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Class of Sheffer stroke BCK-algebras

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Abstract

In this paper, Sheffer stroke BCK-algebra is defined and its features are investigated. It is indicated that the axioms of a Sheffer stroke BCK-algebra are independent. The relationship between a Sheffer stroke BCK-algebra and a BCK-algebra is stated. After describing a commutative, an implicative and an involutory Sheffer stroke BCK-algebras, some of important properties are proved. The relationship of this structures is demonstrated. A Sheffer stroke BCK-algebra with condition (S) is described and the connection with other structures is shown. Finally, it is proved that for a Sheffer stroke BCK-algebra to be a Boolean lattice, it must be an implicative Sheffer stroke BCK-algebra.

1 Introduction

The study of BCK-algebra was initiated by Imai and Iséki in 1966 [6]. This notion is originated from two different ways. One of the motivations is based on set theory. Another motivation is from classical and non-classical propositional calculi. The BCK-operator $*$ is an analogue of the set theoretical difference. BCK-algebras have been applied to many branches of mathematics such as group theory, functional analysis, probability theory and topology. For the general development of BCK-algebras, the ideal theory plays an important role. Since then quite literature has been produced on the theory of BCK-algebras, especially, emphasis seems to have been put on the ideal theory of BCK-algebras.

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The Sheffer stroke operation, which was first introduced by H. M. Sheffer [18], engages many scientists' attention, because any Boolean function or axiom can be expressed by means of this operation [9]. It reducts axiom systems of many algebraic structures. So, many researchers want to use this operation on their studies. For example, Sheffer stroke non-associative MV-algebras [3] and filters [13], (fuzzy) filters of Sheffer stroke BL-algebras [14], Sheffer stroke Hilbert algebras [11] and filters [12], Sheffer stroke UP-algebras [15], Sheffer stroke BG-algebras [16], Sheffer stroke BE-algebras[17] and Sheffer operation in ortholattices [2] are given as some research on Sheffer stroke operation in recent years.

After giving definitions of a Sheffer operation and a BCK-algebra, by using Sheffer stroke operation, we reduce the axioms of BCK-algebra. This axioms make easier our work. It is proved that the axiom system of a Sheffer stroke BCK-algebra is independent and presented its some properties. Then a partial order on a Sheffer stroke BCK-algebra is determined and it is stated that this algebra has the greatest element 1 and the least element 0. It is demonstrated the relationships between a Sheffer stroke BCK-algebra and a (bounded) BCK-algebra. It is proved that every Sheffer stroke BCK-algebra is a Sheffer stroke BE-algebra. A commutative, an implicative and an involutory Sheffer stroke BCK-algebras are defined, respectively. Some of their properties are shown and the connection of this structures is given. It is indicated that every implicative Sheffer stroke BCK-algebra is a commutative and positive implicative Sheffer stroke BCK-algebra. A Sheffer stroke BCK-algebra with condition (S) is identified and it is stated that every involutory Sheffer stroke BCK-algebra A is with the condition (S). It is presented that if a positive implicative Sheffer stroke BCK-algebra with condition (S) is a lattice, then it must be distributive. The necessary condition for a Sheffer stroke BCK-algebra to be a Boolean lattice is shown.

2 Preliminaries

In this section, we give the fundamental concepts of a Sheffer stroke and a BCK-algebra.

Definition 2.1. *[2] Let $\mathcal{A} = (A, |)$ be a groupoid. The operation $|$ is said to be Sheffer stroke if it satisfies the following conditions:*

(S1) $x|y = y|x$,
(S2) $(x|x)|(x|y) = x$,
(S3) $x|((y|z)|(y|z)) = ((x|y)|(x|y))|z$,
(S4) $(x|((x|x)|(y|y)))|(x|((x|x)|(y|y))) = x$.

Lemma 2.1. *[2] Let $\mathcal{A} = (A, |)$ be a groupoid. The binary relation $\leq$ defined on A as below*

$$x \leq y \Leftrightarrow x|y = x|x$$

is an order on A.

Lemma 2.2. *[2] Let $|$ be Sheffer stroke on A and $\leq$ the induced order of $\mathcal{A} = (A, |)$. Then*
(i) $x \leq y$ if and only if $y|y \leq x|x$,
(ii) $x|(y|(x|x)) = x|x$ is the identity of A,
(iii) $x \leq y$ implies $y|z \leq x|z$,
(iv) $a \leq x$ and $a \leq y$ imply $x|y \leq a|a$.

Definition 2.2. *[7] Let A be a set with a binary operation $*$ and a constant 0. Then $(A, *, 0)$ is called a BCK-algebra if it satisfies the following axioms:*
(BCK-1) $((x * y) * (x * z)) * (z * y) = 0$,
(BCK-2) $(x * (x * y)) * y = 0$,
(BCK-3) $x * x = 0$,
(BCK-4) $x * y = 0$ and $y * x = 0$ imply $x = y$,
(BCK-5) $0 * x = 0$,
for all $x, y, z \in A$.

A partial order $\leq$ on A can be defined by $x \leq y$ if and only if $x * y = 0$.

Definition 2.3. *[1, 7, 10, 19] Let A be a BCK-algebra. Then*
(i) A is called a positive implicative BCK-algebra if $(x*y)*z = (x*z)*(y*z)$,
(ii) A is called an implicative BCK-algebra if $x * (y * x) = x$,
(iii) A is called a commutative BCK-algebra if $x * (x * y) = y * (y * x)$,
(iv) A is called a bounded BCK-algebra, if there exists the greatest element 1 of A and $1 * x$ is denoted by Nx for any $x \in A$,
(v) A is called involutory BCK-algebra, if $NNx = x$ for all $x \in A$.

Definition 2.4. *[5, 8] Let A be a BCK-algebra. Then*
(i) A is said to have condition (S), if the set $A(x, y) = \{t \in A : t * x \leq y\}$ has the greatest element which is denoted by x o y for any $x, y \in A$,
(ii) $(A, *, \leq)$ is called a BCK-lattice, if $(A, \leq)$ is a lattice, where $\leq$ is the partial order on A, which has been introduced in Definition 2.2.

Definition 2.5. *[4] Let P be a set. An order (or partial order) on P is a binary relation $\leq$ on P such that:*
(i) $x \leq x$,
(ii) $x \leq y$ and $y \leq x$ imply $x = y$,
(iii) $x \leq y$ and $y \leq z$ imply $x \leq z$.
for all $x, y, z \in P$. A set P equipped with an order relation $\leq$ is said to be an ordered set.

Definition 2.6. *[4] Let P be a non-empty ordered set. If $x \vee y$ and $x \wedge y$ exist for all $x, y \in P$, then P is called a lattice.*

3 Sheffer stroke BCK-algebras

In this paper, we introduce a Sheffer stroke BCK-algebra and give some properties.

Definition 3.1. *A Sheffer stroke BCK-algebra is a structure $(A, |, 0)$ of type $(2, 0)$ such that 0 is the constant in A, $|$ is a Sheffer operation on A and the following axioms are satisfied for all $x, y, z \in A$*
(sBCK-1) $((((x|(y|y))|(x|(y|y)))|(x|(z|z)))|(((x|(y|y))|(x|(y|y)))|(x|(z|z))))|$ $(z|(y|y)) = 0|0$,
(sBCK-2) $(x|(y|y))|(x|(y|y)) = 0$ *and* $(y|(x|x))|y|(x|x)) = 0$ *imply* $x = y$.

A partial order $\leq$ on A can be defined by

$$x \leq y \Leftrightarrow (x|(y|y))|(x|(y|y)) = 0.$$

A Sheffer stroke BCK-algebra is called bounded if it has the greatest element.

Remark 3.1. *The axioms (sBCK-1) and (sBCK-2) are independent:*

To prove this claim, we construct a model for each axiom in which this axiom is true while the other is false.

Let $U = \{0, 1\}$ be the universe of our model. The symbol $|$ is interpreted as a binary operation on U. Let $(U, |)$ be an algebra.

(1) Independence of (sBCK-1):
We define the operation $|$ on U as in the following Cayley table:

Table 1:

$\vert$	0	1
0	1	0
1	1	0

Then (sBCK-2) holds while (sBCK-1) does not when $x = 0$, $y = 1$ and $z = 1$.
(2) Independence of (sBCK-2):
We define the operation $|$ on U as in the following Cayley table:

Table 2:

$\mid$	0	1
0	0	1
1	0	0

Then (sBCK-1) holds while (sBCK-2) does not when $x = 0$ and $y = 1$. We get $(0|(1|1))|(0|(1|1)) = 0|0 = 0$ and $(1|(0|0))|(1|(0|0)) = 0|0 = 0$ and then $0 \neq 1$.□

Example 3.1. *Consider* $(A, |, 0)$ *with the following Hasse diagram, where* $A = \{0, x, y, z, 1\}$*:*

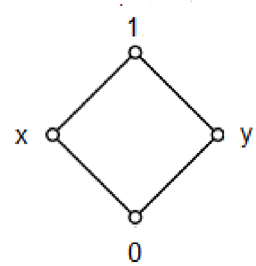


Figure 1:

The binary operation $|$ *has Cayley table as follow:*

Table 3:

$\mid$	0	x	y	1
0	1	1	1	1
x	1	y	1	y
y	1	1	x	x
1	1	y	x	0

Then $(A, |, 0)$ *is a Sheffer stroke BCK-algebra.*

Lemma 3.1. *Let* A *be a Sheffer stroke BCK-algebra. Then the following features hold for all* $x, y, z \in A$*:*
(1) $(x|(x|x))|(x|x) = x$,
(2) $(x|(x|x))|(x|(x|x)) = 0$,

(3) $x|(((x|(y|y))|(y|y))|((x|(y|y))|(y|y))) = 0|0$,
(4) $(0|0)|(x|x) = x$,
(5) $x|0 = 0|0$,
(6) $(x|(0|0))|(x|(0|0)) = x$,
(7) $(0|(x|x))|(0|(x|x)) = 0$,
(8) $x|((y|(z|z))|(y|(z|z))) = y|((x|(z|z))|(x|(z|z)))$,
(9) $((x|(((y|(z|z))|(y|(z|z)))))|((y|(x|(z|z))|(x|(z|z)))|(y|(x|(z|z))|(x|(z|z))))) = 0|0$,
(10) $((x|(x|(y|y)))|(x|(x|(y|y))))|(y|y) = 0|0$.

Proof. (1) Substituting $[y := (x|x)]$ in (S2), we obtain $(x|x)|(x|(x|x)) = x$. Then $(x|(x|x))|(x|x) = x$ from (S1).

(2) In (sBCK-1), by substituting $[y := x|x]$ and $[z := x]$ simultaneously and using (S2), (S3) and (1), we have

$$\begin{aligned}
0|0 &= ((((x|((x|x)|(x|x)))|(x|((x|x)|(x|x))))|(x|(x|x)))|(((x|((x|x)|(x|x)))|(x|((x|x)|(x|x))))|(x|(x|x)))|(x|((x|x)|(x|x))) \\
&= ((((x|x)|(x|x))|(x|(x|x)))|((x|x)|(x|x))|(x|(x|x)))|(x|x) \\
&= ((x|(x|(x|x)))|(x|(x|(x|x))))|(x|x) \\
&= x|(((x|(x|x))|(x|x))|((x|(x|x))|(x|x))) \\
&= x|(x|x).
\end{aligned}$$

From (S2), we obtain $(x|(x|x))|(x|(x|x)) = 0$.

(3) In (S3), by substituting $[y := x|(y|y)]$ and $[z := y|y]$ and applying (S1), (S3) and (2), we obtain

$$\begin{aligned}
x|((x|(y|y))|(y|y))|((x|(y|y))|(y|y)) &= x|(((y|y)|(x|(y|y)))|((y|y)|(x|(y|y)))) \\
&= ((x|(y|y))|(x|(y|y)))|(x|(y|y)) \\
&= (x|(y|y))|((x|(y|y))|(x|(y|y))) \\
&= 0|0.
\end{aligned}$$

(4) $(0|0)|(x|x) = (x|(x|x))|(x|x) = x$ from (1), (2) and (S2).

(5) By using (4), (S1) and (S2),

$$\begin{aligned}
x|0 &= x|((0|0)|(0|0)) \\
&= ((0|0)|(x|x))|((0|0)|(0|0)) \\
&= ((0|0)|(0|0))|((0|0)|(x|x)) \\
&= (0|0).
\end{aligned}$$

(6) By using (S1), (S2) and (4),

$$\begin{aligned}(x|(0|0))|(x|(0|0)) &= ((0|0)|((x|x)|(x|x)))|((0|0)|((x|x)|(x|x))) \\ &= (x|x)|(x|x) \\ &= x.\end{aligned}$$

(7) From (5), (S1) and (S2), we have $(0|(x|x))|(0|(x|x)) = (0|0)|(0|0) = 0$.
(8) By using (S1) and (S3), we have

$$\begin{aligned}x|((y|(z|z))|(y|(z|z))) &= (((x|y)|(x|y))|(z|z)) \\ &= (((y|x)|(y|x))|(z|z)) \\ &= y|((x|(z|z))|(x|(z|z))).\end{aligned}$$

(9) It is obtained from (2) and (8).
(10) It is obtained from (3) and (S3). □

Lemma 3.2. *Let $(A, |, 0)$ be a Sheffer stroke BCK-algebra. A binary relation $\leq$ is defined on A as follows:*

$$x \leq z \quad if\ and\ only\ if \quad (x|(z|z))|(x|(z|z)) = 0.$$

Then the binary relation $\leq$ is a partial order on A such that $0 \leq x$ for each $x \in A$. Moreover, we have

$$y \leq (x|(y|y)) \text{ and } x \leq z \text{ implies } (x|(y|y))|(x|(y|y)) \leq (z|(y|y))|(z|(y|y))$$

for all $x, y, z \in A$.

Proof. • Reflexivity follows from Lemma 3.1 (2).

• Assume that $x \leq y$ and $y \leq x$. Then $(x|(y|y))|(x|(y|y)) = 0$ and $(y|(x|x))|(y|(x|x)) = 0$. We obtain from (sBCK-2) that $x = y$.

• Assume that $x \leq z$ and $z \leq y$. Then $(x|(z|z))|(x|(z|z)) = 0$ and $(z|(y|y))|(z|(y|y)) = 0$. Using (S1), (S2), (sBCK-1) and Lemma 3.1 (4), we get

$$\begin{aligned}0|0 &= ((((x|(y|y))|(x|(y|y)))|(x|(z|z)))|(((x|(y|y))|(x|(y|y)))|(x|(z|z))))| \\ &\quad (z|(y|y)) \\ &= ((((x|(y|y))|(x|(y|y)))|(0|0))|(((x|(y|y))|(x|(y|y)))|(0|0)))|(z|(y|y)) \\ &= ((x|(y|y))|(x|(y|y)))|(0|0) \\ &= (x|(y|y)).\end{aligned}$$

Then $x \leq y$ and so $\leq$ is a partial order on A. From Lemma 3.1 (7), we get $0 \leq x$ for each $x \in A$.

Moreover, assume that $x \leq z$ and $y \in A$. Then

$$\begin{aligned} 0|0 &= ((((x|(y|y))|(x|(y|y)))|(x|(z|z)))|(((x|(y|y))|(x|(y|y)))|(x|(z|z))))| \\ &\quad (z|(y|y)) \\ &= ((((x|(y|y))|(x|(y|y)))|(0|0))|(((x|(y|y))|(x|(y|y)))|(0|0)))|(z|(y|y)) \\ &= (((0|0)|((x|(y|y))|(x|(y|y))))|((0|0)|((x|(y|y))|(x|(y|y)))))|(z|(y|y)) \\ &= ((x|(y|y))|(x|(y|y)))|(z|(y|y)) \\ &= (z|(y|y))|((x|(y|y))|(x|(y|y))), \end{aligned}$$

which means $z|(y|y) \leq x|(y|y)$. From Lemma 2.2 (i), we have $(x|(y|y))|(x|(y|y)) \leq (z|(y|y))|(z|(y|y))$. Putting here $[z := 0|0]$, we obtain $y = (0|0)|(y|y) \leq x|(y|y)$. □

Let A be a Sheffer stroke BCK-algebra. Then $1 = 0|0$ is the greatest element and $0 = 1|1$ is the least element of A.

Proposition 3.1. *Let $(A, |, 0)$ be a Sheffer stroke BCK-algebra. Then the following features are hold for all $x, y, z \in A$:*
(i) $x \leq z$ implies $(y|(z|z))|(y|(z|z)) \leq (y|(x|x))|(y|(x|x))$,
(ii) $((x|(y|y))|(x|(y|y)))|(z|z) = ((x|(z|z))|(x|(z|z)))|(y|y)$,
(iii) $((x|(y|y))|(x|(y|y))) \leq z \Leftrightarrow ((x|(z|z))|(x|(z|z))) \leq y$,
(iv) $(x|(y|y))|(x|(y|y)) \leq x$,
(v) $x \leq y|(x|x)$,
(vi) $x \leq (x|(y|y))|(y|y)$,
(vii) If $x \leq y$, then $z|(x|x) \leq z|(y|y)$.

Proof. (i): Let $x \leq z$. Then by (sBCK-1), we have $(((y|(z|z))|(y|(z|z)))(y|(x|x)))|(((y|(z|z))|(y|(z|z)))|(y|(x|x))) \leq (x|(z|z))|(x|(z|z))$. Hence, $(((y|(z|z))|(y|(z|z)))|(y|(x|x)))|(((y|(z|z))|(y|(z|z)))|(y|(x|x))) \leq 0$. By using Lemma 3.1 (6) and (S2), we have $(((y|(z|z))|(y|(z|z)))(y|(x|x)))|(((y|(z|z))|(y|(z|z)))|(y|(x|x))) = 0$. Therefore, $(y|(z|z))|(y|(z|z)) \leq (y|(x|x))|(y|(x|x))$.
(ii): By Lemma 3.1 (3) and (S3), we have $(x|(x|(z|z)))|(x|(x|(z|z))) \leq z$. Making use of (sBCK-1) and (i), we get $(((x|(y|y))|(x|(y|y)))|(z|z))|(((x|(y|y))|(x|(y|y)))|(z|z)) \leq (((x|(y|y))|(x|(y|y)))|(x|(x|(z|z))))|(((x|(y|y))|(x|(y|y)))|(x|(x|(z|z)))) \leq (((x|(z|z))|(x|(z|z)))|(y|y))|(((x|(z|z))|(x|(z|z)))|(y|y))$. By using Lemma 2.2 (i) and (S2), we have $((x|(z|z))|(x|(z|z)))|(y|y) \leq ((x|(y|y))|(x|(y|y)))|(z|z)$. Interchanging y and z in the above inequality, we obtain $((x|(y|y))|(x|(y|y)))|(z|z) \leq ((x|(z|z))|(x|(z|z)))|(y|y)$. By (sBCK-2), we have $((x|(y|y))|(x|(y|y)))|(z|z) = ((x|(z|z))|(x|(z|z)))|(y|y)$.
(iii): This is a straightforward consequence of (ii).

(iv): By (ii), Lemma 3.1 (2), (7) and (S2), we have $((x|(y|y))|(x|(y|y)))|(x|x) = ((x|(x|x))|(x|(x|x)))|(y|y) = (0|(y|y)) = 0|0$. Consequently, $(x|(y|y))|(x|(y|y)) \leq x$.
(v): By using (S1), (S2), (S3), Lemma 3.1 (2) and (5), we have

$$\begin{aligned} x|((y|(x|x))|(y|(x|x))) &= x|(((x|x)|y)|((x|x)|y)) \\ &= ((x|(x|x))|(x|(x|x)))|y \\ &= 0|y \\ &= y|0 \\ &= 0|0, \end{aligned}$$

that is, $x \leq y(x|x)$.
(vi): By using (S1), (S2), (S3) and Lemma 3.1 (2), we have

$$\begin{aligned} x|(((x|(y|y))|(y|y))|((x|(y|y))|(y|y))) &= x|(((y|y)|(x|(y|y))) \\ &\quad |((y|y)|(x|(y|y)))) \\ &= ((x|(y|y))|(x|(y|y)))|(x|(y|y)) \\ &= (x|(y|y))|((x|(y|y))|(x|(y|y))) \\ &= 0|0, \end{aligned}$$

that is, $x \leq (x|(y|y))|(y|y)$.
(vii): By using (S1), Lemma 2.2 (i) and (iii), we have

$$\begin{aligned} x \leq y &\Leftrightarrow y|y \leq x|x \\ &\Leftrightarrow (x|x)|z \leq (y|y)|z \\ &\Leftrightarrow z|(x|x) \leq z|(y|y). \end{aligned}$$

□

Theorem 3.1. *Let $(A, |, 0)$ be a Sheffer stroke BCK-algebra. If we define*

$$x * y := (x|(y|y))|(x|y|y),$$

then $(A, *, 0)$ is a BCK-algebra.

Proof. By using (S1), (S2), (sBCK-1), (sBCK-2), Lemma 3.1 (2), (7) and (10), we have:

$(BCK-1)$:

$$\begin{aligned}((x*y)*(x*z))*(z*y) &= (((((x|(y|y))|(x|(y|y)))|(x|(z|z)))|((((x|\\ &\quad (y|y)))|(x|(y|y)))|(x|(z|z))))|(z|(y|y)))|\\ &\quad (((((x|(y|y))|(x|(y|y)))|(x|(z|z)))|((((x|\\ &\quad (y|y)))|(x|(y|y)))|(x|(z|z))))|(z|(y|y)))\\ &= (0|0)|(0|0)\\ &= 0.\end{aligned}$$

$(BCK-2)$:

$$\begin{aligned}(x*(x*y))*y &= ((x|(x|(y|y)))|(x|(x|(y|y))))|(y|y)\\ &= (0|0)|(0|0)\\ &= 0.\end{aligned}$$

$(BCK-3)$: $x*x = (x|(x|x))|(x|(x|x)) = 0$.
$(BCK-4)$: $x*y = (x|(y|y))|(x|(y|y)) = 0$ and $y*x = (y|(x|x))|(y|(x|x)) = 0$ imply $x = y$.
$(BCK-5)$: $0*x = (0|(x|x))|(0|(x|x)) = 0$. □

Example 3.2. *Consider the Sheffer stroke BCK-algebra $(A, |, 0)$ in Example 3.1. Then a BCK-algebra $(A, *, 0)$ defined by this Sheffer stroke BCK-algebra has the following Cayley table:*

Table 4:

$*$	0	x	y	1
0	0	0	0	0
x	x	0	x	0
y	y	y	0	0
1	1	y	x	0

Theorem 3.2. *Let $(A, *, 0, 1)$ be a bounded BCK-algebra. If we define $x|y := (x*y^0)^0$ and $x^0 = 1*x$, where $x*(1*x) = x$ and $1*(1*x) = x$, then $(A, |, 0)$ is a Sheffer stroke BCK-algebra.*

Proof. From (BCK-3), we have $1^0 = 1*1 = 0$ and $0^0 = (1^0)^0 = 1*(1*1) = 1$.
$(sBCK-1)$: By using (BCK-1), we have
$((((x|(y|y))|(x|(y|y)))|(x|(z|z)))|((((x|(y|y)))|(x|(y|y)))|(x|(z|z))))|(z|(y|y))$
$= (((x*y)|(x*z)^0)|((x*y)|(x*z)^0))|(z*y)^0$

$= (((x * y) * (x * z))^0 | ((x * y) * (x * z))^0) | (z * y)^0$
$= (((x * y) * (x * z)) * (z * y))^0$
$= 0^0$
$= 1$
$= 0|0.$

$(sBCK-2)$: By using (BCK-2), we get $(x|(y|y))|(x|(y|y)) = x * y = 0$ and $(y|(x|x))|(y|(x|x)) = y * x = 0$ imply $x = y$. □

Example 3.3. *Consider a bounded BCK-algebra $(A, *, 0, 1)$ with $A = \{0, x, y, z, t, u, v, 1\}$ and the binary operation $*$ on A defined as follows:*

Table 5:

$*$	0	x	y	z	t	u	v	1
0	0	0	0	0	0	0	0	0
x	x	0	x	x	0	0	x	0
y	y	y	0	y	0	y	0	0
z	z	z	z	0	z	0	0	0
t	t	y	x	t	0	y	x	0
u	u	z	u	x	z	0	x	0
v	v	v	z	y	z	y	0	0
1	1	v	u	t	z	y	x	0

Then a Sheffer stroke BCK-algebra $(A, |, 0)$ defined by this bounded BCK-algebra $(A, *, 0, 1)$ has the following Cayley table:

Table 6:

$\vert$	0	x	y	z	t	u	v	1
0	1	1	1	1	1	1	1	1
x	1	v	1	1	v	v	1	v
y	1	1	u	1	u	1	u	u
z	1	1	1	t	1	t	t	t
t	1	v	u	1	z	v	u	z
u	1	v	1	t	v	y	t	y
v	1	1	u	t	u	t	x	x
1	1	v	u	t	z	y	x	0

Definition 3.2. *[17] A Sheffer stroke BE-algebra is a structure $(S, |, 1)$ of type $(2, 0)$ such that 1 is the constant in S, $|$ is a Sheffer operation on S and the*

following axioms are satisfied for all $x, y, z \in S$:
$(SBE-1)$ $x|(x|x) = 1$,
$(SBE-2)$ $x|((y|(z|z))|(y|(z|z)) = y|((x|(z|z))|(x|(z|z)))$.

Example 3.4. *[17] Consider a structure $(S, |, 1)$ where $S = \{0, u, v, w, t, 1\}$ and a binary operation $|$ with the following Cayley table:*

Table 7:

$\|$	0	u	v	w	t	1
0	1	1	1	1	1	1
u	1	v	1	1	1	v
v	1	1	u	1	1	u
w	1	1	1	t	1	t
t	1	1	1	1	w	w
1	1	v	u	t	w	0

Then this structure is a Sheffer stroke BE-algebra.

Theorem 3.3. *Every Sheffer stroke BCK-algebra is a Sheffer stroke BE-algebra.*

Proof. It is obtained from Lemma 3.1 (2), (8) and (S2). □

Remark 3.2. *The converse of Theorem 3.3 is not true as in the following example.*

Example 3.5. *Consider the Sheffer stroke BE-algebra $(A, |, 1)$ in Example 3.4. Then S is not a Sheffer stroke BCK-algebra when $x = u, y = v$ and $z = t$, since $((((u|(v|v))|(u|(v|v)))|(u|(t|t)))|(((u|(v|v))|(u|(v|v)))|(u|(t|t))))|(t|(v|v))$ $= v \neq 0|0$.*

Definition 3.3. *Let A be a Sheffer stroke BCK-algebra. Then*
(i) A is called a positive implicative Sheffer stroke BCK-algebra if $((x|(y|y))|(x|(y|y)))|(z|z) = ((x|(z|z))|(x|(z|z)))|(y|(z|z))$,
(ii) A is called an implicative Sheffer stroke BCK-algebra if $x|(y|(x|x)) = x|x$,
(iii) A is called a commutative Sheffer stroke BCK-algebra if $x|(x|(y|y)) = y|(y|(x|x))$,
(iv) A is called a bounded Sheffer stroke BCK-algebra, if there exists the greatest element 1 of A and $(1|(x|x))|(1|(x|x))$ is denoted by Nx for any $x \in A$,
(v) A is called an involutory Sheffer stroke BCK-algebra, if $NNx = x$ for all $x \in A$.

Example 3.6. *Consider the Sheffer stroke BCK-algebra* $(A, |, 0)$ *in Example 3.1. Then* A *is a positive implicative, an implicative, a commutative, a bounded and a involutory Sheffer stroke BCK-algebra.*

Proposition 3.2. *Let* A *be a bounded Sheffer stroke BCK-algebra. Then the following features hold for all* $x, y \in A$*:*
(i) $N1 = 0$ *and* $N0 = 1$*,*
(ii) $(Nx|(Ny|Ny))|(Nx|(Ny|Ny)) \leq (y|(x|x))|(y|(x|x))$*,*
(iii) $y \leq x$ *implies* $Nx \leq Ny$*,*
(iv) $Nx|(y|y) = Ny|(x|x)$*.*

Proof. (i) By using (S1), (S2) and Lemma 3.1 (2), we obtain

$$\begin{aligned} N1 &= (1|(1|1))|(1|(1|1)) \\ &= ((0|0)|((0|0)|(0|0)))|((0|0)|((0|0)|(0|0))) \\ &= ((0|0)|0)|((0|0)|0) \\ &= (0|(0|0))|(0|(0|0)) \\ &= 0. \end{aligned}$$

$$\begin{aligned} N0 &= (1|(0|0))|(1|(0|0)) \\ &= ((0|0)|(0|0))|((0|0)|(0|0)) \\ &= 0|0 \\ &= 1. \end{aligned}$$

(ii) By using (S1), (S2), Lemma 3.1 (2) and (4), we have

$$\begin{aligned} (Nx|(Ny|Ny))|(Nx|(Ny|Ny))|(y|(x|x)) &= (((((0|0)|(x|x))|((0|0)|(x|\\ &\quad x)))|((0|0)|(y|y)))|((((0|\\ &\quad 0)|(x|x))|((0|0)|(x|x)))|\\ &\quad ((0|0)|(y|y))))|(y|(x|x)) \\ &= (((x|x)|y)|((x|x)|y))|(y|(x|x)) \\ &= ((y|(x|x))|(y|(x|x)))|(y|(x|x)) \\ &= 0|0, \end{aligned}$$

which means, $(Nx|(Ny|Ny))|(Nx|(Ny|Ny)) \leq (y|(x|x))|(y|(x|x))$.
(iii) Assume that $y \leq x$. We get $(y|(x|x))|(y|(x|x)) = 0$. By using Lemma 3.1 (4) and (S1), we obtain $(Nx|(Ny|Ny))|(Nx|(Ny|Ny)) = ((x|x)|y)|((x|x)|y) = (y|(x|x))|(y|(x|x)) = 0$. Therefore, $Nx \leq Ny$.
(iv) $Nx|(y|y) = (x|x)|(y|y) = (y|y)|(x|x) = Ny|(x|x)$ from (S1) and Lemma 3.1 (4). □

Theorem 3.4. *Let A be a bounded Sheffer stroke BCK-algebra. Then the following are equivalent for any $x, y \in A$:*
(i) A is involutory,
(ii) $x|(y|y) = Ny|(Nx|Nx)$,
(iii) $x|(Ny|Ny) = y|(Nx|Nx)$,
(iv) $x \leq Ny$ implies $y \leq Nx$.

Proof. (i)⇒ (ii): Since A is involutory, we have $NNx = x$ for all $x \in A$. Then Proposition 3.2 (iv) implies that $x|(y|y) = NNx|(y|y) = Ny|(Nx|Nx)$.
(ii)⇒ (iii): By (ii), $x|(Ny|Ny) = NNy|(Nx|Nx)$ and $y|(Nx|Nx) = NNx|(Ny|Ny)$. Also, by Proposition 3.2 (iv), $NNy|(Nx|Nx) = NNx|(Ny|Ny)$. Therefore, $x|(Ny|Ny) = y|(Nx|Nx)$.
(iii)⇒ (iv): If $x \leq Ny$ then $(x|(Ny|Ny))|(x|(Ny|Ny)) = 0$. So, $(y|(Nx|Nx))|(y|(Nx|Nx)) = 0$ by (iii). Therefore, $y \leq Nx$.
(iv)⇒ (i): It is clear that $NNx \leq x$. Also it is obvious that $Nx \leq Nx$ then (iv) gives $x \leq NNx$. Comparison gives $NNx = x$ for all $x \in A$. Therefore, A is involutory. □

Theorem 3.5. *Let A be an implicative Sheffer stroke BCK-algebra. Then*
(a) A is a commutative Sheffer stroke BCK-algebra.
(b) A is a positive implicative Sheffer stroke BCK-algebra.

Proof. (a): By using Proposition 3.1 (ii), Lemma 3.1 (10), Definition 3.3 (ii), we get

$$\begin{aligned} x|(x|(y|y)) &= ((x|(y|(x|x)))|(x|(y|(x|x))))|(x|(y|y)) \\ &= ((x|(x|(y|y)))|(x|(x|(y|y))))|(y|(x|x)) \\ &= y|(y|(x|x)) \end{aligned}$$

Therefore, A is a commutative Sheffer stroke BCK-algebra.
(b): Substituting $[x := (x|(y|y))|(x|(y|y))]$ in the identity $x|x = x|(y|(x|x))$ and by using (S1), (S2), Lemma 3.1 (2) and (4), we have,

$$\begin{aligned} x|(y|y) &= ((x|(y|y))|(x|(y|y)))|(y|(x|(y|y))) \\ &= ((x|(y|y))|(x|(y|y)))|(y|y) \end{aligned}$$

Therefore, A is a positive implicative Sheffer stroke BCK-algebra. □

Theorem 3.6. *Let A be a both commutative and positive implicative Sheffer stroke BCK-algebra. Then A is an implicative Sheffer stroke BCK-algebra.*

Proof. From Proposition 3.1 (v), we have $x \leq y|(x|x)$.
By using Proposition 3.1 (vii), Lemma 2.2 (i) and (S2), we get

$\Rightarrow (y|(x|x))|(y|(x|x)) \leq x|x$.
$\Rightarrow x|(((y|(x|x))|(y|(x|x)))|((y|(x|x))|(y|(x|x)))) \leq x|((x|x)|(x|x))$
$\Rightarrow x|(y|(x|x)) \leq x|x$.
Therefore, $(x|(y|(x|x)))|x = x|(x|(y|(x|x))) = 0|0$.
By using Lemma 3.1 (2), (4), (5), (6), (S2), Definition 3.3 (i), (ii), (iii), we obtain

$$\begin{aligned}
(x|(y|(x|x)))|(x|(y|(x|x))) &= (x|(y|(x|x)))|(0|0) \\
&= ((x|(y|(x|x)))|((x|(y|(x|x)))|x) \\
&= ((x|(y|(x|x)))|((x|(y|(x|x)))|((x|x)|(x|x))) \\
&= (x|x)|((x|x)|((x|(y|(x|x)))|(x|(y|(x|x))))) \\
&= (x|x)|(((x|(y|(x|x)))|(x|(y|(x|x))))|(x|x)) \\
&= (x|x)|(((x|(x|x))|(x|(x|x)))|((y|(x|x))|(x|x))) \\
&= (x|x)|(0|((y|(x|x))|(x|x))) \\
&= (x|x)|(0|0) \\
&= x.
\end{aligned}$$

From (S2), $x|(y|(x|x)) = x|x$. Then A is an implicative Sheffer stroke BCK-algebra.

□

Definition 3.4. *Let A be a Sheffer stroke BCK-algebra. Then*
(i) A is said to have condition (S) if the set $A(x, y) = \{t \in A : (t|(x|x))|(t|(x|x)) \leq y\}$ has the greatest element which is denoted by $x \circ y$ for any $x, y \in A$. Moreover,

$$((x|(y|y))|(x|(y|y)))|(z|z) = (x|((y \circ z)|(y \circ z))),$$

for all $x, y, z \in A$,
(ii) $(A, |, \leq)$ is called a Sheffer stroke BCK-lattice, if $(A, \leq)$ is a lattice, where $\leq$ is the partial order on A defined as in Definition 3.1.

Example 3.7. *Consider the Sheffer stroke BCK-algebra $(A, |, 0)$ in Example 3.1. Then $(A, |, 0)$ is a Sheffer stroke BCK-algebra with condition (S) where $x \circ y = x \vee y$. Moreover, $(A, |, \leq)$ is a Sheffer stroke BCK-lattice.*

Proposition 3.3. *If A is a bounded Sheffer stroke BCK-algebra. Then A satisfies condition (S). In this case $x \circ y = (1|(((1|(x|x))|(1|(x|x)))|(y|y)))|(1|(((1|(x|x))|(1|(x|x)))|(y|y)))$.*

Proof. Define $x \circ y = (1|(((1|(x|x))|(1|(x|x)))|(y|y)))|(1|(((1|(x|x))|(1|(x|x)))|(y|y)))$, for all $x, y \in A$. Then by using Proposition 3.1 (ii) and Lemma 3.1

(3), we have

$$\begin{aligned}((x\circ y)|(x|x))|((x\circ y)|(x|x)) &= (((1|(((1|(x|x))|(1|(x|x)))|(y|y)))|(1\\ &\quad |(((1|(x|x))|(1|(x|x)))|(y|y))))|(x|x))\\ &\quad |(((1|(((1|(x|x)))|(1|(x|x))|(y|y)))|(1|\\ &\quad (((1|(x|x))|(1|(x|x)))|(y|y))))|(x|x))\\ &= (((1|(x|x))|(1|(x|x)))|(((1|(x|x))|\\ &\quad (1|(x|x)))|(y|y)))|(((1|(x|x))|(1|(x\\ &\quad |x)))|(((1|(x|x))|(1|(x|x)))|(y|y)))\\ &\leq y.\end{aligned}$$

For $z \in A$, by using (S2), (S3) and Lemma 3.1 (4), we have

$$\begin{aligned}z|((x\circ y)|(x\circ y)) &= z|(1|(((1|(x|x))|(1|(x|x)))|(y|y)))\\ &= z|((0|0)|((((0|0)|(x|x))|((0|0)|(x|x)))|(y|y)))\\ &= z|(((x|x)|(y|y))|(x|x)|(y|y))\\ &= ((z|(x|x))|(z|(x|x)))|(y|y).\end{aligned}$$

Hence, A satisfies condition (S). □

Theorem 3.7. *Every involutory Sheffer stroke BCK-algebra A is with the condition (S).*

Proof. Suppose that 1 is the greatest element of A and $x, y, z \in A$. Because A is involutory, we have

$$Nx|(Ny|Ny) = y|(x|x) \tag{1}$$

by Theorem 3.4. We define " $\circ$ " as follows:

$$x\circ y = N((Nx|(y|y))|(Nx|(y|y))).$$

Using the involutory property of x and Equation (1) as well as Proposition 3.1 (ii), we obtain

$$\begin{aligned}(x|((y\circ z)|(y\circ z))) &= NNx|(N((Ny|(z|z))|(Ny|(z|z)))\\ &\quad |(N((Ny|(z|z))|(Ny|(z|z)))))\\ &= ((Ny|(z|z))|(Ny|(z|z)))|(Nx|Nx)\\ &= ((Ny|(Nx|Nx))|(Ny|(Nx|Nx)))|(z|z)\\ &= ((x|(y|y))|(x|(y|y)))|(z|z).\end{aligned}$$

Therefore, A is with condition (S). □

Remark 3.3. *Let A be a bounded Sheffer stroke BCK-algebra. Then every commutative Sheffer stroke BCK-algebra is an involutory Sheffer stroke BCK-algebra.*

Corollary 3.1. *Every bounded commutative Sheffer stroke BCK-algebra satisfies condition (S).*

Proof. It is obtained from Theorem 3.7 and Remark 3.3. □

Corollary 3.2. *Any bounded implicative Sheffer stroke BCK-algebra satisfies condition (S) and*

$$x \circ y = x \vee y.$$

Indeed, it is possible to show that a least upper bound of x and y, $x \vee y$, exists in A and $x \vee y = (1|(((1|(x|x))|(1|(x|x)))|(y|y)))|(1|(((1|(x|x))|(1|(x|x)))|(y|y)))$.

Theorem 3.8. *Let A be a positive implicative Sheffer stroke BCK-algebra with condition (S). If $(A, \leq)$ is a lattice, it must be distributive.*

Proof. From the theory of lattices, a lattice is distributive if and only if it contains neither a rhombus sublattice nor a pentagon sublattice. Assume that the lattice $(A, \leq)$ contains either a rhombus sublattice or a pentagon sublattice whose Hasse diagrams are respectively assumed as follows:

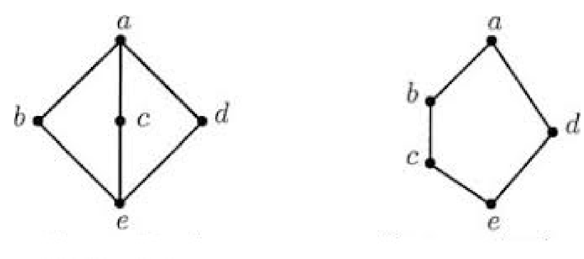


Figure 2:

For the first diagram, we have $b \vee c = a$ and $b \vee d = a$, which means from Corollary 3.2 that $b \circ c = a$ and $b \circ d = a$. Then we have from Definition 3.4 and Lemma 3.1 (2) that

$$\begin{aligned} ((a|(b|b))|(a|(b|b)))|(c|c) &= a|((b \circ c)|(b \circ c)) \\ &= a|(a|a) \\ &= 0|0. \end{aligned}$$

Namely, $((a|(b|b))|(a|(b|b))) \leq c$. Likewise, $((a|(b|b))|(a|(b|b))) \leq d$. So, $((a|(b|b))|(a|(b|b))) \leq c \wedge d$. Noticing, $c \wedge d = e$, it follows $((a|(b|b))|(a|(b|b))) \leq e$. Also, since $e \leq b$ by Corollary 3.2, $b \circ e = b \vee e = b$. Now, Definition 3.4 gives

$$\begin{aligned} ((a|(b|b))|(a|(b|b))) &= (a|((b \circ e)|(b \circ e)))|(a|((b \circ e)|(b \circ e))) \\ &= (((a|(b|b))|(a|(b|b)))|(e|e))|(((a|(b|b))|(a|(b|b)))|(e|e)) \\ &\leq e|(e|e))|(e|(e|e)) \\ &= 0. \end{aligned}$$

Therefore, $a \leq b$ which is a contradiction with $a > b$.

For the second diagram, we have $c \vee d = a$. Then Corollary 3.2 implies that $c \circ d = a$. Applying Definition 3.4 and the fact that $b \leq a$, we derive

$$\begin{aligned} ((b|(c|c))|(b|(c|c)))|(d|d) &= b|((c \circ d)|(c \circ d)) \\ &= (b|(a|a)) \\ &= 0|0. \end{aligned}$$

That is $((b|(c|c))|(b|(c|c))) \leq d$. Also, by Proposition 3.1 (iv), $((b|(c|c))|(b|(c|c))) \leq b$. Then $((b|(c|c))|(b|(c|c))) \leq b \wedge d = e$ and so $((b|(c|c))|(b|(c|c)))|(e|e) = 0|0$. Using Corollary 3.2 again, it follows $b|((c \circ e)|(c \circ e)) = 0|0$. Next, because $e \leq c$, $c \circ e = e \vee c = c$. Hence

$$\begin{aligned} ((b|(c|c))|(b|(c|c))) &= (b|((c \circ e)|(c \circ e)))|(b|((c \circ e)|(c \circ e))) \\ &= (((b|(c|c))|(b|(c|c)))|(e|e))|(((b|(c|c))|(b|(c|c)))|(e|e)) \\ &\leq (e|(e|e))|(e|(e|e)) \\ &= 0. \end{aligned}$$

Therefore, $b \leq c$, which is impossible since $b > c$. The proof is complete. □

Lemma 3.3. *Let A be a Sheffer stroke BCK-lattice. Then* $(x|((y \wedge z)|(y \wedge z)))|(x|((y \wedge z)|(y \wedge z))) = ((x|(y|y))|(x|(y|y))) \vee ((x|(z|z))|(x|(z|z)))$*, for any* $x, y, z \in A$.

Proof. Suppose that A is a Sheffer stroke BCK-lattice and $x, y, z \in A$. Since $y \wedge z \leq y$ and $y \wedge z \leq z$, by Proposition 3.1 (i), we obtain $((x|(z|z))|(x|(z|z))) \leq (x|((y \wedge z)|(y \wedge z)))|(x|((y \wedge z)|(y \wedge z)))$ and $((x|(y|y))|(x|(y|y))) \leq (x|((y \wedge z)|(y \wedge z)))|(x|((y \wedge z)|(y \wedge z)))$. Hence, $((x|(y|y))|(x|(y|y))) \vee ((x|(z|z))|(x|(z|z))) \leq (x|((y \wedge z)|(y \wedge z)))|(x|((y \wedge z)|(y \wedge z)))$.
Now, since $(x|(y|y))|(x|(y|y)) \leq ((x|(y|y))|(x|(y|y))) \vee ((x|(z|z))|(x|(z|z)))$, by Proposition 3.1 (i) and Lemma 3.1 (3), we get that $(x|((((x|(y|y))|(x|(y|y))) \vee((x|(z|z))|(x|(z|z))))|(((x|(y|y))|(x|(y|y))) \vee ((x|(z|z))|(x|(z|z))))))|(x|((((x|$

$(y|y))|(x|(y|y))) \vee ((x|(z|z))|(x|(z|z))))|(((x|(y|y))|(x|(y|y))) \vee ((x|(z|z))|(x|(z|z)))))) \leq (x|(x|(y|y)))|(x|(x|(y|y))) \leq y$ and z. Hence $(x|((((x|(y|y))|(x|(y|y))) \vee ((x|(z|z))|(x|(z|z))))|(((x|(y|y))|(x|(y|y))) \vee ((x|(z|z))|(x|(z|z))))))|(x|((((x|(y|y))|(x|(y|y))) \vee ((x|(z|z))|(x|(z|z))))|(((x|(y|y))|(x|(y|y))) \vee ((x|(z|z))|(x|(z|z)))))) \leq y \wedge z$. By Proposition 3.1 (iii), we conclude that $(x|((y \wedge z)|(y \wedge z)))|(x|((y \wedge z)|(y \wedge z))) \leq ((x|(y|y))|(x|(y|y))) \vee ((x|(z|z))|(x|(z|z)))$. Therefore, $(x|((y \wedge z)|(y \wedge z)))|(x|((y \wedge z)|(y \wedge z))) = ((x|(y|y))|(x|(y|y))) \vee ((x|(z|z))|(x|(z|z)))$. □

Lemma 3.4. *Let A be a bounded Sheffer stroke BCK-algebra and $x, y \in A$.*
(1) If the greatest lower bound $x \wedge y$ of x and y exists, then least upper bound $Nx \vee Ny$ of Nx and Ny exists and $Nx \vee Ny = N(x \wedge y)$.
(2) If A is involutory and if the least upper bound $x \vee y$ exists, then the greatest lower bound $Nx \wedge Ny$ exists and $Nx \wedge Ny = N(x \vee y)$.

Proof. It is known from Lemma 3.3 that if the greatest lower bound $x \wedge y$ of x and y exists, then for any $z \in A$, the least upper bound $((z|(x|x))|(z|(x|x))) \vee ((z|(y|y))|(z|(y|y)))$ exists and $(z|(x|x))|(z|(x|x))) \vee ((z|(y|y))|(z|(y|y))) = ((z|((x \wedge y)|(x \wedge y)))|(z|((x \wedge y)|(x \wedge y))))$.
(1) Assume that z is the greatest element of A. If $x \wedge y$ exists, then $((z|(x|x))|(z|(x|x))) \vee ((z|(y|y))|(z|(y|y)))$ exists and $(z|(x|x))|(z|(x|x))) \vee ((z|(y|y))|(z|(y|y))) = ((z|((x \wedge y)|(x \wedge y)))|(z|((x \wedge y)|(x \wedge y))))$. Because $((z|(x|x))|(z|(x|x))) = Nx$, it yields that $Nx \vee Ny$ exists and $Nx \vee Ny = N(x \wedge y)$.
(2) If $x \vee y$ exists, since $x \leq x \vee y$ and $y \leq x \vee y$, it follows from Proposition 3.2 (iii) that $N(x \vee y) \leq Nx$ and $N(x \vee y) \leq Ny$. Hence, $N(x \vee y)$ is a lower bound of Nx and Ny. Also let z be any lower bound of Nx and Ny. Since $z \leq Nx$ and $z \leq Ny$ by A being involutory, Theorem 3.4 (iv) gives $x \leq Nz$ and $y \leq Nz$. So, $x \vee y \leq Nz$. Using Theorem 3.4 (iv) once more, we get $z \leq N(x \vee y)$. Hence, $N(x \vee y)$ is the greatest lower bound of Nx and Ny. Therefore, $Nx \wedge Ny$ exists and $Nx \wedge Ny = N(x \vee y)$. □

Theorem 3.9. *Let A be an involutory Sheffer stroke BCK-algebra. Then the following are equivalent:*
(1) $(A, \leq)$ is a lower semilattice,
(2) $(A, \leq)$ is an upper semilattice,
(3) $(A, \leq)$ is a lattice.

Moreover, Sheffer stroke BCK-lattice $(A, \leq)$ is a distributive lattice, where

$$x \wedge y = N(Nx \vee Ny) \text{ and } x \vee y = N(Nx \wedge Ny).$$

Proof. (1) ⇒ (2): Since $(A, \leq)$ is a lower semilattice, $Nx \wedge Ny$ exists for any $x, y \in A$. Then Lemma 3.4 (1) gives that $NNx \vee NNy$ exists. Also, since A is involutory, we have $NNx \vee NNy = x \vee y$. Hence $x \vee y$ exists and $(A, \leq)$ is

an upper semilattice.
(2) $\Rightarrow$ (3): Because $(A, \leq)$ is an upper semilattice by using Lemma 3.4 (2) and following the preceding proof, we obtain that $(A, \leq)$ is a lower semilattice. Therefore, $(A, \leq)$ is a lattice.
(3) $\Rightarrow$ (1): Obvious.
Moreover, if $(A, \leq)$ is a lattice, then we have from Lemma 3.4 that

$$NNx \wedge NNy = N(Nx \vee Ny) \text{ and } NNx \vee NNy = N(Nx \wedge Ny).$$

for all $x, y \in A$. So, we derive

$$x \wedge y = N(Nx \vee Ny) \text{ and } x \vee y = N(Nx \wedge Ny)$$

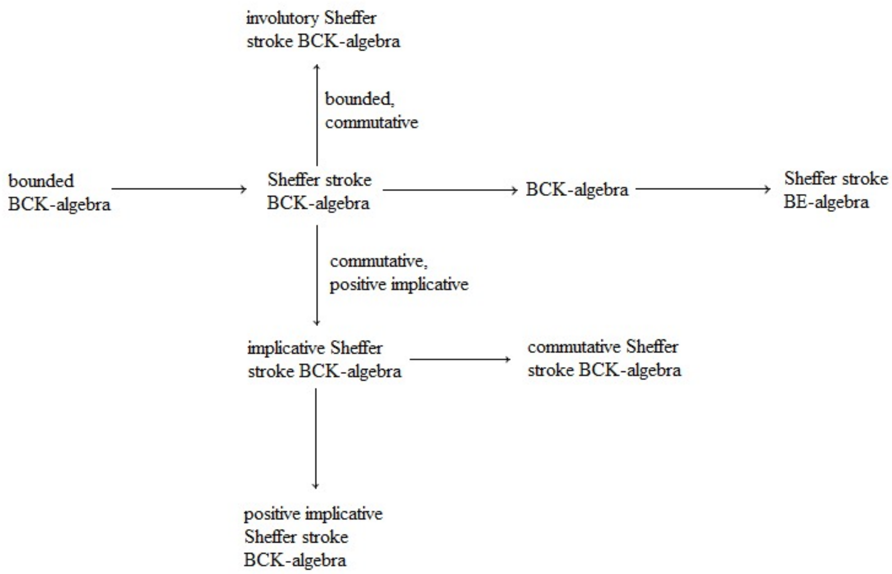
by the involution. □

Corollary 3.3. *Let A be a bounded Sheffer stroke BCK-algebra. Then any implicative Sheffer stroke BCK-algebra is a Boolean lattice.*

Proof. The proof is obtained from Theorem 3.5, Remark 3.3, Theorem 3.8 and Theorem 3.9. □

4 Conclusion

In this study, a Sheffer stroke BCK-algebra, a partial order, a commutative, an implicative, an involutory Sheffer stroke BCK-algebra and their some properties are investigated. By presenting definitions of a Sheffer stroke and a BCK-algebra, a Sheffer stroke BCK-algebra is introduced and related notions are given. It is proved that the axiom system of a Sheffer stroke BCK-algebra is independent. It is stated the relationships between a Sheffer stroke BCK-algebra and a (bounded) BCK-algebra. It is proved that every Sheffer stroke BCK-algebra is a Sheffer stroke BE-algebra. A commutative, an implicative and an involutory Sheffer stroke BCK-algebras are defined and the relationship of this structures is given. It is indicated that every implicative Sheffer stroke BCK-algebra is a commutative and a positive implicative Sheffer stroke BCK-algebra. A Sheffer stroke BCK-algebra with condition (S) is identified. It is presented that if a positive implicative Sheffer stroke BCK-algebra with condition (S) is a lattice, then it must be distributive. Finally, it is shown that any implicative Sheffer stroke BCK-algebra is a Boolean lattice.



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