 7 and 8.

$\otimes$	0	a	b	c	d	e	f	m	1
0	0	0	0	0	0	0	0	0	0
a	0	0	0	0	0	0	0	0	a
b	0	0	0	0	0	0	0	0	b
c	0	0	0	0	0	0	0	0	c
d	0	0	0	0	0	0	0	0	d
e	0	0	0	0	0	0	0	0	e
f	0	0	0	0	0	0	0	0	f
m	0	0	0	0	0	0	0	0	m
1	0	a	b	c	d	e	f	m	1

Table 7

$\sim$	0	a	b	c	d	e	f	m	1
0	1	m	m	m	m	m	m	m	0
a	m	1	m	m	m	m	m	m	a
b	m	m	1	m	m	m	m	m	b
c	m	m	m	1	m	m	m	m	c
d	m	m	m	m	1	m	m	m	d
e	m	m	m	m	m	1	m	m	e
f	m	m	m	m	m	m	1	m	f
m	m	m	m	m	m	m	m	1	m
1	0	a	b	c	d	e	f	m	1

Table 8

$\to$	0	a	b	c	d	e	f	m	1
0	1	1	1	1	1	1	1	1	1
a	m	1	m	1	m	1	m	1	1
b	m	m	1	1	m	m	1	1	1
c	m	m	m	1	m	m	m	1	1
d	m	m	m	m	1	1	1	1	1
e	m	m	m	m	m	1	m	1	1
f	m	m	m	m	m	m	1	1	1
m	m	m	m	m	m	m	m	1	1
1	0	a	b	c	d	e	f	m	1

Table 9

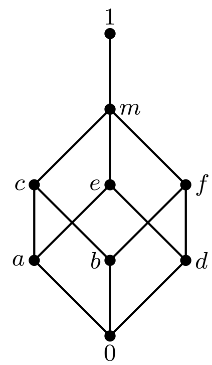


Figure 3

Then $\mathcal{E} = (E, \wedge, \otimes, \sim, 1)$ is a non-involutive BEQ-algebra and the operation $\to$ is as Table 9. It is easy to check that, $I = \{0\}$ is the only proper preideal of $\mathcal{E}$ and so I is $\cap$-prime and $\cap$-irreducible. But I is not $\wedge$-prime, because $a \wedge b = 0 \in I$, but $a, b \notin I$.

Theorem 3.19. *Let $\mathcal{E}$ be prelinear and involutive and $I \in \mathcal{PI}(\mathcal{E})$. If I is $\cap$-irreducible, then I is $\wedge$-prime.*

Proof. Let $a \wedge b \in I$ such that $a, b \notin I$. Then by Proposition 3.6(ii), $(I \cup \{a\}]_P \cap (I \cup \{b\}]_P = I$. Thus $(I \cup \{a\}]_P = I$ or $(I \cup \{b\}]_P = I$. Hence, $a \in I$ or $b \in I$. □

Corollary 3.20. *In prelinear IEQ-algebras, $\wedge$-prime, $\cap$-prime, and $\cap$-irreducible preideals are coincide.*

Proof. By Theorems 3.17 and 3.19, the proof is clear. □

Lemma 3.21. *Let $\mathcal{E}$ be involutive and prelinear and $I, J, K \in \mathcal{PI}(\mathcal{E})$. Then $I \cap J \subseteq K$ if and only if $J \subseteq I \to K$.*

Proof. Let $I \cap J \subseteq K$. If $a \in J$, then $(a]_P \subseteq J$ and so, $(a]_P \cap I \subseteq J \cap I \subseteq K$. Thus, $a \in I \to K$ and so $J \subseteq I \to K$. Conversely, let $a \in I \cap J$. Then $a \in J \subseteq I \to K$ and so $a \in I \to K$. Thus $(a]_P \cap I \subseteq K$ and $(a]_p = (a]_p \cap I \subseteq K$. Hence, $a \in K$ and so $I \cap J \subseteq K$. □

Lemma 3.22. *Let $\mathcal{E}$ be involutive and prelinear and $I, K \in \mathcal{PI}(\mathcal{E})$. Then*

$$I \to K = \sup\{J \in \mathcal{PI}(\mathcal{E}) | I \cap J \subseteq K\}.$$

Proof. The proof is straightforward. □

Theorem 3.23. *Let $\mathcal{E}$ be involutive and prelinear and $P \in \mathcal{PI}(\mathcal{E})$. Then the following statements are equivalent:*
(i) P is $\wedge$-prime,
(ii) for any $a, b \in E \setminus P$, there exists $c \in E \setminus P$ such that $c \leqslant a$ and $c \leqslant b$,
(iii) for any $a, b \in E$, if $(a]_P \cap (b]_P \subseteq P$, then $a \in P$ or $b \in P$,
(iv) for any $I \in \mathcal{PI}(\mathcal{E})$, $I \to P = P$ or $I \subseteq P$.

Proof. (i) $\Rightarrow$ (ii) By the contrary, suppose that for any $c \in E$, such that $c \leqslant a$ and $c \leqslant b$, we consider $c \in P$. Since $a \wedge b \leqslant a, b$, we get $a \wedge b \in P$ and so $a \in P$ or $b \in P$. This is a contradiction.
(ii) $\Rightarrow$ (i) Suppose P is not $\wedge$-prime. Then there exist $I_1, I_2 \in \mathcal{PI}(\mathcal{E})$ such that $I_1 \cap I_2 = P$ and $P \neq I_1$, $P \neq I_2$. There exist $a \in I_1 \setminus P$ and $b \in I_2 \setminus P$. By (ii), there exists $c \in E \setminus P$ such that $c \leqslant a$ and $c \leqslant b$. Thus, $c \in I_1$ and $c \in I_2$ and so $c \in I_1 \cap I_2 = P$. Which is a contradiction. Therefore, P is $\wedge$-prime.
(ii) $\Rightarrow$ (iii) Let $a, b \in E$ such that $(a]_P \cap (b]_P \subseteq P$. By contrary, suppose $a, b \notin P$. Then there exists $c \in E \setminus P$ such that $c \leqslant a$ and $c \leqslant b$. Thus, $c \in (a]_P \cap (b]_P \subseteq P$, which is a contradiction. Hence, $a \in P$ or $b \in P$.
(iii) $\Rightarrow$ (i) Suppose $a \wedge b \in P$. Then $(a \wedge b]_P \subseteq P$ and by Proposition 2.12(vi), we get $(a]_P \cap (b]_P = (a \wedge b]_P \subseteq P$. Hence, by assumption, we have $a \in P$ or $b \in P$ and so P is $\wedge$-prime.
(i) $\Rightarrow$ (iv) Let P is $\wedge$-prime and $I \in \mathcal{PI}(\mathcal{E})$. By lemma 3.22, we have $I \to P = \sup\{J \in \mathcal{PI}(\mathcal{E}) | I \cap J \subseteq P\}$. Since P is $\wedge$-prime, by Theorem 3.17, we have $I \to P = \sup\{J \in \mathcal{PI}(\mathcal{E}) | I \subseteq P$ or $J \subseteq P\}$. Thus, we have $I \subseteq P$ or $I \to P = P$.
(iv) $\Rightarrow$ (i) Let $I, J \in \mathcal{PI}(\mathcal{E})$ such that $I \cap J = P$. By Lemma 3.21, $I \subseteq J \to P$. By (iv), $I \subseteq J \to P = P$ or $J \subseteq P$. Thus, $I \subseteq P$ or $J \subseteq P$ and by Theorem 3.17, P is $\wedge$-prime. □

Finally, we introduce the concept of *maximal preideals* of BEQ-algebras and show the quotient structure induced by the maximal ideal is simple.

Definition 3.24. Let I be a proper preideal of $\mathcal{E}$. Then I is called a *maximal preideal* of $\mathcal{E}$, if I is not strictly contained in a proper preideal of $\mathcal{E}$.

Note. The notion of *maximal ideal* of BEQ-algebras can be defined, similarly.

Example 3.25. Let $\mathcal{E}$ be a BEQ-algebra as in Example 3.2 and $I_1 = \{0\}$, $I_2 = \{0, a, b\}$, and $I_3 = \{0, c, f\}$. Then I_2 and I_3 are maximal preideals of $\mathcal{E}$, but I_1 is not maximal because, $I_1 \subsetneqq I_2$ and $I_1 \subsetneqq I_3$.

Proposition 3.26. *For any proper preideal I, there exists a unique maximal preideal of $\mathcal{E}$ which contains I.*

Proof. Let $\mathcal{I}_I = \{M \in \mathcal{PI}(\mathcal{E}) | M \neq E, I \subseteq M\}$. Since $I \in \mathcal{I}_I$, we have $\mathcal{I}_I \neq \emptyset$. By Zorn's Lemma we get $\mathcal{I}_I$ has a maximal element M which is a maximal preideal of $\mathcal{E}$ and contains I. □

Theorem 3.27. *Let $\mathcal{E}$ be good and $M \in \mathcal{I}(\mathcal{E})$. Then M is maximal if and only if $|\mathcal{I}(\mathcal{E}/\mathrm{M})| = 2$.*

Proof. Let M be a maximal ideal of $\mathcal{E}$. Then for any $I \in \mathcal{PI}(\mathcal{E})$ such that $M \subsetneqq I$, we have $I/M \in \mathcal{I}(\mathcal{E}/\mathrm{M})$. Since M is maximal and $M \subsetneqq I$, we have $I = E$ and so $\mathcal{E}/M$ has only trivial ideals, which are $0/M$ and $\mathcal{E}/M$. Thus, $|\mathcal{I}(\mathcal{E}/\mathrm{M})| = 2$.
Conversely, let $|\mathcal{I}(\mathcal{E}/\mathrm{M})| = 2$. Suppose $I \in \mathcal{I}(\mathcal{E})$ such that $M \subsetneqq I$. If $I \neq E$, then $[0] = M/M \subsetneqq I/M \subsetneqq \mathcal{E}/M$. Thus, $|\mathcal{I}(\mathcal{E}/\mathrm{M})| > 2$ which is a contradiction. Hence, M is a maximal ideal of $\mathcal{E}$. □

Corollary 3.28. *Let $\mathcal{E}$ be good and prelinear. Then any maximal ideal of $\mathcal{E}$ is $\wedge$-prime.*

Proof. Let M be a maximal ideal of $\mathcal{E}$. Then by Theorem 3.27, $\mathcal{E}/I$ has only trivial ideals and so $\mathcal{E}/I$ is a chain. Thus, by Theorem 3.5, I is $\wedge$-prime. □

In the following example, we show the condition in Corollary 3.28, is necessary.

Example 3.29. Let $\mathcal{E}$ be a BEQ-algebra as in Example 3.18. Then $I = \{0\}$ is a maximal preideal of $\mathcal{E}$ and it is not $\wedge$-prime. Also, it is not an ideal of $\mathcal{E}$. Because, $1 \to 0 = 0 \in \{0\}$ but $1 \otimes m \to 0 \otimes m = m \to 0 = m \notin \{0\}$.

Remark 3.30. Let $I \in \mathcal{PI}(\mathcal{E})$. Then for any $a \in E$ and $n \in \mathbb{N}$, $a \in I$ if and only if $na \in I$.

Proposition 3.31. *Let $\mathcal{E}$ be involutive. If M is a proper preideal of $\mathcal{E}$, then the following statements are equivalent:*
(i) M is maximal,
(ii) if $x \notin M$, then there exist $m \in M$ and $n \in \mathbb{N}$ such that $nx \oplus m = 1$,
(iii) for any $x \in E$, $x \notin M$ if and only if for some $n \in \mathbb{N}$, $\neg(nx) \in M$.

Proof. $(i) \Rightarrow (ii)$ If $x \notin M$, then $M \subseteq (M \cup \{x\}]_P$. Since M is maximal, we get $(M \cup \{x\}]_P = E$ and so $1 \in (M \cup \{x\}]_P$. Thus, by Proposition 2.12(iii), there exist $n \in \mathbb{N}$ and $m \in M$ such that $1 \leqslant nx \oplus m$. Hence $nx \oplus m = 1$.
$(ii) \Rightarrow (iii)$ Let $x \notin M$. By (ii), there exist $n \in \mathbb{N}$ and $m \in M$ such that $nx \oplus m = 1$. Thus, by Proposition 2.6(vi), we obtain

$$1 = 1 \to (\neg nx \to m) \leqslant (1 \otimes (\neg nx)) \to m = (\neg nx) \to m.$$

Thus by Proposition 2.6(v), $\neg nx \leqslant m$ and so $\neg nx \in M$.
Now, suppose that for some $n \in \mathbb{N}$, $\neg nx \in M$. Since M is proper, $nx \oplus (\neg nx) = 1 \notin M$ and so $nx \notin M$. Hence, by Remark 3.30, $x \notin M$.
$(iii) \Rightarrow (i)$ Let M' be a proper preideal of $\mathcal{E}$ such that $M \subseteq M'$. If $M \neq M'$, then there exists $x \in M' \setminus M$. From (iii), there exists $n \in \mathbb{N}$, such that $\neg nx \in M \subseteq M'$. By Remark 3.30, $nx \in M'$ and so $1 = (\neg nx) \to (\neg nx) \in M'$. Hence, $M' = E$, which is a contradiction. □

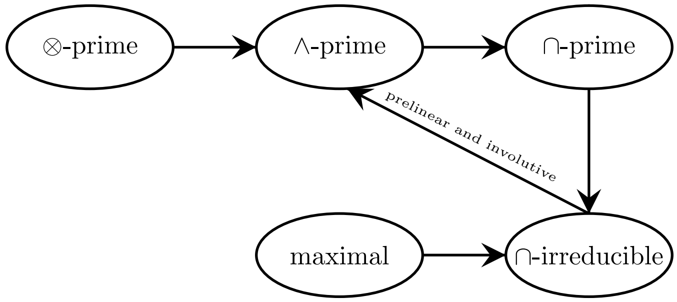
Proposition 3.32. *Let M be a maximal preideal of $\mathcal{E}$. Then the following statements hold:*
(i) M is $\cap$-irreducible.
(ii) If $\mathcal{E}$ is prelinear and involutive, then every maximal preideal of $\mathcal{E}$ is $\wedge$-prime.

Proof. (i) Let M be a maximal preideal of $\mathcal{E}$. If there exist $I, J \in \mathcal{PI}(\mathcal{E})$ such that $M = I \cap J$, then $M \subseteq I$ and $M \subseteq J$. By maximality of M, we have $M = I = J$. Thus, M is $\cap$-irreducible.
(ii) By (i) and Corollary 3.20, we get M is $\wedge$-prime. □

In the following example, we show that the involutive condition in Proposition 3.32(ii), is necessary.

Example 3.33. Let $\mathcal{E}$ be a BEQ-algebra as in Example 3.18. Then $I = \{0\}$ is the only proper preideal of $\mathcal{E}$ and so I is maximal. But I is not a $\wedge$-prime preideal of $\mathcal{E}$. Because $a \wedge b = 0 \in I$, but $a, b \notin I$.

The following diagram shows the relation among maximal, $\wedge$-prime, $\otimes$-prime, $\cap$-prime, and $\cap$-irreducible preideals of an EQ-algebra:



4 Spectrum Topology on BEQ-algebras

We denote the set of all $\wedge$-prime and maximal preideals of $\mathcal{E}$ by $Spec_P(\mathcal{E})$ and $Max_{PI}(\mathcal{E})$, respectively. By this notions, we introduce a spectrum topology on good EQ-algebras and show that $Spec_P(\mathcal{E})$ with this topology is a compact T_0-space. Moreover, we prove that under some conditions $Max_{PI}(\mathcal{E})$ is a Urysohn space.

Recall that a set E with a family τ of subsets of E is called a *topological space*, denoted by (E,τ), if $E,\emptyset \in \tau$, the intersection of any finite members of τ is in τ, and the arbitrary union of members of τ is in τ. The members of τ are called *open sets* of E, and the complement of an open set U, $E \setminus U$, is a *closed set*. A subfamily $\{U_\alpha\}_{\alpha\in I}$ of τ is called a *base* of τ if for each $x \in U \in \tau$ there is an $\alpha \in I$ such that $x \in U_\alpha \subseteq U$. A collection $\{U_\alpha\}_{\alpha\in I}$ of subsets of E is said to be an *open covering* if its elements are open subsets of E and the union of the elements of it is equal to E. The set $X \subseteq E$ is said to be *compact* if every open covering of X contains a finite sub-collection that also covers X. Consider the topological space (E,τ). Then it is called *compact space* if each open covering of E is reducible to a finite open cover, called T_0, if for all $x, y \in E$ and $x \neq y$, there is an open set in E that contains x or y, but not both of them, is called T_1, if for all $x, y \in E$ and $x \neq y$, there are open sets U_1 and U_2 in E such that $x \in U_1$ and $y \in U_2$ but $y \notin U_1$ and $x \notin U_2$, is called T_2, if for all $x, y \in E$ and $x \neq y$, there are two distinct open sets U_1 and U_2 in E such that $x \in U_1$, $y \in U_2$ and $U_1 \cap U_2 = \emptyset$, is called $T_{2\frac{1}{2}}$, if for all $x, y \in E$ and $x \neq y$, there are two distinct close sets V_1 and V_2 in E such that $x \in V_1$, $y \in V_2$ and $V_1 \cap V_2 = \emptyset$, is called T_3, if for all closed subset A and $x \in E \setminus A$, there are two distinct open sets V_1 and V_2 in E such that $x \in V_1$, $A \subseteq V_2$ and $V_1 \cap V_2 = \emptyset$, is called T_4, if for all disjoint closed subsets A, B there are two distinct open sets V_1 and V_2 in E such that $A \subseteq V_1$, $B \subseteq V_2$ and $V_1 \cap V_2 = \emptyset$. The T_2, $T_{2\frac{1}{2}}$, T_3 and T_4-spaces are also known as a *Hausdorff*, *Urysohn*, *regular Hausdorff*, and *normal Hausdorff* spaces, respectively. A topological space (E,τ) is said to be *disconnected* if it is the union of two disjoint non-empty open sets. Otherwise,

E is said to be *connected*.

Definition 4.1. Let $X \subseteq E$. The set of all $\wedge$-prime preideals of $\mathcal{E}$ containing X is denoted by $V(X) = \{P \in Spec_P(\mathcal{E}) | X \subseteq P\}$. For any $a \in E$, we denote $V(\{a\})$ by $V(a)$ and $V(a) = \{P \in Spec_P(\mathcal{E}) | a \in P\}$.

Example 4.2. Let $\mathcal{E}$ be the EQ-algebra as in Example 3.2. Then $Spec_P(\mathcal{E}) = \{\underbrace{\{0, a, b\}}_{I_2}, \underbrace{\{0, c, f\}}_{I_3}\}$. If $X = \{a, b\}$, then $V(X) = \{I_2\}$. Also, $V(d) = \emptyset$ and $V(0) = \{I_2, I_3\}$.

Definition 4.3. Let $X \subseteq E$. Then the complement of $V(X)$ in $Spec_P(\mathcal{E})$ is denoted by $D(X)$. Then

$$D(X) = \{P \in Spec_P(\mathcal{E}) | X \nsubseteq P\}.$$

For any $a \in E$, we denote $D(\{a\})$ by $D(a)$ and $D(a) = \{P \in Spec_P(\mathcal{E}) | a \notin P\}$.

Proposition 4.4. *Let $\mathcal{E}$ be good and $I \in \mathcal{PI}(\mathcal{E})$. Then for any $a, b \in E$, the following statements hold:*
(i) If $a, b \in I$ and $a \vee b$ exists, then $a \vee b \in I$.
(ii) If $\mathcal{E}$ is prelinear, then for any $n \in \mathbb{N}$, $n(a \oplus b) \leqslant 2n(a \vee b)$.
(iii) If $\mathcal{E}$ is prelinear, then $(a \oplus b]_P = (a \vee b]_P$.

Proof. (i) By Proposition 2.6(ii), $b \leqslant \neg a \to b = a \oplus b$. Moreover, since $\mathcal{E}$ is good, by Propositions 2.6(iv) and 2.4(iii), $a \leq \neg\neg a \leqslant \neg a \to b = a \oplus b$. Thus, $a \vee b \leqslant a \oplus b$ and so by (I_1) we have $a \vee b \in I$.
(ii) First, we show $a \oplus b \leqslant 2(a \vee b)$. By Propositions 2.3 and 2.6(vii) and $(viii)$, we have

$$\begin{aligned} 2(a \vee b) = (\neg(a \vee b)) \to (a \vee b) &= (\neg a \wedge \neg b) \to (a \vee b) \\ &= (\neg a \to (a \vee b)) \vee (\neg b \to (a \vee b)) \\ &= (\neg a \to (a \vee b)) \vee ((\neg b \to a) \vee (\neg b \to b)). \end{aligned}$$

By Proposition 2.6(ii), we have $a \oplus b = \neg a \to b \leqslant \neg a \to (a \vee b)$. Thus, $a \oplus b \leqslant 2(a \vee b)$ and so by induction on n, we get $n(a \oplus b) \leqslant 2n(a \vee b)$.
(iii) By (ii), the proof is clear. □

Proposition 4.5. *Let $X, X_i \subseteq E$, for any $i \in \Gamma$. Then for any $i, j \in \Gamma$, the following statements hold:*
(i) If $X_i \subseteq X_j$, then $D(X_i) \subseteq D(X_j)$.
(ii) $D(\{1\}) = D(E) = Spec_P(\mathcal{E})$ and $D(\{0\}) = D(\emptyset) = \emptyset$.
(iii) $\bigcup_{i \in \Gamma} D(X_i) = D(\bigcup_{i \in \Gamma} X_i)$.

(iv) $D(X) = D((X]_P)$.
(v) Let $\mathcal{E}$ *be prelinear and involuitve. If* $D(X) = Spec_P(\mathcal{E})$*, then* $(X]_P = E$.
(vi) Let $\mathcal{E}$ *be prelinear and involutive. If* $D(X_i) = D(X_j)$*, then* $(X_i]_P = (X_j]_P$.
(vii) For any $a, b \in E$, $D(a \wedge b) = D(a) \cap D(b)$.
(viii) $D(X_i) \cap D(X_j) = D((X_i]_P \cap (X_j]_P)$*. Also, if* $I, J \in \mathcal{PI}(\mathcal{E})$*, then* $D(I) \cap D(J) = D(I \cap J)$.
(ix) Let $\mathcal{E}$ *be involutive and prelinear. Then for any* $a, b \in E$,

$$D(a \vee b) = D(a) \cup D(b) = D(a \oplus b).$$

Proof. *(i)* Let $X_i \subseteq X_j$ and $P \in V(X_i)$. Then $X_i \nsubseteq P$ and so $X_j \nsubseteq P$. Thus $P \in D(X_j)$.
(ii) Let $P \in Spec_P(\mathcal{E})$. Since P is a proper preideal of $\mathcal{E}$, we have $1 \notin P$ and so $P \in D(\{1\})$. Also, for any $P \in Spec_P(\mathcal{E})$, we have $E \nsubseteq P$ and so $D(E) \subseteq Spec_P(\mathcal{E})$, For any $P \in Spec(\mathcal{E})$, we have $0 \in P$. Thus $P \notin D(\{0\})$ and $D(\{0\}) = D(\emptyset) = \emptyset$.
(iii) Let $P \in \bigcup\limits_{i \in \Gamma} D(X_i)$. Then there exists $j \in \Gamma$, such that $P \in D(X_j)$ and so $X_j \nsubseteq P$. Thus, $\bigcup\limits_{i \in \Gamma} X_i \nsubseteq P$ and $P \in D(\bigcup\limits_{i \in \Gamma} X_i)$.
Conversely, let $P \in D(\bigcup\limits_{i \in \Gamma} X_i)$. Then $\bigcup\limits_{i \in \Gamma} X_i \nsubseteq P$ and there exists $j \in \Gamma$ such that $X_j \nsubseteq P$. Hence, $P \in D(X_J)$ and so $P \in \bigcup\limits_{I \in \Gamma} D(X_i)$.
(iv) Since $X \subseteq (X]_P$, by *(i)* we have $D(X) \subseteq D((x]_P)$. Let $P \in D((X]_P)$. Then $(X]_P \nsubseteq P$ and so $X \nsubseteq P$. Thus, $P \in D(X)$ and so $D(X) = D((X]_P)$.
(v) Let $D(X) = Spec_P(\mathcal{E})$. Then for any $P \in Spec_P(\mathcal{E})$, $X \nsubseteq P$. Suppose by contrary $(X]_P \neq E$. Thus, there exists $a \in E \setminus (X]_P$ and by Corollary 3.10(*i*), there exists $P \in Spec_P(\mathcal{E})$ such that $(X]_P \subseteq P$, which is a contradiction. Therefore, $(X]_P = E$.
(vi) Let $D(X_i) = D(X_j)$. It is clear that $V(X_i) = V(X_j)$. If $(X_i]_P = E$, then $D(X_i) = D(X_j) = Spec(\mathcal{E})$. By *(v)*, we get $(X_j]_P = E$. If $(X_i]_P$ is a proper preideal of $\mathcal{E}$, then by Corollary 3.10(*ii*),

$$\begin{aligned}(X_i]_P &= \bigcap\{P \in Spec_P(\mathcal{E}) | (X_i]_P \subseteq P\}\\ &= \bigcap\{P \in Spec_P(\mathcal{E}) | P \in V((X_i]_P)\}\\ &= \bigcap\{P \in Spec_P(\mathcal{E}) | (X_j]_P \subseteq P\}\\ &= (X_j]_P.\end{aligned}$$

(vii) Since $(a \wedge b]_P \subseteq (a]_P$ and $(a \wedge b]_P \subseteq (b]_P$, by *(i)* and *(iv)* we have $D(a \wedge b) \subseteq D(a) \cap D(b)$. Conversely, let $P \in D(a) \cap D(b)$. Then $a \notin P$ and $b \notin P$, and so $a \wedge b \notin P$. Thus, $P \in D(a \wedge b)$ and $D(a \wedge b) = D(a) \cap D(b)$.

$(viii)$ Since $(X_i]_P \cap (X_j]_P \subseteq (X_i]_P, (X_j]_P$, by (i) we have $D((X_i]_P \cap (X_j]_P) \subseteq D(X_i) \cap D(X_j)$.
Conversely, let $P \in D(X_i) \cap D(X_j)$. Then $X_i \nsubseteq P$ and $X_j \nsubseteq P$ and so $(X_i]_P \nsubseteq P$ and $(X_j]_P \nsubseteq P$. By Theorem 3.17, we have $(X_i]_P \cap (X_j]_P \nsubseteq P$ and so $P \in D((X_i]_P \cap (X_j]_P)$. Therefore, $D(X_i) \cap D(X_j) = D((X_i]_P \cap (X_j]_P)$. The rest of proof is similar.
(ix) By Proposition 4.4(iii), we have $(a \vee b]_P = (a \oplus b]_P$. Thus, $D(a \vee b) = D(a \oplus b)$. Since $a, b \leqslant a \vee b$, by Proposition 2.12(ii), $(a]_P, (b]_P \subseteq (a \vee b]_P$ and so by (i), we have $D(a) \cup D(b) \subseteq D(a \vee b)$. Let $P \in D(a \vee b)$. Then $a \vee b \notin P$ and by Proposition 4.4(i), we have $a \notin P$ or $b \notin P$. Thus $P \in D(a)$ or $P \in D(b)$ and so $P \in D(a) \cup D(b)$. Hence, $D(a \vee b) = D(a) \cup D(b) = D(a \oplus b)$.
Since $a, b \leqslant a \vee b$, we have $D(a), D(b) \subseteq D(a \vee b)$. Thus, $D(a) \cup D(b) \subseteq D(a \vee b)$. □

Theorem 4.6. *Let $\tau_{\mathcal{E}} = \{D(X)\}_{X \subseteq E}$. Then $\tau_{\mathcal{E}}$ is a topology on $Spec_P(\mathcal{E})$.*

Proof. By Proposition 4.5(ii), (iii), and $(viii)$ the proof is clear. □

Theorem 4.7. *Let $\mathcal{E}$ be good. Then the family $\{D(x)\}_{x \in E}$ is a basis for the topology of $Spec_P(\mathcal{E})$.*

Proof. Let $X \subseteq E$ and $D(X)$ be an open subset of $Spec_P(\mathcal{E})$. Then by Proposition 4.5(iii), $D(X) = D(\bigcup\limits_{x \in X} \{x\}) = \bigcup\limits_{x \in X} D(x)$. Hence, any open subset of $Spec_P(\mathcal{E})$ is the union of subsets from the family $\{D(x)\}_{x \in E}$. □

Example 4.8. Let $\mathcal{E}$ be an BEQ-algebra as in Example 3.2. Then $\tau_{\mathcal{E}} = \{\emptyset, \{I_2\}, \{I_3\}, \{I_2, I_3\}\}$.

Proposition 4.9. *Let $\mathcal{E}$ be involutive and prelinear. Then the following statements hold:*
(i) For any $a \in E$, $D(a)$ is compact in $Spec_P(\mathcal{E})$.
(ii) The compact open subsets of $Spec_P(\mathcal{E})$ are exactly the finite unions of basic open sets.
(iii) The $(Spec_P(\mathcal{E}), \tau_E)$ is compact.

Proof. (i) Let $a \in E$. By Theorem 4.7, there exist $\{a_i\}_{i \in \Gamma} \subseteq E$ such that $D(a) = \bigcup\limits_{i \in \Gamma} D(a_i) = D(\bigcup\limits_{i \in \Gamma} \{a_i\})$. By Theorem 2.5 and Proposition 4.5(vi), we get $(a]_P = (\bigcup\limits_{i \in \Gamma} a_i]_P$ and so $a \in (\bigcup\limits_{i \in \Gamma} a_i]_P$. By Proposition 2.12$(i)$, there exist $i_1, i_2, \cdots, i_n \in \Gamma$ such that $a \leqslant a_{i_1} \oplus a_{i_2} \oplus \cdots \oplus a_{i_n}$. Thus by Proposition 4.5(i) and (ix), we have

$$D(a) \subseteq D(a_{i_1} \oplus a_{i_2} \oplus \cdots \oplus a_{i_n}) = D(a_{i_1}) \cup D(a_{i_2}) \cup \cdots \cup D(a_{i_n})$$

Moreover, since

$$D(a_{i_1} \oplus a_{i_2} \oplus \cdots \oplus a_{i_n}) = D(a_{i_1}) \cup D(a_{i_2}) \cup \cdots \cup D(a_{i_n}) \subseteq \bigcup_{i\in\Gamma} D(a_i) = D(a),$$

we have

$$D(a) = D(a_{i_1} \oplus a_{i_2} \oplus \cdots \oplus a_{i_n}) = D(a_{i_1}) \cup D(a_{i_2}) \cup \cdots \cup D(a_{i_n})$$

and so $D(a)$ is compact.
(ii) Since any basic open set is compact open, we get a finite union of basic open sets is compact open, too. Now, let $D(X)$ be a compact open subset of $Spec_P(\mathcal{E})$. Since $D(X)$ is open, $D(X)$ is a union of basic open sets.
(iii) By Proposition 4.5(ii), $Spec_P(\mathcal{E}) = D(\{1\})$. From (i), we have $Spec_P(\mathcal{E})$ is compact. □

Theorem 4.10. *The* $(Spec_P(\mathcal{E}), \tau)$ *is a* T_0*-topological space.*

Proof. Let $P, Q \in Spec_P(\mathcal{E})$ such that $P \neq Q$. Then $P \nsubseteq Q$ or $Q \nsubseteq P$. Without loss of generality, we can suppose $P \nsubseteq Q$. Thus there exists $a \in P$ such that $a \notin Q$. Let $D = D(a)$. Then $Q \in D$ and $P \notin D$. Hence, $Spec_P(\mathcal{E})$ is T_0-topological space. □

Example 4.11. Let $\mathcal{E}$ be the BEQ-algebra as in Example 4.8. Then $I_2, I_3 \in Spec_P(\mathcal{E})$. Since there is not any open subset $D \in \tau_{\mathcal{E}}$ such that $I_2 \in D$ and $I_3 \notin D$, then $Spec_P(\mathcal{E})$ is not a T_1-space. Also, it is not a Hausdorff space.

Lemma 4.12. *Let* $B(\mathcal{E}) = E$ *and* $P \in Spec_P(\mathcal{E})$. *Then* $a \in P$ *if and only if* $\neg a \notin P$.

Proof. Let $a \in P$. By contrary, suppose $\neg a \in P$. Then $\neg a \to \neg a \in P$, which is a contradiction. Conversely, suppose $\neg a \notin P$. Since for any $a \in E$, $0 = a \wedge \neg a \in P$, we get $a \in P$. □

Theorem 4.13. *Let* $\mathcal{E}$ *be involutive and prelinear. If* $B(\mathcal{E}) = \{0, 1\}$, *then* $(Spec_P(\mathcal{E}), \tau)$ *is connected.*

Proof. Let $(Spec_P(\mathcal{E}), \tau)$ be connected and there exists $a \in B(\mathcal{E})$ such that $a \neq 0, 1$. Since $\mathcal{E}$ is involutive, $\neg a \neq 0, 1$. By Proposition 4.5(ii) and (ix), $D(a), D(\neg a) \neq \emptyset$ and $D(a), D(\neg a) \neq Spec_P(\mathcal{E})$. By Proposition 4.5$(vii)$ and (ix), we get $D(a) \cap D(\neg a) = D(a \wedge \neg a) = \emptyset$ and $D(a) \cup D(\neg a) = D(a \vee \neg a) = Spec_P(\mathcal{E})$. Since $(Spec_P(\mathcal{E}), \tau)$ is connected, we should have $D(a) = \emptyset$ or $D(\neg a) = \emptyset$, which is a contradiction. Therefore, $B(\mathcal{E}) = \{0, 1\}$.

□

Open Problem. In what conditions, the converse of Theorem 4.13 will be true?

Form Proposition 3.32, we know that if $\mathcal{E}$ is involutive and prelinear, then $Max_{PI}(\mathcal{E}) \subseteq Spec_P(\mathcal{E})$. Thus, we can endow $Max_{PI}(\mathcal{E})$ with the topology induced by the topology $\tau_{\mathcal{E}}$ on $Spec_P(\mathcal{E})$. The maximal preideals space of $\mathcal{E}$ is a topological space and denoted by $(\mathcal{M}(\mathcal{E}), \tau_{\mathcal{M}(\mathcal{E})})$. The open and closed sets of $\mathcal{M}(\mathcal{E})$for any $X \subseteq E$ are as follows:

$$D_{max}(X) = D(X) \cap Max_{PI}(\mathcal{E}) = \{P \in Max_{PI}(\mathcal{E}) | X \nsubseteq P\},$$
$$V_{max}(X) = V(X) \cap Max_{PI}(\mathcal{E}) = \{P \in Max_{PI}(\mathcal{E}) | X \subseteq P\}.$$

Also, for any $a \in E$, $D_{max}(a) = D(a) \cap Max_{PI}(\mathcal{E}) = \{P \in Max_{PI}(\mathcal{E}) | a \notin P\}$. The family $\{D_{max}(a)\}_{a \in E}$ is a basis for the induced topology on $\mathcal{M}(\mathcal{E})$. Hence, all the results of Propositions 4.5 hold. Therefore, $\mathcal{M}(\mathcal{E})$ is a compact T_0-space.

Proposition 4.14. *Let $\mathcal{E}$ be prelinear and involutive and $P \in Spec_P(\mathcal{E})$. Then the set $\{P\}$ is closed if and only if $P \in Max_{PI}(\mathcal{E})$.*

Proof. Let $\{P\}$ be a closed in $Spec_P(\mathcal{E})$. Then there exists a proper subset $X \subseteq E$ such that $V(X) = \{P\}$. By Proposition 3.26, there exists a maximal preideal M such that $P \subseteq M$. Thus, $X \subseteq P \subseteq M$ and so by Proposition 3.32(ii), $M \in V(X) = \{P\}$. Hence, $P = M \in Max_{PI}(\mathcal{E})$. Conversely, let $P \in Max_{PI}(\mathcal{E})$. Then $V(P) = \{Q \in Spec_P(\mathcal{E}) | P \subseteq Q \subsetneqq E\} = \{P\}$. Therefore, $\{P\}$ is a closed in $Spec_P(\mathcal{E})$ □

Theorem 4.15. *Let $\mathcal{E}$ be involutive and prelinear. Then the following statements hold:*
(i) *The $\mathcal{M}(\mathcal{E})$ space is Hausdorff.*
(ii) *The $\mathcal{M}(\mathcal{E})$ space is Urysohn.*

Proof. (i) Let $P, Q \in \mathcal{M}(\mathcal{E})$ such that $P \neq Q$. Then $P \nsubseteq Q$ or $Q \nsubseteq P$ and so there exist $a \in P \setminus Q$ or $b \in Q \setminus P$. Let $x = \neg(\neg a \to \neg b)$ and $y = \neg(\neg b \to \neg a)$. Since $a \in P$ and $b \notin P$, then $x \notin P$. Analogously, $y \notin Q$. By Proposition 2.3, we have

$$x \wedge y = \neg(\neg a \to \neg b) \wedge \neg(\neg b \to \neg a) = \neg\big((\neg a \to \neg b) \vee (\neg b \to \neg a)\big) = \neg 1 = 0.$$

Thus, $D_{max}(x) \cap D_{max}(y) = D_{max}(0) = \emptyset$. Also, since $P \in D_{max}(x)$ and $Q \in D_{max}(y)$, we have $\mathcal{M}(\mathcal{E})$ is Hausdorff.
(ii) By (i) and Proposition 4.14, the proof is clear. □

Theorem 4.16. [13] *Every compact subset of a Hausdorff space is closed.*

Proposition 4.17. *Let $\mathcal{E}$ be involutive and prelinear. If $\mathcal{E}$ is finite, then the following statements hold:*
(i) Every open subset in $\mathcal{M}(\mathcal{E})$ is closed.
(ii) Every closed subset in $\mathcal{M}(\mathcal{E})$ is open.

Proof. (*i*) For any $a \in E$, $D(a)$ is compact. Since $\mathcal{E}$ is Hausdorff, by Theorem 4.16 $D(a)$ is closed. Thus for any $X \subseteq E$, $D(X)$ is finite union of closed subset and so $D(X)$ is closed.
(*ii*) Let F be a closed subset of $\mathcal{M}(\mathcal{E})$. Then there exists an open subset D such that $F = \mathcal{M}(\mathcal{E}) \setminus D$. Thus $\mathcal{M}(\mathcal{E}) \setminus F = D$ is open. By (*i*), D is closed and so F is open. □

Theorem 4.18. *Let $\mathcal{E}$ be finite. Then $(\mathcal{M}(\mathcal{E}), \tau_{\mathcal{M}(\mathcal{E})})$ is a normal Hausdorff space.*

Proof. Let F, H be two disjoint closed subsets of $\mathcal{M}(\mathcal{E})$. By Proposition 4.17, F and H are open and so $(\mathcal{M}(\mathcal{E}), \tau_{\mathcal{M}(\mathcal{E})})$ is a normal Hausdorff space. □

Open Problem. We use Proposition 4.17 for proving Theorem 4.18. Unfortunately, we can not either prove Proposition 4.17 or give an counter example for infinite *EQ*-algebras. Thus, we state an open problem: Is there any conditions for an infinite *EQ*-algebra $\mathcal{E}$ such that $(\mathcal{M}(\mathcal{E}), \tau_{\mathcal{M}(\mathcal{E})})$ be a normal Hausdorff space?

The categories of *EQ*-algebras and topological spaces are denoted by $\mathcal{EQ}$ and $\mathcal{T}op$, respectively.

Theorem 4.19. *Let $f : \mathcal{E} \to \mathcal{G}$ be an EQ-homomorphism. Then the following statements hold:*
(i) If $P \in Spec_P(\mathcal{G})$, then $f^{-1}(P) \in Spec_P(\mathcal{E})$.
(ii) Let $\mathcal{S} : \mathcal{EQ} \to \mathcal{T}op$ be a map such that $\mathcal{S}(\mathcal{E}) = Spec_P(\mathcal{E})$. If $\mathcal{S}(f) = f^{-1} : Spec_P(\mathcal{G}) \to Spec_P(\mathcal{E})$, then $\mathcal{S}$ is a contravariant functor.

Proof. (*i*) By Proposition 2.9, $f^{-1}(P) \in \mathcal{PI}(\mathcal{E})$. Since $1 \notin P$, we have $1 \notin f^{-1}(P)$. For any $a, b \in E$, if $a \wedge b \in f^{-1}(P)$, then $f(a \wedge b) = f(a) \wedge f(b) \in P$ and so $f(a) \in P$ or $f(b) \in P$. Thus, $a \in f^{-1}(P)$ or $b \in f^{-1}(P)$. Therefore, $f^{-1}(P) \in Spec_P(\mathcal{E})$.
(*ii*) We prove $\mathcal{S}(f)$ is continuous. Let $D(X)$ be an open set in $Spec_P(\mathcal{E})$. Then

$$\begin{aligned}
(\mathcal{S}(f))^{-1}(D(X)) &= \{Q \in Spec_P(\mathcal{G}) | \mathcal{S}(f)(Q) \in D(X)\} \\
&= \{Q \in Spec_P(\mathcal{G}) | f^{-1}(Q) \in D(X)\} \\
&= \{Q \in Spec_P(\mathcal{G}) | X \nsubseteq f^{-1}(Q)\} \\
&= \{Q \in Spec_P(\mathcal{E}) | f(X) \nsubseteq Q\} \\
&= D(f(X))
\end{aligned}$$

which is an open set in $Spec_P(\mathcal{G})$. Thus, $\mathcal{S}(f)$ is continuous. Also, we show that the following diagram is commutative.

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\quad f \quad} & \mathcal{G} \\ {\scriptstyle \mathcal{S}}\downarrow & & \downarrow{\scriptstyle \mathcal{S}} \\ Spec_P(\mathcal{E}) & \xleftarrow[\mathcal{S}(f)]{} & Spec_P(\mathcal{G}) \end{array}$$

Let $a \in E$. We prove $\mathcal{S}(f)\big(D(f(a))\big) = D(a)$. Suppose $P \in D(a)$, then $a \notin P$ and so $f(a) \notin f(P)$. Thus, $f(P) \in D(f(a))$ and $P \in f^{-1}\big(D(f(a))\big)$. Hence, $D(a) \subseteq \mathcal{S}(f)\big(D(f(a))\big)$. Conversely, let $P \in \mathcal{S}(f)\big(D(f(a))\big)$. Then $P \in f^{-1}(D(f(a)))$ and so $f(P) \in D(f(a))$. Thus, $f(a) \notin f(P)$ and $a \notin P$. Hence, $P \in D(a)$ and so $\mathcal{S}(f)\big(D(f(a))\big) \subseteq D(a)$. Therefore, the diagram is commutative. □

5 Conclusions and future works

In this paper, notions of various preideals in EQ-algebras such as $\wedge$-prime, $\cap$-prime, and maximal are introduced and some properties and relations between them are investigated. It is proved that an ideal in perlinear EQ-algebra is prime (maximal) if and only if the quotient structure induced by it, is chain (simple). In prelinear IEQ-algebras, every maximal preideal is $\wedge$-prime. For any EQ-algebra $\mathcal{E}$, the set of all $\wedge$-prime preideals of $\mathcal{E}$, which is denoted by $Spec_P(\mathcal{E})$ is a T_0-topological space and if $\mathcal{E}$ is a prelinear IEQ-algebra, then $Spec_P(\mathcal{E})$ is compact. Under some conditions, $Spec_P(\mathcal{E})$ is connected or Hausdorff. The set of all maximal preideals of a prelinear IEQ-algebra, which is denoted by $\mathcal{M}(\mathcal{E})$, is a Hausdorff topological space. If $\mathcal{E}$ is a finite and prelinear IEQ-algebra, then $\mathcal{M}(\mathcal{E})$ is a normal Hausdorff space. Finally, a contravariant functor from the category of EQ-algebras to the category of topological spaces is introduced. In the future works, we will study the other topological properties in $Spec_P(\mathcal{E})$ and $Max_{PI}(\mathcal{E})$. Also, we will try to yield a sheaf representations of EQ-algebras.

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