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# Qualitative Analysis of Coupled Fractional Differential Equations involving Hilfer Derivative

**Kanika Dhawan, Ramesh Kumar Vats and Ravi P. Agarwal**

**Abstract**

In this manuscript, we have studied the coupled system of Hilfer fractional differential equations with nonlocal conditions. We have used the Leray-alternative Schauder's and the Contraction principle to obtain the results on the existence and uniqueness of the solution of the proposed problem in the weighted space of continuous functions. For the defined problem, sufficient conditions have also been developed to determine the Ulam stability of the solution. The key conclusions are well-illustrated with examples.

## 1 Introduction

Fractional calculus is one of the important areas in mathematics. In recent years, many researchers paid attention to the fractional calculus (see for instance [15, 17, 20, 22]). It has numerous applications in various fields of science and technology such as viscoelasticity, physics, biology, control hypothesis, chemistry, fluid dynamics, etc. see references [1, 4]. There are many definitions of fractional integrals and derivatives available in the literature, but Riemann-Liouville, Caputo, Hadamard, etc. are widely used. Later, Hilfer [10] introduced a new fractional derivative (known as Hilfer fractional derivative) which

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is considered as generalized Riemann-Liouville fractional derivative. The main advantage of utilizing fractional-order derivative over integer-order derivative is that the integer-order derivatives are local in nature, whereas fractional-order derivatives are global in nature.

There are many real-world problems that can not be modelled by a single differential equation. In this case, the coupled differential equations help us to overcome this difficulty. These type of system can often exist in the various model such as pesticides in soil and trees, brine tanks, predator-prey, irregular heartbeats and lidocaine, chemical kinetics, chemostats and microorganism culturing, etc., see for instances [3, 24, 25, 27, 33] and references therein.

The nonlocal conditions are useful for explaining certain peculiarities of physical, chemical, or other processes occurring at different points within the domain, instead of using end-point (initial/boundary) conditions. For the historical background of these conditions see [6, 21, 35]. Recently, the authors studied the fractional-order differential systems using nonlocal conditions, see references [16, 29, 30].

The most interesting topics in the field of differential equations are to find sufficient conditions for establishing the existence and uniqueness of the solution and to study the stability analysis of a system. It is well known that it is almost impossible to find an analytical solution to any differential equation. Hence, the study of qualitative theory is helpful for us to study the behavior of any differential system when an analytical solution does not exist for the system. In recent years, the existence and uniqueness of solutions of the different types of fractional differential equations (Riemann -Liouville, Caputo and Hilfer fractional differential equations) were studied extensively by many researchers, see for the instance [7, 8, 10, 14, 34, 36] and references cited therein.

The stability theory studies the solutions of differential equations using small perturbations. In 1940, Ulam [28] introduced Ulam's stability and then he studied different mathematical problems. Afterwards, Hyers [11] extended the results of Ulam's stability for the linear functional equation in 1941. Ulam and Hyers also established the various results on the stability of the differential equation. In the last two decades, stability results were analyzed and extended by many researchers for fractional-order differential equations (see for instance [5, 12, 18, 23] and references therein). By utilizing the $\psi$-Hilfer operator, Sousa et al. [26] studied Hyers-Ulam-Rassias stability for fractional integral differential equations. In 2017, Vivek et al. [31] derived the existence and stability analysis for nonlinear neutral Hilfer fractional pantograph differential equations. In 2018, Harikrishnan et al. [9] investigated the existence, uniqueness and generalized Ulam-Hyers-Rassias stability for coupled differential equations via the Hilfer-Katugampola fractional derivative. In 2019, Aziz

Khan et al. [13] derived the existence results and Hyers-Ulam stability for a nonlinear singular fractional differential equation with Mittag-Leffler kernel. In 2020, Ankit et al. [19] found sufficient conditions for the existence, uniqueness and Ulam-Hyers stability of the solutions for the coupled boundary value problem of nonlinear Caputo-Hadamard fractional differential equations associating with nonlocal integral boundary conditions. Recently, in 2021, Wang [32] established Hyers-Ulam-Rassias stability of the solution for Generalized fractional systems by the $p$-Laplace transform method.

As per our knowledge, there is no paper that discusses sufficient condition for the existence, uniqueness and stability of the solution for the coupled Hilfer fractional differential equations involving nonlocal conditions. In this article, we consider the coupled Hilfer fractional differential equations as

$$\begin{cases} D_{0^+}^{\kappa_1,\eta_1}\mathfrak{r}(\mathfrak{t}) = \pounds(\mathfrak{t},\mathfrak{r}(\mathfrak{t}),\mathfrak{p}(\mathfrak{t})),\ \kappa_1 < \upsilon_1 = \kappa_1+\eta_1-\kappa_1\eta_1, \\ D_{0^+}^{\kappa_2,\eta_2}\mathfrak{p}(\mathfrak{t}) = \mathfrak{N}(\mathfrak{t},\mathfrak{r}(\mathfrak{t}),\mathfrak{p}(\mathfrak{t})),\ \kappa_2 < \upsilon_2 = \kappa_2+\eta_2-\kappa_2\eta_2,\ \mathfrak{t}\in\mathcal{J}=[0,T], \end{cases} \tag{1}$$

involving nonlocal conditions

$$\begin{cases} I_{0+}^{1-\upsilon_1}\mathfrak{r}(0) = \sum_{i=1}^{m} c_i\mathfrak{r}(\xi_i),\ \xi_i\in[0,T], \\ I_{0+}^{1-\upsilon_2}\mathfrak{p}(0) = \sum_{i=1}^{m} d_i\mathfrak{p}(\zeta_i),\ \zeta_i\in[0,T], \end{cases} \tag{2}$$

where $D_{0^+}^{\kappa_i,\eta_i}$, represents the Hilfer fractional derivative of order $\kappa_i$ and type $\eta_i$, $(i=1,2)$, and $\kappa_i\in(0,1),\ \eta_i\in[0,1]$. $I_{0+}^{1-\upsilon_i}$ is the left-side Riemann-Liouville integral of order $1-\upsilon_i,\ i=1,2$. Also $\pounds,\ \mathfrak{N}:[0,T]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are given continuous nonlinear functions and $T>0$. $c_i,\ d_i\ (i=1,2,...,m)$ are real numbers, $\xi_i,\ \zeta_i\ (i=1,2,...,m)$ are prefixed points satisfying $0<\xi_1\le\xi_2\le ...\le\xi_m<T$ and $0<\zeta_1\le\zeta_2\le ...\le\zeta_m<T$ respectively.

The rest of the paper is organized as follows: In section 2, we introduce some basic definitions, notations, and preliminaries results. In Section 3, we prove the existence and uniqueness results using Leray-alternative Schauder's theorem and Banach contraction principle. In Section 4, we investigate Ulam's stability of the couple Hilfer fractional differential equation. In Section 5, the validation of our results is given through some suitable examples. Finally, a summary is given in the last section.

## 2 Preliminaries

This section is devoted for some basic notations, definitions related to the fractional calculus, and lemmas before stating our main results.

**Definition 2.1.** *[22] The Riemann-Liouville fractional integral of order $\beta > 0$ for a function $\ell$ is given by*

$$I^{\beta}\ell(s) = \frac{1}{\Gamma(\beta)}\int_0^s (s-\mathfrak{t})^{\beta-1}\ell(\mathfrak{t})d\mathfrak{t}, \ s > 0,$$

*where $\Gamma$ is the gamma function, and $\ell \in L^1(\mathcal{J}, \mathbb{R}^n)$.*

**Definition 2.2.** *[22] The Riemann-Liouville fractional derivative of order $0 \leq n-1 < \beta < n$ for a function $\ell$ is defined as*

$$D^{\beta}\ell(s) = \frac{1}{\Gamma(n-q)}\frac{d^n}{ds^n}\int_0^s \frac{\ell(\mathfrak{t})}{(s-\mathfrak{t})^{q+1-n}}d\mathfrak{t}, \ s > 0.$$

**Definition 2.3.** *[10] The Hilfer fractional derivative of type $0 \leq \alpha \leq 1$ and of order $0 < \beta < 1$ with lower limit 0 for a function $\ell$ is defined as*

$$D_{0+}^{\alpha,\beta}\ell(s) = \left(I_{0+}^{\alpha(1-\beta)}\frac{d}{ds}(I_{0+}^{(1-\nu)}\ell)\right)(s) = \left(I_{0+}^{\alpha(1-\beta)}(D^{\nu}\ell)\right)(s),$$

*provided that the right hand side exists.*

**Remark 2.4.** *If $\alpha = 0$ and $0 < \beta < 1$, then Hilfer fractional derivative becomes Riemann-Liouville fractional derivative of order $\beta$ and if $\alpha = 1$ and $0 < \beta < 1$, then Hilfer fractional derivative becomes Caputo fractional derivative of order $\beta$*

Throughout this paper we assume that $\mathcal{C} = \mathcal{C}(\mathcal{J}, \mathbb{R})$ is a Banach Space of all continuous functions from $\mathcal{J}$ into $\mathbb{R}$ with the sup-norm $\|\cdot\|_{\mathcal{C}}$ and $L^1(\mathcal{J}, \mathbb{R})$ is a Banach space of Lebesgue-integral functions from $\mathcal{J}$ into $\mathbb{R}$ endowed with the norm

$$\|\mathcal{P}\|_1 = \int_0^T \|\mathcal{P}(\mathfrak{t})\|\, d\mathfrak{t}.$$

Define $\mathcal{C}_{\upsilon}(\mathcal{J}, \mathbb{R})$ and $\mathcal{C}'_{\upsilon}(\mathcal{J}, \mathbb{R})$ are the weighted spaces of continuous functions as

$$\mathcal{C}_{\upsilon}(\mathcal{J}, \mathbb{R}) = \{x : [0,T] \to \mathbb{R} : x : (0,T] \to \mathbb{R} \text{ is continuous and } \lim_{\mathfrak{t}\to 0} \mathfrak{t}^{1-\upsilon}x(\mathfrak{t}) \text{ exists and finite}\}$$

with the norm

$$\|x\|_{\mathcal{C}_{\upsilon}} = \sup_{\mathfrak{t}\in\mathcal{J}} |\mathfrak{t}^{1-\upsilon}x(\mathfrak{t})|,$$

and

$$\mathcal{C}'_{\upsilon}(\mathcal{J}, \mathbb{R}) = \{x \in \mathcal{C} : \frac{dx}{dt} \in \mathcal{C}_{\upsilon}\}$$

with the norm

$$\|x\|_{\mathcal{C}'_v} = \|x\|_\infty + \|x'\|_{\mathcal{C}_v}.$$

By $\mathfrak{C} = \mathcal{C}_{v_1} \times \mathcal{C}_{v_2}$ we denote the product weighted space with the norm

$$\|(\mathfrak{r}, \mathfrak{p})\|_{\mathfrak{C}} = \|\mathfrak{r}\|_{\mathcal{C}_{v_1}} + \|\mathfrak{p}\|_{\mathcal{C}_{v_2}}.$$

**Lemma 2.5.** *[2]* ***Leray-Schauder alternative:*** *Let $E$ is a Banach space and $F : E \to E$ be a completely continuous operator and let*

$$\varepsilon(F) = \{x \in E : x = \lambda F(x) \text{ for some } 0 < \lambda < 1\}.$$

*Then either the set $\varepsilon(F)$ is unbounded, or $F$ has at least one fixed point.*

**Lemma 2.6.** *([7], Theorem 23) Let $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ be a continuous function. A function $x \in C_v^v[\mathfrak{J}, \mathbb{R}]$ is a solution of the fractional initial value problem:*

$$\begin{cases} D_{0+}^{\alpha,\beta} x(t) = f(t, x(t)), \quad \alpha \in (0,1), \ \beta \in [0,1], \\ I_{0+}^{1-v} x(0) = x_0, \quad v = \alpha + \beta - \alpha\beta, \end{cases}$$

*if and only if $x$ satisfies the following Volterra integral equation:*

$$x(t) = \frac{x_0 t^{v-1}}{\Gamma(v)} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s)) ds.$$

## 3 Existence and Uniqueness

In this section, we shall discuss the existence and uniqueness of the solution of the nonlinear Hilfer fractional couple differential system (1) with nonlocal conditions (2).

In view of Lemma 2.6, the solution $(\mathfrak{r}, \mathfrak{p}) \in \mathfrak{C}$ of system (1)-(2) can be written as follows

$$\begin{aligned} \mathfrak{r}(\mathfrak{t}) = & \frac{\mathfrak{z}_1 \mathfrak{t}^{v_1-1}}{\Gamma(\kappa_1)} \sum_{i=1}^m c_i \int_0^{\xi_i} (\xi_i - s)^{\kappa_1-1} \pounds(s, \mathfrak{r}(s), \mathfrak{p}(s)) ds \\ & + \frac{1}{\Gamma(\kappa_1)} \int_0^{\mathfrak{t}} (\mathfrak{t} - s)^{\kappa_1-1} \pounds(s, \mathfrak{r}(s), \mathfrak{p}(s)) ds \end{aligned}$$

and

$$\mathfrak{p}(\mathfrak{t}) = \frac{\mathfrak{z}_2 \mathfrak{t}^{v_2-1}}{\Gamma(\kappa_2)} \sum_{i=1}^m d_i \int_0^{\zeta_i} (\zeta_i - s)^{\kappa_2-1} \mathfrak{N}(s, \mathfrak{r}(s), \mathfrak{p}(s)) ds$$

$$+\frac{1}{\Gamma(\kappa_2)}\int_0^{\mathfrak{t}}(\mathfrak{t}-s)^{\kappa_2-1}\mathfrak{N}(s,\mathfrak{r}(s),\mathfrak{p}(s))ds,$$

where

$$\mathfrak{z}_1=\frac{1}{\Gamma(\upsilon_1)-\sum_{i=1}^m c_i\xi_i^{\upsilon_1-1}} \text{ if } \Gamma(\upsilon_1)\neq\sum_{i=1}^m c_i\xi_i^{\upsilon_1-1}$$

and

$$\mathfrak{z}_2=\frac{1}{\Gamma(\upsilon_2)-\sum_{i=1}^m d_i\zeta_i^{\upsilon_2-1}} \text{ if } \Gamma(\upsilon_2)\neq\sum_{i=1}^m d_i\zeta_i^{\upsilon_2-1}.$$

Now we state the following assumptions which are required to establish the existence and uniqueness results:

(A1) For $s^{1-\upsilon_1}\mathcal{L},\ s^{1-\upsilon_2}\mathfrak{N}$, the maps are continuous and there exist real constants $f_i,\ g_i\geq 0\ (i=1,2)$ and $f_0>0,\ g_0>0$ such that $\forall\ \mathfrak{r},\ \mathfrak{p}\in\mathbb{R}$, we have

$$|\mathcal{L}(\mathfrak{t},\mathfrak{r},\mathfrak{p})|\leq f_0+f_1|\mathfrak{r}|+f_2|\mathfrak{p}|$$

and

$$|\mathfrak{N}(\mathfrak{t},\mathfrak{r},\mathfrak{p})|\leq g_0+g_1|\mathfrak{r}|+g_2|\mathfrak{p}|.$$

(A2) The function $\mathcal{L},\mathfrak{N}:[0,T]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are measurable functions and there exist continuous functions $p_i:\mathcal{J}\to(0,\infty),\ (i=1,2)$ such that

$$\|\mathcal{L}(\mathfrak{t},\mathfrak{r},\mathfrak{p})-\mathcal{L}(\mathfrak{t},\mathfrak{r}',\mathfrak{p}')\|\leq p_1(\mathfrak{t})(\|\mathfrak{r}-\mathfrak{r}'\|+\|\mathfrak{p}-\mathfrak{p}'\|)$$

and

$$\|\mathfrak{N}(\mathfrak{t},\mathfrak{r},\mathfrak{p})-\mathfrak{N}(\mathfrak{t},\mathfrak{r}',\mathfrak{p}')\|\leq p_2(\mathfrak{t})(\|\mathfrak{r}-\mathfrak{r}'\|+\|\mathfrak{p}-\mathfrak{p}'\|)$$

with

$$p_i^*=\sup_{\mathfrak{t}\in\mathcal{J}}p_i(\mathfrak{t}),\ \ i=1,2.$$

(A3)

$$\begin{aligned}\mathfrak{G}=&\Bigg[\frac{p_1^*|\mathfrak{z}_1|}{\Gamma(\kappa_1)}\sum_{i=1}^m c_i\xi^{\kappa_1+\upsilon_1-1}B(\upsilon_1,\kappa_1)+\frac{p_1^*T^{\kappa_1}}{\Gamma(\kappa_1)}B(\upsilon_1,\kappa_1)\\&+\frac{p_2^*|\mathfrak{z}_2|}{\Gamma(\kappa_2)}\sum_{i=1}^m d_i\zeta^{\kappa_2+\upsilon_2-1}B(\upsilon_2,\kappa_2)+\frac{p_2^*T^{\kappa_2}}{\Gamma(\kappa_2)}B(\upsilon_2,\kappa_2)\Bigg]\\&<1.\end{aligned}$$

For brevity, we shall take

$$\mathfrak{A}_0 = \frac{|\mathfrak{Z}_1|}{\Gamma(\kappa_1+1)} \sum_{i=1}^{m} c_i \xi_i^{\kappa_1} f_0 + \frac{T^{1-\upsilon_1+\kappa_1}}{\Gamma(\kappa_1+1)} f_0,$$

$$\mathfrak{B}_0 = \frac{|\mathfrak{Z}_1|}{\Gamma(\kappa_1)} \sum_{i=1}^{m} c_i \xi_i^{\kappa_1+\upsilon_1-1} B(\upsilon_1,\kappa_1) f_1 + \frac{T^{\kappa_1}}{\Gamma(\kappa_1)} B(\upsilon_1,\kappa_1) f_1,$$

$$\mathfrak{D}_0 = \frac{|\mathfrak{Z}_1|}{\Gamma(\kappa_1)} \sum_{i=1}^{m} c_i \xi_i^{\kappa_1+\upsilon_1-1} B(\upsilon_1,\kappa_1) f_2 + \frac{T^{\kappa_1}}{\Gamma(\kappa_1)} B(\upsilon_1,\kappa_1) f_2,$$

$$\mathfrak{A}_1 = \frac{|\mathfrak{Z}_2|}{\Gamma(\kappa_2+1)} \sum_{i=1}^{m} d_i \zeta_i^{\kappa_2} g_0 + \frac{T^{1-\upsilon_2+\kappa_2}}{\Gamma(\kappa_2+1)} g_0,$$

$$\mathfrak{B}_1 = \frac{|\mathfrak{Z}_2|}{\Gamma(\kappa_2)} \sum_{i=1}^{m} d_i \zeta_i^{\kappa_2+\upsilon_2-1} B(\upsilon_2,\kappa_2) g_1 + \frac{T^{\kappa_2}}{\Gamma(\kappa_2)} B(\upsilon_2,\kappa_2) g_1$$

and

$$\mathfrak{D}_1 = \frac{|\mathfrak{Z}_2|}{\Gamma(\kappa_2)} \sum_{i=1}^{m} d_i \zeta_i^{\kappa_2+\upsilon_2-1} B(\upsilon_2,\kappa_2) g_2 + \frac{T^{\kappa_2}}{\Gamma(\kappa_2)} B(\upsilon_2,\kappa_2) g_2.$$

**Theorem 3.1.** *Suppose that assumption* $(A1)$ *is satisfied along with the condition* $\max\{\mathfrak{U},\mathfrak{T}\} < 1$, *where* $\mathfrak{U} = \mathfrak{B}_0 + \mathfrak{B}_1$, $\mathfrak{T} = \mathfrak{D}_0 + \mathfrak{D}_1$. *Then the couple system* (1)-(2) *has at least one solution on* $[0,T]$.

*Proof.* Define operators $\mathfrak{F}_1 : \mathcal{C}_{\upsilon_1} \to \mathcal{C}_{\upsilon_1}$ and $\mathfrak{F}_2 : \mathcal{C}_{\upsilon_2} \to \mathcal{C}_{\upsilon_2}$ as

$$(\mathfrak{F}_1\mathfrak{r})(\mathfrak{t}) = \frac{\mathfrak{Z}_1 \mathfrak{t}^{\upsilon_1-1}}{\Gamma(\kappa_1)} \sum_{i=1}^{m} c_i \int_0^{\xi_i} (\xi_i - s)^{\kappa_1-1} \pounds(s,\mathfrak{r}(s),\mathfrak{p}(s))ds + \frac{1}{\Gamma(\kappa_1)} \int_0^{\mathfrak{t}} (\mathfrak{t}-s)^{\kappa_1-1} \pounds(s,\mathfrak{r}(s),\mathfrak{p}(s))ds$$

and

$$(\mathfrak{F}_2\mathfrak{p})(\mathfrak{t}) = \frac{\mathfrak{Z}_2 \mathfrak{t}^{\upsilon_2-1}}{\Gamma(\kappa_2)} \sum_{i=1}^{m} d_i \int_0^{\zeta_i} (\zeta_i - s)^{\kappa_2-1} \mathfrak{N}(s,\mathfrak{r}(s),\mathfrak{p}(s))ds + \frac{1}{\Gamma(\kappa_2)} \int_0^{\mathfrak{t}} (\mathfrak{t}-s)^{\kappa_2-1} \mathfrak{N}(s,\mathfrak{r}(s),\mathfrak{p}(s))ds.$$

We also define the continuous operator $\mathfrak{F} : \mathfrak{C} \to \mathfrak{C}$ as

$$\mathfrak{F}(\mathfrak{r},\mathfrak{p})(\mathfrak{t}) = (\mathfrak{t}^{1-\upsilon_1}\mathfrak{F}_1(\mathfrak{r},\mathfrak{p})(\mathfrak{t}), \mathfrak{t}^{1-\upsilon_2}\mathfrak{F}_2(\mathfrak{r},\mathfrak{p})(\mathfrak{t})). \tag{3}$$

It is clear that any fixed points of the operator $\mathfrak{F}$ is a solution of the system (1)-(2).
The proof is divided into following steps.
**Step 1** $\mathfrak{F}$ is continuous.

Let $(\mathfrak{r}_n, \mathfrak{p}_n)$ be a sequence in $\mathfrak{C}$ such that $(\mathfrak{r}_n, \mathfrak{p}_n) \to (\mathfrak{r}, \mathfrak{p}) \in \mathfrak{C}$, we have

$$
\begin{aligned}
&|\mathfrak{F}(\mathfrak{r}_n, \mathfrak{p}_n)(\mathfrak{t}) - \mathfrak{F}(\mathfrak{r}, \mathfrak{p})(\mathfrak{t})| \\
&\leq \frac{|\mathfrak{Z}_1|}{\Gamma(\kappa_1)} \sum_{i=1}^{m} c_i \int_0^{\xi_i} (\xi_i - s)^{\kappa_1 - 1} |\pounds(s, \mathfrak{r}_n(s), \mathfrak{p}_n(s)) - \pounds(s, \mathfrak{r}(s), \mathfrak{p}(s))| ds \\
&+ \frac{\mathfrak{t}^{1-\upsilon_1}}{\Gamma(\kappa_1)} \int_0^{\mathfrak{t}} (\mathfrak{t} - s)^{\kappa_1 - 1} |\pounds(s, \mathfrak{r}_n(s), \mathfrak{p}_n(s)) - \pounds(s, \mathfrak{r}(s), \mathfrak{p}(s))| ds \\
&+ \frac{|\mathfrak{Z}_2|}{\Gamma(\kappa_2)} \sum_{i=1}^{m} d_i \int_0^{\zeta_i} (\zeta_i - s)^{\kappa_2 - 1} |\mathfrak{N}(s, \mathfrak{r}_n(s), \mathfrak{p}_n(s)) - \mathfrak{N}(s, \mathfrak{r}(s), \mathfrak{p}(s))| ds \\
&+ \frac{\mathfrak{t}^{1-\upsilon_2}}{\Gamma(\kappa_2)} \int_0^{\mathfrak{t}} (\mathfrak{t} - s)^{\kappa_2 - 1} |\mathfrak{N}(s, \mathfrak{r}_n(s), \mathfrak{p}_n(s)) - \mathfrak{N}(s, \mathfrak{r}(s), \mathfrak{p}(s))| ds \\
&\leq \frac{|\mathfrak{Z}_1|}{\Gamma(\kappa_1)} \sum_{i=1}^{m} c_i \int_0^{\xi_i} (\xi_i - s)^{\kappa_1 - 1} s^{\upsilon_1 - 1} | s^{1-\upsilon_1} (\pounds(s, \mathfrak{r}_n(s), \mathfrak{p}_n(s)) \\
&- \pounds(s, \mathfrak{r}(s), \mathfrak{p}(s)))| ds + \frac{\mathfrak{t}^{1-\upsilon_1}}{\Gamma(\kappa_1)} \int_0^{\mathfrak{t}} (\mathfrak{t} - s)^{\kappa_1 - 1} s^{\upsilon_1 - 1} | s^{1-\upsilon_1} (\pounds(s, \mathfrak{r}_n(s), \mathfrak{p}_n(s)) \\
&- \pounds(s, \mathfrak{r}(s), \mathfrak{p}(s)))| ds + \frac{|\mathfrak{Z}_2|}{\Gamma(\kappa_2)} \sum_{i=1}^{m} d_i \int_0^{\zeta_i} (\zeta_i - s)^{\kappa_2 - 1} s^{\upsilon_2 - 1} \\
&\times | s^{1-\upsilon_2} (\mathfrak{N}(s, \mathfrak{r}_n(s), \mathfrak{p}_n(s)) - \mathfrak{N}(s, \mathfrak{r}(s), \mathfrak{p}(s)))| ds \\
&+ \frac{\mathfrak{t}^{1-\upsilon_2}}{\Gamma(\kappa_2)} \int_0^{\mathfrak{t}} (\mathfrak{t} - s)^{\kappa_2 - 1} s^{\upsilon_2 - 1} | s^{1-\upsilon_2} (\mathfrak{N}(s, \mathfrak{r}_n(s), \mathfrak{p}_n(s)) - \mathfrak{N}(s, \mathfrak{r}(s), \mathfrak{p}(s)))| ds \\
&\leq \left[ \frac{|\mathfrak{Z}_1|}{\Gamma(\kappa_1)} \sum_{i=1}^{m} c_i \xi_i^{\kappa_1 + \upsilon_1 - 1} B(\upsilon_1, \kappa_1) + \frac{T^{\kappa_1}}{\Gamma(\kappa_1)} B(\upsilon_1, \kappa_1) \right] \| \pounds(s, \mathfrak{r}_n(s), \mathfrak{p}_n(s)) \\
&- \pounds(s, \mathfrak{r}(s), \mathfrak{p}(s)) \|_{\mathfrak{C}_{\upsilon_1}} + \left[ \frac{|\mathfrak{Z}_2|}{\Gamma(\kappa_2)} \sum_{i=1}^{m} d_i \zeta_i^{\kappa_2 + \upsilon_2 - 1} B(\upsilon_2, \kappa_2) + \frac{T^{\kappa_2}}{\Gamma(\kappa_2)} B(\upsilon_2, \kappa_2) \right] \\
&\times \| \mathfrak{N}(s, \mathfrak{r}_n(s), \mathfrak{p}_n(s)) - \mathfrak{N}(s, \mathfrak{r}(s), \mathfrak{p}(s)) \|_{\mathfrak{C}_{\upsilon_2}}.
\end{aligned}
$$

where we use

$$
\begin{aligned}
\int_a^t (t - s)^{(\alpha - 1)} |x(s)| ds &\leq \left( \int_a^t (t - a)^{(\alpha - 1)} (s - a)^{(\gamma - 1)} ds \right) \|x\|_{\mathfrak{C}} \\
&= (t - a)^{(\alpha + \gamma - 1)} B(\gamma, \alpha) \|x\|_{\mathfrak{C}}.
\end{aligned}
$$

As $s^{1-\upsilon_1}\pounds,\ s^{1-\upsilon_2}\mathfrak{N}$ are continuous, we obtain that $\|\mathfrak{F}(\mathfrak{r}_n,\mathfrak{p}_n)-\mathfrak{F}(\mathfrak{r},\mathfrak{p})\|_{\mathfrak{C}}\to 0$ as $n\to\infty$. Hence $\mathfrak{F}$ is continuous.

**Step 2** $\mathfrak{F}$ maps bounded set to bounded set.

Let $R>0$ be any constant and also let $B_R=B(R,0)=\{(\mathfrak{r},\mathfrak{p})\in\mathfrak{C}:\|(\mathfrak{r},\mathfrak{p})\|\le R\}$. Using assumption (A1), it is easy to find $\mathfrak{D}_\pounds$, $\mathfrak{D}_\mathfrak{N}$ such that $\|\pounds(\mathfrak{t},\mathfrak{r}(\mathfrak{t}),\mathfrak{p}(\mathfrak{t}))\|_{\mathfrak{C}_{\upsilon_1}}\le\mathfrak{D}_\pounds$, $\|\mathfrak{N}(\mathfrak{t},\mathfrak{r}(\mathfrak{t}),\mathfrak{p}(\mathfrak{t}))\|_{\mathfrak{C}_{\upsilon_2}}\le\mathfrak{D}_\mathfrak{N}$. For $(\mathfrak{r},\mathfrak{p})\in B_R$ and $\mathfrak{t}\in\mathfrak{J}$, we have

$$
\begin{aligned}
\|\mathfrak{F}(\mathfrak{r},\mathfrak{p})\|_{\mathfrak{C}} \le{}& \frac{|\mathfrak{Z}_1|}{\Gamma(\kappa_1)}\sum_{i=1}^{m}c_i\int_0^{\xi_i}(\xi_i-s)^{\kappa_1-1}\|\pounds(s,\mathfrak{r}(s),\mathfrak{p}(s))\|ds\\
&+\frac{\mathfrak{t}^{1-\upsilon_1}}{\Gamma(\kappa_1)}\int_0^{\mathfrak{t}}(\mathfrak{t}-s)^{\kappa_1-1}\|\pounds(s,\mathfrak{r}(s),\mathfrak{p}(s))\|ds\\
&+\frac{|\mathfrak{Z}_2|}{\Gamma(\kappa_2)}\sum_{i=1}^{m}d_i\int_0^{\zeta_i}(\zeta_i-s)^{\kappa_2-1}\|\mathfrak{N}(s,\mathfrak{r}(s),\mathfrak{p}(s))\|ds\\
&+\frac{\mathfrak{t}^{1-\upsilon_2}}{\Gamma(\kappa_2)}\int_0^{\mathfrak{t}}(\mathfrak{t}-s)^{\kappa_2-1}\|\mathfrak{N}(s,\mathfrak{r}(s),\mathfrak{p}(s))\|ds\\
\le{}& \frac{|\mathfrak{Z}_1|}{\Gamma(\kappa_1)}\sum_{i=1}^{m}c_i\int_0^{\xi_i}(\xi_i-s)^{\kappa_1-1}s^{\upsilon_1-1}\|s^{1-\upsilon_1}\pounds(s,\mathfrak{r}(s),\mathfrak{p}(s))\|ds\\
&+\frac{\mathfrak{t}^{1-\upsilon_1}}{\Gamma(\kappa_1)}\int_0^{\mathfrak{t}}(\mathfrak{t}-s)^{\kappa_1-1}s^{\upsilon_1-1}\|s^{1-\upsilon_1}\pounds(s,\mathfrak{r}(s),\mathfrak{p}(s))\|ds\\
&+\frac{|\mathfrak{Z}_2|}{\Gamma(\kappa_2)}\sum_{i=1}^{m}d_i\int_0^{\zeta_i}(\zeta_i-s)^{\kappa_2-1}s^{\upsilon_2-1}\|s^{1-\upsilon_2}\mathfrak{N}(s,\mathfrak{r}(s),\mathfrak{p}(s))\|ds\\
&+\frac{\mathfrak{t}^{1-\upsilon_2}}{\Gamma(\kappa_2)}\int_0^{\mathfrak{t}}(\mathfrak{t}-s)^{\kappa_2-1}s^{\upsilon_2-1}\|s^{1-\upsilon_2}\mathfrak{N}(s,\mathfrak{r}(s),\mathfrak{p}(s))\|ds\\
\le{}& \frac{|\mathfrak{Z}_1|B(\upsilon_1,\kappa_1)}{\Gamma(\kappa_1)}\sum_{i=1}^{m}c_i\xi_i^{\kappa_1+\upsilon_1-1}\mathfrak{D}_\pounds+\frac{T^{\kappa_1}}{\Gamma(\kappa_1)}B(\upsilon_1,\kappa_1)\mathfrak{D}_\pounds\\
&+\frac{|\mathfrak{Z}_2|B(\upsilon_2,\kappa_2)}{\Gamma(\kappa_2)}\sum_{i=1}^{m}d_i\zeta_i^{\kappa_2+\upsilon_2-1}\mathfrak{D}_\mathfrak{N}+\frac{T^{\kappa_2}}{\Gamma(\kappa_2)}B(\upsilon_2,\kappa_2)\mathfrak{D}_\mathfrak{N}\\
<{}&\infty.
\end{aligned}
$$

Hence $\mathfrak{F}$ maps bounded set to bounded set.

**Step 3** $\mathfrak{F}$ is equi-continuous.

Let $0\le\mathfrak{t}_1<\mathfrak{t}_2\le 1$ and $(\mathfrak{r},\mathfrak{p})\in B_R$. Then we have

$$\|\mathfrak{t}_1^{1-\upsilon_1}\mathfrak{F}_1(\mathfrak{t}_1)-\mathfrak{t}_2^{1-\upsilon_1}\mathfrak{F}_1(\mathfrak{t}_2)\|$$

$$\begin{aligned}
&\leq \left\| \frac{\mathfrak{t}_1^{1-\upsilon_1}}{\Gamma(\kappa_1)} \int_0^{\mathfrak{t}_1} (\mathfrak{t}_1 - s)^{\kappa_1 - 1} \pounds(s, \mathfrak{r}(s), \mathfrak{p}(s)) ds \right.\\
&\left. - \frac{\mathfrak{t}_2^{1-\upsilon_1}}{\Gamma(\kappa_1)} \int_0^{\mathfrak{t}_2} (\mathfrak{t}_2 - s)^{\kappa_1 - 1} \pounds(s, \mathfrak{r}(s), \mathfrak{p}(s)) ds \right\|\\
&\leq \frac{1}{\Gamma(\kappa_1)} \int_0^{\mathfrak{t}_1} |\mathfrak{t}_2^{1-\upsilon_1} (\mathfrak{t}_2 - s)^{\kappa_1 - 1} - \mathfrak{t}_1^{1-\upsilon_1} (\mathfrak{t}_1 - s)^{\kappa_1 - 1}| s^{\upsilon_1 - 1}\\
&\times \| s^{1-\upsilon_1} \pounds(s, \mathfrak{r}(s), \mathfrak{p}(s)) \| ds\\
&+ \frac{\mathfrak{t}_2^{1-\upsilon_1}}{\Gamma(\kappa_1)} \int_{\mathfrak{t}_1}^{\mathfrak{t}_2} |(\mathfrak{t}_2 - s)^{\kappa_1 - 1}| s^{\upsilon_1 - 1} \| s^{1-\upsilon_1} \pounds(s, \mathfrak{r}(s), \mathfrak{p}(s)) \| ds\\
&\leq \frac{\mathfrak{D}_{\pounds}}{\Gamma(\kappa_1)} \int_0^{\mathfrak{t}_1} |\mathfrak{t}_2^{1-\upsilon_1} (\mathfrak{t}_2 - s)^{\kappa_1 - 1} - \mathfrak{t}_1^{1-\upsilon_1} (\mathfrak{t}_1 - s)^{\kappa_1 - 1}| s^{\upsilon_1 - 1} ds\\
&+ \frac{\mathfrak{D}_{\pounds} \mathfrak{t}_2^{1-\upsilon_1}}{\Gamma(\kappa_1)} \int_{\mathfrak{t}_1}^{\mathfrak{t}_2} |(\mathfrak{t}_2 - s)^{\kappa_1 - 1}| s^{\upsilon_1 - 1} ds\\
&\leq \frac{\mathfrak{D}_{\pounds} B(\upsilon_1, \kappa_1)}{\Gamma(\kappa_1)} [\mathfrak{t}_2^{\kappa_1} - \mathfrak{t}_1^{\kappa_1} + \mathfrak{t}_2^{1-\upsilon_1} (\mathfrak{t}_2 - \mathfrak{t}_1)^{\kappa_1 + \upsilon_1 + 1}].
\end{aligned}$$

Similarly, we get

$$\|\mathfrak{t}_1^{1-\upsilon_2} \mathfrak{F}_2(\mathfrak{t}_1) - \mathfrak{t}_2^{1-\upsilon_2} \mathfrak{F}_2(\mathfrak{t}_2)\| \leq \frac{\mathfrak{D}_{\mathfrak{N}} B(\upsilon_2, \kappa_2)}{\Gamma(\kappa_2)} [\mathfrak{t}_2^{\kappa_2} - \mathfrak{t}_1^{\kappa_2} + \mathfrak{t}_2^{1-\upsilon_2} (\mathfrak{t}_2 - \mathfrak{t}_1)^{\kappa_2 + \upsilon_2 + 1}].$$

Thus, we can now say that

$$\|\mathfrak{F}(\mathfrak{t}_1) - \mathfrak{F}(\mathfrak{t}_2)\|_{\mathfrak{C}} \to 0 \quad \text{as} \quad \mathfrak{t}_1 \to \mathfrak{t}_2.$$

Therefore, $\mathfrak{F}$ is equi-continuous and from Arzelá-Ascoli theorem, $\mathfrak{F}$ is completely continuous.

**Step 4** $\Theta$ is bounded.

Let $\Theta = \{(\mathfrak{r}, \mathfrak{p}) \in \mathfrak{C} : (\mathfrak{r}, \mathfrak{p}) = \theta \mathfrak{F}(\mathfrak{r}, \mathfrak{p}) \text{ for some } \theta \in (0,1)\}$. Let $(\mathfrak{r}, \mathfrak{p}) = \theta \mathfrak{F}(\mathfrak{r}, \mathfrak{p})$, then for arbitrary $\mathfrak{t} \in [0, T]$,

$$\mathfrak{r}(\mathfrak{t}) = \theta \mathfrak{F}_1(\mathfrak{r}, \mathfrak{p})(\mathfrak{t}), \;\; \mathfrak{p}(\mathfrak{t}) = \theta \mathfrak{F}_2(\mathfrak{r}, \mathfrak{p})(\mathfrak{t}).$$

Since

$$\begin{aligned}
\|\mathfrak{r}(\mathfrak{t})\|_{\mathcal{C}_{\upsilon_1}} &= \|\theta \mathfrak{F}_1(\mathfrak{r}, \mathfrak{p})(\mathfrak{t})\|_{\mathcal{C}_{\upsilon_1}}\\
&\leq \frac{\theta |\mathfrak{Z}_1|}{\Gamma(\kappa_1)} \sum_{i=1}^{m} c_i \int_0^{\xi_i} (\xi_i - s)^{\kappa_1 - 1} \|\pounds(s, \mathfrak{r}(s), \mathfrak{p}(s))\| ds\\
&+ \frac{\theta \mathfrak{t}^{1-\upsilon_1}}{\Gamma(\kappa_1)} \int_0^{\mathfrak{t}} (\mathfrak{t} - s)^{\kappa_1 - 1} \|\pounds(s, \mathfrak{r}(s), \mathfrak{p}(s))\| ds
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{|\mathfrak{Z}_1|}{\Gamma(\kappa_1)}\sum_{i=1}^{m} c_i \int_0^{\xi_i} (\xi_i - s)^{\kappa_1-1}[f_0 + f_1\|\mathfrak{r}\| + f_2\|\mathfrak{p}\|]ds \\
&+\frac{\mathfrak{t}^{1-\upsilon_1}}{\Gamma(\kappa_1)}\int_0^{\mathfrak{t}} (\mathfrak{t} - s)^{\kappa_1-1}[f_0 + f_1\|\mathfrak{r}\| + f_2\|\mathfrak{p}\|]ds \\
&\leq \left(\frac{|\mathfrak{Z}_1|}{\Gamma(\kappa_1+1)}\sum_{i=1}^{m} c_i\xi_i^{\kappa_1} + \frac{T^{1-\upsilon_1+\kappa_1}}{\Gamma(\kappa_1+1)}\right) f_0 \\
&+\left(\frac{|\mathfrak{Z}_1|}{\Gamma(\kappa_1)}\sum_{i=1}^{m} c_i\xi_i^{\kappa_1+\upsilon_1-1}B(\upsilon_1,\kappa_1)f_1 + \frac{T^{\kappa_1}}{\Gamma(\kappa_1)}B(\upsilon_1,\kappa_1)f_1\right)\|\mathfrak{r}\|_{\mathcal{C}_{\upsilon_1}} \\
&+\left(\frac{|\mathfrak{Z}_1|}{\Gamma(\kappa_1)}\sum_{i=1}^{m} c_i\xi_i^{\kappa_1+\upsilon_1-1}B(\upsilon_1,\kappa_1)f_2 + \frac{T^{\kappa_1}}{\Gamma(\kappa_1)}B(\upsilon_1,\kappa_1)f_2\right)\|\mathfrak{p}\|_{\mathcal{C}_{\upsilon_2}} \\
&\leq \mathfrak{A}_0 + \mathfrak{B}_0\|\mathfrak{r}\|_{\mathcal{C}_{\upsilon_1}} + \mathfrak{D}_0\|\mathfrak{p}\|_{\mathcal{C}_{\upsilon_2}}.
\end{aligned}
$$

Similarly, we obtain

$$
\begin{aligned}
\|\mathfrak{p}(\mathfrak{t})\|_{\mathcal{C}_{\upsilon_2}} \leq& \left(\frac{|\mathfrak{Z}_2|}{\Gamma(\kappa_2+1)}\sum_{i=1}^{m} d_i\zeta_i^{\kappa_2} + \frac{T^{1-\upsilon_2+\kappa_2}}{\Gamma(\kappa_2+1)}\right) g_0 \\
&+\left(\frac{|\mathfrak{Z}_2|}{\Gamma(\kappa_2)}\sum_{i=1}^{m} d_i\zeta_i^{\kappa_2+\upsilon_2-1}B(\upsilon_2,\kappa_2)g_1 + \frac{T^{\kappa_2}}{\Gamma(\kappa_2)}B(\upsilon_2,\kappa_2)g_1\right)\|\mathfrak{r}\|_{\mathcal{C}_{\upsilon_1}} \\
&+\left(\frac{|\mathfrak{Z}_2|}{\Gamma(\kappa_2)}\sum_{i=1}^{m} d_i\zeta_i^{\kappa_2+\upsilon_2-1}B(\upsilon_2,\kappa_2)g_2 + \frac{T^{\kappa_2}}{\Gamma(\kappa_2)}B(\upsilon_2,\kappa_2)g_2\right)\|\mathfrak{p}\|_{\mathcal{C}_{\upsilon_2}} \\
&\leq \mathfrak{A}_1 + \mathfrak{B}_1\|\mathfrak{r}\|_{\mathcal{C}_{\upsilon_1}} + \mathfrak{D}_1\|\mathfrak{p}\|_{\mathcal{C}_{\upsilon_2}}.
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
\|(\mathfrak{r},\mathfrak{p})\|_{\mathfrak{C}} &= \|\mathfrak{r}\|_{\mathcal{C}_{\upsilon_1}} + \|\mathfrak{p}\|_{\mathcal{C}_{\upsilon_2}} \\
&\leq \mathfrak{A}_0 + \mathfrak{B}_0\|\mathfrak{r}\|_{\mathcal{C}_{\upsilon_1}} + \mathfrak{D}_0\|\mathfrak{p}\|_{\mathcal{C}_{\upsilon_2}} + \mathfrak{A}_1 + \mathfrak{B}_1\|\mathfrak{r}\|_{\mathcal{C}_{\upsilon_1}} + \mathfrak{D}_1\|\mathfrak{p}\|_{\mathcal{C}_{\upsilon_2}} \\
&\leq \mathfrak{A}_0 + \mathfrak{A}_1 + (\mathfrak{B}_0 + \mathfrak{B}_1)\|\mathfrak{r}\|_{\mathcal{C}_{\upsilon_1}} + (\mathfrak{D}_0 + \mathfrak{D}_1)\|\mathfrak{p}\|_{\mathcal{C}_{\upsilon_2}} \\
&\leq \mathfrak{R} + \max\{\mathfrak{U},\mathfrak{T}\}\|(\mathfrak{r},\mathfrak{p})\|_{\mathfrak{C}} \\
&\leq \frac{\mathfrak{R}}{1-\max\{\mathfrak{U},\mathfrak{T}\}}
\end{aligned}
$$

where $\mathfrak{R} = \mathfrak{A}_0 + \mathfrak{A}_1$, $\mathfrak{U} = \mathfrak{B}_0 + \mathfrak{B}_1$, $\mathfrak{T} = \mathfrak{D}_0 + \mathfrak{D}_1$.
This implies $\Theta$ is bounded. Hence, from Leray-Schauder alternative, the operator $\mathfrak{F}$ has at least one fixed point. Therefore the problem (1) and (2) has at least one solution on $[0,T]$. □

**Theorem 3.2.** *Suppose that assumption* (A2) *and* (A3) *are satisfied. Then the couple system* (1)-(2) *has unique solution on* $[0, T]$.

*Proof.* We consider the map $\mathfrak{F}$ which is defined by (3). From the assumption (A2), it is easy to check that the assumption (A1) is satisfied. Now we say from Theorem (3.1) that there exists a constant $l > 0$ such that the map $\mathfrak{F} : B_l \to B_l,\ (B_l \subset \mathfrak{C})$ where $B_l = \{(\mathfrak{r}, \mathfrak{p}) \in \mathfrak{C} : \|(\mathfrak{r}, \mathfrak{p})\| \leq l\}$.
For this we have to prove that there exists a constant $l_1 > 0$ such that $\mathfrak{F}_1 : B_{l_1} \to B_{l_1}$ is contraction map where $B_{l_1} = \{\mathfrak{r} \in \mathcal{C}_{\upsilon_1} : \|\mathfrak{r}\| \leq l_1\}$ and there exists a constant $l_2 > 0$ such that $\mathfrak{F}_2 : B_{l_2} \to B_{l_2}$ is contraction map where $B_{l_2} = \{\mathfrak{p} \in \mathcal{C}_{\upsilon_2} : \|\mathfrak{p}\| \leq l_2\}$.
Let $(\mathfrak{r}, \mathfrak{p}), (\bar{\mathfrak{r}}, \bar{\mathfrak{p}}) \in \mathfrak{C}$ and $\mathfrak{t} \in [0, T]$ then we have

$$
\begin{aligned}
&\|\mathfrak{t}^{1-\upsilon_1}[(\mathfrak{F}_1(\mathfrak{r}, \mathfrak{p}))(\mathfrak{t}) - (\mathfrak{F}_1(\bar{\mathfrak{r}}, \bar{\mathfrak{p}}))(\mathfrak{t})]\| \\
&\quad \leq \frac{|\mathfrak{Z}_1|}{\Gamma(\kappa_1)} \sum_{i=1}^{m} c_i \int_0^{\xi_i} (\xi_i - s)^{\kappa_1 - 1} \|\mathcal{L}(s, \mathfrak{r}(s), \mathfrak{p}(s)) - \mathcal{L}(s, \bar{\mathfrak{r}}(s), \bar{\mathfrak{p}}(s))\| ds \\
&\quad + \frac{\mathfrak{t}^{1-\upsilon_1}}{\Gamma(\kappa_1)} \int_0^{\mathfrak{t}} (\mathfrak{t} - s)^{\kappa_1 - 1} \|\mathcal{L}(s, \mathfrak{r}(s), \mathfrak{p}(s)) - \mathcal{L}(s, \bar{\mathfrak{r}}(s), \bar{\mathfrak{p}}(s))\| ds \\
&\quad \leq \frac{|\mathfrak{Z}_1|}{\Gamma(\kappa_1)} \sum_{i=1}^{m} c_i \int_0^{\xi_i} (\xi_i - s)^{\kappa_1 - 1} s^{\upsilon_1 - 1} p_1(s) \Big( s^{1-\upsilon_1} \big[ \|\mathfrak{r}(s) - \bar{\mathfrak{r}}(s)\| \\
&\quad + \|\mathfrak{p}(s) - \bar{\mathfrak{p}}(s)\| \big] \Big) ds + \frac{\mathfrak{t}^{1-\upsilon_1}}{\Gamma(\kappa_1)} \int_0^{\mathfrak{t}} (\mathfrak{t} - s)^{\kappa_1 - 1} s^{\upsilon_1 - 1} p_1(s) \\
&\quad \times \Big( s^{1-\upsilon_1} \big[ \|\mathfrak{r}(s) - \bar{\mathfrak{r}}(s)\| + \|\mathfrak{p}(s) - \bar{\mathfrak{p}}(s)\| \big] \Big) ds \\
&\quad \leq \left[ \frac{p_1^* |\mathfrak{Z}_1|}{\Gamma(\kappa_1)} \sum_{i=1}^{m} c_i \xi^{\kappa_1 + \upsilon_1 - 1} B(\upsilon_1, \kappa_1) + \frac{p_1^* T^{\kappa_1}}{\Gamma(\kappa_1)} B(\upsilon_1, \kappa_1) \right] \\
&\quad \times [\|\mathfrak{r} - \bar{\mathfrak{r}}\|_{\mathcal{C}_{\upsilon_1}} + \|\mathfrak{p} - \bar{\mathfrak{p}}\|_{\mathcal{C}_{\upsilon_2}}].
\end{aligned}
$$

Similarly, we get

$$
\begin{aligned}
&\|\mathfrak{t}^{1-\upsilon_2}[(\mathfrak{F}_2(\mathfrak{r}, \mathfrak{p}))(\mathfrak{t}) - (\mathfrak{F}_2(\bar{\mathfrak{r}}, \bar{\mathfrak{p}}))(\mathfrak{t})]\| \\
&\quad \leq \left[ \frac{p_2^* |\mathfrak{Z}_2|}{\Gamma(\kappa_2)} \sum_{i=1}^{m} d_i \zeta^{\kappa_2 + \upsilon_2 - 1} B(\upsilon_2, \kappa_2) + \frac{p_2^* T^{\kappa_2}}{\Gamma(\kappa_2)} B(\upsilon_2, \kappa_2) \right] \\
&\quad \times [\|\mathfrak{r} - \bar{\mathfrak{r}}\|_{\mathcal{C}_{\upsilon_1}} + \|\mathfrak{p} - \bar{\mathfrak{p}}\|_{\mathcal{C}_{\upsilon_2}}].
\end{aligned}
$$

Therefore, we have

$$
\|[(\mathfrak{F}(\mathfrak{r}, \mathfrak{p}))(\mathfrak{t}) - (\mathfrak{F}(\bar{\mathfrak{r}}, \bar{\mathfrak{p}}))(\mathfrak{t})]\|_{\mathfrak{C}}
$$

$$
\begin{aligned}
&\leq\left[\frac{p_1^*\|\mathfrak{Z}_1\|}{\Gamma(\kappa_1)}\sum_{i=1}^{m}c_i\xi^{\kappa_1+\upsilon_1-1}B(\upsilon_1,\kappa_1)+\frac{p_1^*T^{\kappa_1}}{\Gamma(\kappa_1)}B(\upsilon_1,\kappa_1)\right]\\
&\times[\|\mathfrak{r}-\bar{\mathfrak{r}}\|_{\mathcal{C}_{\upsilon_1}}+\|\mathfrak{p}-\bar{\mathfrak{p}}\|_{\mathcal{C}_{\upsilon_2}}]\\
&+\left[\frac{p_2^*\|\mathfrak{Z}_2\|}{\Gamma(\kappa_2)}\sum_{i=1}^{m}d_i\zeta^{\kappa_2+\upsilon_2-1}B(\upsilon_2,\kappa_2)+\frac{p_2^*T^{\kappa_2}}{\Gamma(\kappa_2)}B(\upsilon_2,\kappa_2)\right]\\
&\times[\|\mathfrak{r}-\bar{\mathfrak{r}}\|_{\mathcal{C}_{\upsilon_1}}+\|\mathfrak{p}-\bar{\mathfrak{p}}\|_{\mathcal{C}_{\upsilon_2}}]\\
&\leq\ \mathfrak{G}\ \ [\|\mathfrak{r}-\bar{\mathfrak{r}}\|_{\mathcal{C}_{\upsilon_1}}+\|\mathfrak{p}-\bar{\mathfrak{p}}\|_{\mathcal{C}_{\upsilon_2}}].
\end{aligned}
$$

Hence as an application of Banach contraction principle, the operator $\mathfrak{F}$ has a fixed point. Therefore the couple system (1)-(2) has a unique solution on $[0,T]$. □

## 4 Stability

In this section, we shall derive the various type of Ulam's stability results for system (1)-(2).

**Definition 4.1.** *[23] The coupled system* (1)-(2) *is said to be Hyers-Ulam stable, if there exist* $\mathcal{M}_{1,2}=max\{\mathcal{M}_1,\ \mathcal{M}_2\}>0$ *such that for any* $\delta_1>0,\ \delta_2>0$ *and* $\delta=max\{\delta_1,\delta_2\}$ *and for every solution* $(\mathfrak{r},\mathfrak{p})\in\mathcal{C}\times\mathcal{C}$ *of the inequalities*

$$
\begin{aligned}
|D_{0^+}^{\kappa_1,\eta_1}\mathfrak{r}(\mathfrak{t})-\pounds(\mathfrak{t},\mathfrak{r}(\mathfrak{t}),\mathfrak{p}(\mathfrak{t}))|\leq&\delta_1,\\
|D_{0^+}^{\kappa_2,\eta_2}\mathfrak{p}(\mathfrak{t})-\mathfrak{N}(\mathfrak{t},\mathfrak{r}(\mathfrak{t}),\mathfrak{p}(\mathfrak{t}))|\leq&\delta_2,
\end{aligned}
\tag{4}
$$

*there exists a unique solution* $(\mathfrak{r}',\mathfrak{p}')\in\mathcal{C}\times\mathcal{C}$ *of the problem* (1) *and* (2) *with*

$$\|(\mathfrak{r},\mathfrak{p})-(\mathfrak{r}',\mathfrak{p}')\|\leq\mathcal{M}_{1,2}\delta.$$

**Definition 4.2.** *[23] The coupled system* (1)-(2) *is said to be Hyers-Ulam-Rassias stable, if there exist* $\mathcal{N}_{1,2}=max\{\mathcal{N}_1,\ \mathcal{N}_2\}>0$ *and a non-decreasing function* $\varphi=max\{\varphi_1,\varphi_2\}$ *such that for any* $\delta_1>0,\ \delta_2>0$ *and* $\delta=max\{\delta_1,\delta_2\}$ *and for every solution* $(\mathfrak{r},\mathfrak{p})\in\mathcal{C}\times\mathcal{C}$ *of the inequalities*

$$
\begin{aligned}
|D_{0^+}^{\kappa_1,\eta_1}\mathfrak{r}(\mathfrak{t})-\pounds(\mathfrak{t},\mathfrak{r}(\mathfrak{t}),\mathfrak{p}(\mathfrak{t}))|\leq&\varphi_1(\mathfrak{t})\delta_1,\\
|D_{0^+}^{\kappa_2,\eta_2}\mathfrak{p}(\mathfrak{t})-\mathfrak{N}(\mathfrak{t},\mathfrak{r}(\mathfrak{t}),\mathfrak{p}(\mathfrak{t}))|\leq&\varphi_2(\mathfrak{t})\delta_2,
\end{aligned}
\tag{5}
$$

*there exists a unique solution* $(\mathfrak{r}',\mathfrak{p}')\in\mathcal{C}\times\mathcal{C}$ *of the problem* (1) *and* (2) *with*

$$\|(\mathfrak{r},\mathfrak{p})-(\mathfrak{r}',\mathfrak{p}')\|\leq\mathcal{N}_{1,2}\varphi(\mathfrak{t})\delta.$$

**Remark 4.3.** *[23] Any function* $(\mathfrak{r},\mathfrak{p})\in\mathcal{C}\times\mathcal{C}$ *satisfies the inequalities*

$$\begin{aligned}|D_{0^+}^{\kappa_1,\eta_1}\mathfrak{r}(\mathfrak{t})-\pounds(\mathfrak{t},\mathfrak{r}(\mathfrak{t}),\mathfrak{p}(\mathfrak{t}))|\leq&\delta_1,\\ |D_{0^+}^{\kappa_2,\eta_2}\mathfrak{p}(\mathfrak{t})-\mathfrak{N}(\mathfrak{t},\mathfrak{r}(\mathfrak{t}),\mathfrak{p}(\mathfrak{t}))|\leq&\delta_2,\ \ \mathfrak{t}\in\mathcal{J}=[0,T]\end{aligned}$$

*if and only if there are functions* $g_1,g_2\in\mathcal{C}$ *such that*

*(i)* $|g_1(\mathfrak{t})|\leq\delta_1$, $|g_2(\mathfrak{t})|\leq\delta_2$,

*(ii)* $D_{0^+}^{\kappa_1,\eta_1}\mathfrak{r}(\mathfrak{t})=\pounds(\mathfrak{t},\mathfrak{r}(\mathfrak{t}),\mathfrak{p}(\mathfrak{t}))+g_1(\mathfrak{t})$, $D_{0^+}^{\kappa_2,\eta_2}\mathfrak{p}(\mathfrak{t})=\mathfrak{N}(\mathfrak{t},\mathfrak{r}(\mathfrak{t}),\mathfrak{p}(\mathfrak{t}))+g_2(\mathfrak{t})$.

**Lemma 4.4.** *If* $(\mathfrak{r},\mathfrak{p})\in\mathcal{C}\times\mathcal{C}$ *satisfies the inequalities* (4)*, then* $(\mathfrak{r},\mathfrak{p})$ *is a solution of the inequalities*

$$\begin{aligned}\Bigg|\mathfrak{r}(\mathfrak{t})-\Bigg(&\frac{\mathfrak{Z}_1\mathfrak{t}^{\upsilon_1-1}}{\Gamma(\kappa_1)}\sum_{i=1}^{m}c_i\int_0^{\xi_i}(\xi_i-s)^{\kappa_1-1}\pounds(s,\mathfrak{r}(s),\mathfrak{p}(s))ds\\ &+\frac{1}{\Gamma(\kappa_1)}\int_0^{\mathfrak{t}}(\mathfrak{t}-s)^{\kappa_1-1}\pounds(s,\mathfrak{r}(s),\mathfrak{p}(s))ds\Bigg)\Bigg|\leq\mathcal{M}_1\delta_1\end{aligned}$$

*and*

$$\begin{aligned}\Bigg|\mathfrak{p}(\mathfrak{t})-\Bigg(&\frac{\mathfrak{Z}_2\mathfrak{t}^{\upsilon_2-1}}{\Gamma(\kappa_2)}\sum_{i=1}^{m}d_i\int_0^{\zeta_i}(\zeta_i-s)^{\kappa_2-1}\mathfrak{N}(s,\mathfrak{r}(s),\mathfrak{p}(s))ds\\ &+\frac{1}{\Gamma(\kappa_2)}\int_0^{\mathfrak{t}}(\mathfrak{t}-s)^{\kappa_2-1}\mathfrak{N}(s,\mathfrak{r}(s),\mathfrak{p}(s))ds\Bigg)\Bigg|\leq\mathcal{M}_2\delta_2\end{aligned}$$

*where*

$$\mathcal{M}_1=\frac{\mathfrak{Z}_1T^{\upsilon_1-1}}{\Gamma(\kappa_1+1)}\sum_{i=1}^{m}c_i\xi_i^{\kappa_1}+\frac{1}{\Gamma(\kappa_1+1)}T^{\kappa_1}$$

*and*

$$\mathcal{M}_2=\frac{\mathfrak{Z}_2T^{\upsilon_2-1}}{\Gamma(\kappa_2+1)}\sum_{i=1}^{m}d_i\zeta_i^{\kappa_2}+\frac{1}{\Gamma(\kappa_2+1)}T^{\kappa_2}.$$

*Proof.* From Remarks (4.3), we obtain

$$\begin{aligned}D_{0^+}^{\kappa_1,\eta_1}\mathfrak{r}(\mathfrak{t})&=\pounds(\mathfrak{t},\mathfrak{r}(\mathfrak{t}),\mathfrak{p}(\mathfrak{t}))+g_1(\mathfrak{t}),\\ D_{0^+}^{\kappa_2,\eta_2}\mathfrak{p}(\mathfrak{t})&=\mathfrak{N}(\mathfrak{t},\mathfrak{r}(\mathfrak{t}),\mathfrak{p}(\mathfrak{t}))+g_2(\mathfrak{t}).\end{aligned}$$

Therefore, by Lemma (2.6), we have

$$\mathfrak{r}(\mathfrak{t})=\frac{\mathfrak{Z}_1\mathfrak{t}^{\upsilon_1-1}}{\Gamma(\kappa_1)}\sum_{i=1}^{m}c_i\int_0^{\xi_i}(\xi_i-s)^{\kappa_1-1}[\pounds(s,\mathfrak{r}(s),\mathfrak{p}(s))+g_1(s)]ds$$

$$+\frac{1}{\Gamma(\kappa_1)}\int_0^{\mathfrak{t}}(\mathfrak{t}-s)^{\kappa_1-1}[\mathcal{L}(s,\mathfrak{r}(s),\mathfrak{p}(s))+g_1(s)]ds$$

and

$$\mathfrak{p}(\mathfrak{t})=\frac{\mathfrak{Z}_2\mathfrak{t}^{\upsilon_2-1}}{\Gamma(\kappa_2)}\sum_{i=1}^{m}d_i\int_0^{\zeta_i}(\zeta_i-s)^{\kappa_2-1}[\mathfrak{N}(s,\mathfrak{r}(s),\mathfrak{p}(s))+g_2(s)]ds$$
$$+\frac{1}{\Gamma(\kappa_2)}\int_0^{\mathfrak{t}}(\mathfrak{t}-s)^{\kappa_2-1}[\mathfrak{N}(s,\mathfrak{r}(s),\mathfrak{p}(s))+g_2(s)]ds.$$

Now

$$\begin{aligned}
&\Bigg|\mathfrak{r}(\mathfrak{t})-\Bigg(\frac{\mathfrak{Z}_1\mathfrak{t}^{\upsilon_1-1}}{\Gamma(\kappa_1)}\sum_{i=1}^{m}c_i\int_0^{\xi_i}(\xi_i-s)^{\kappa_1-1}\mathcal{L}(s,\mathfrak{r}(s),\mathfrak{p}(s))ds+\frac{1}{\Gamma(\kappa_1)}\int_0^{\mathfrak{t}}(\mathfrak{t}-s)^{\kappa_1-1}\\
&\times\mathcal{L}(s,\mathfrak{r}(s),\mathfrak{p}(s))ds\Bigg)\Bigg| \le \Bigg|\frac{\mathfrak{Z}_1\mathfrak{t}^{\upsilon_1-1}}{\Gamma(\kappa_1)}\sum_{i=1}^{m}c_i\int_0^{\xi_i}(\xi_i-s)^{\kappa_1-1}g_1(s)ds\\
&\qquad+\frac{1}{\Gamma(\kappa_1)}\int_0^{\mathfrak{t}}(\mathfrak{t}-s)^{\kappa_1-1}g_1(s)ds\Bigg|\\
&\le\frac{|\mathfrak{Z}_1|\mathfrak{t}^{\upsilon_1-1}}{\Gamma(\kappa_1)}\sum_{i=1}^{m}c_i\int_0^{\xi_i}(\xi_i-s)^{\kappa_1-1}|g_1(s)|ds\\
&\qquad+\frac{1}{\Gamma(\kappa_1)}\int_0^{\mathfrak{t}}(\mathfrak{t}-s)^{\kappa_1-1}|g_1(s)|ds\\
&\le\left(\frac{|\mathfrak{Z}_1|T^{\upsilon_1-1}}{\Gamma(\kappa_1+1)}\sum_{i=1}^{m}c_i\xi_i^{\kappa_1}+\frac{1}{\Gamma(\kappa_1+1)}T^{\kappa_1}\right)\delta_1\\
&\le\mathcal{M}_1\delta_1.
\end{aligned}$$

Simillarly, we get

$$\begin{aligned}
&\Bigg|\mathfrak{p}(\mathfrak{t})-\Bigg(\frac{\mathfrak{Z}_2\mathfrak{t}^{\upsilon_2-1}}{\Gamma(\kappa_2)}\sum_{i=1}^{m}d_i\int_0^{\zeta_i}(\zeta_i-s)^{\kappa_2-1}\mathfrak{N}(s,\mathfrak{r}(s),\mathfrak{p}(s))ds\\
&\qquad+\frac{1}{\Gamma(\kappa_2)}\int_0^{\mathfrak{t}}(\mathfrak{t}-s)^{\kappa_2-1}\mathfrak{N}(s,\mathfrak{r}(s),\mathfrak{p}(s))ds\Bigg)\Bigg|\le\mathcal{M}_2\delta_2.
\end{aligned}$$

□

**Theorem 4.5.** *If the assumptions of Theorem* (3.2) *hold along with the condition* $\mathcal{Q}_{\mathfrak{r}}+\mathcal{Q}_{\mathfrak{p}}<1$, *then the solution of system* (1)-(2) *is Hyers-Ulam stable.*

*Proof.* It is easy to see that all the hypothesis of the Theorem (3.2) are satisfied. Also let $(\mathfrak{r}, \mathfrak{p}) \in \mathcal{C} \times \mathcal{C}$ is the approximate solution of inequalities (4) and $(\mathfrak{r}', \mathfrak{p}') \in \mathcal{C} \times \mathcal{C}$ be the unique solution of the couple system.

$$\begin{cases} D_{0^+}^{\kappa_1,\eta_1}\mathfrak{r}'(\mathfrak{t}) = \pounds(\mathfrak{t}, \mathfrak{r}'(\mathfrak{t}), \mathfrak{p}'(\mathfrak{t})), \\ D_{0^+}^{\kappa_2,\eta_2}\mathfrak{p}'(\mathfrak{t}) = \mathfrak{N}(\mathfrak{t}, \mathfrak{r}'(\mathfrak{t}), \mathfrak{p}'(\mathfrak{t})), \\ I^{1-\upsilon_1}\mathfrak{r}'(0) = \sum_{i=1}^m c_i\mathfrak{r}'(\xi_i),\ \xi_i \in [0, T], \\ I^{1-\upsilon_2}\mathfrak{p}'(0) = \sum_{i=1}^m d_i\mathfrak{p}'(\zeta_i),\ \zeta_i \in [0, T]. \end{cases}$$

In view of the Lemma (2.6), the solution of the above couple system are as follows

$$\mathfrak{r}'(\mathfrak{t}) = \frac{\mathfrak{Z}_1\mathfrak{t}^{\upsilon_1-1}}{\Gamma(\kappa_1)}\sum_{i=1}^m c_i\int_0^{\xi_i}(\xi_i - s)^{\kappa_1-1}\pounds(s, \mathfrak{r}'(s), \mathfrak{p}'(s))ds \\ + \frac{1}{\Gamma(\kappa_1)}\int_0^{\mathfrak{t}}(\mathfrak{t} - s)^{\kappa_1-1}\pounds(s, \mathfrak{r}'(s), \mathfrak{p}'(s))ds$$

and

$$\mathfrak{p}'(\mathfrak{t}) = \frac{\mathfrak{Z}_2\mathfrak{t}^{\upsilon_2-1}}{\Gamma(\kappa_2)}\sum_{i=1}^m d_i\int_0^{\zeta_i}(\zeta_i - s)^{\kappa_2-1}\mathfrak{N}(s, \mathfrak{r}'(s), \mathfrak{p}'(s))ds \\ + \frac{1}{\Gamma(\kappa_2)}\int_0^{\mathfrak{t}}(\mathfrak{t} - s)^{\kappa_2-1}\mathfrak{N}(s, \mathfrak{r}'(s), \mathfrak{p}'(s))ds.$$

Using Lemma (4.4), we get

$$\begin{aligned} |\mathfrak{r}(\mathfrak{t}) - \mathfrak{r}'(\mathfrak{t})| = &\Bigg|\mathfrak{r}(\mathfrak{t}) - \frac{\mathfrak{Z}_1\mathfrak{t}^{\upsilon_1-1}}{\Gamma(\kappa_1)}\sum_{i=1}^m c_i\int_0^{\xi_i}(\xi_i - s)^{\kappa_1-1}\pounds(s, \mathfrak{r}'(s), \mathfrak{p}'(s))ds \\ &+ \frac{1}{\Gamma(\kappa_1)}\int_0^{\mathfrak{t}}(\mathfrak{t} - s)^{\kappa_1-1}\pounds(s, \mathfrak{r}'(s), \mathfrak{p}'(s))ds\Bigg| \\ \leq &\Bigg|\mathfrak{r}(\mathfrak{t}) - \Bigg(\frac{\mathfrak{Z}_1\mathfrak{t}^{\upsilon_1-1}}{\Gamma(\kappa_1)}\sum_{i=1}^m c_i\int_0^{\xi_i}(\xi_i - s)^{\kappa_1-1}\pounds(s, \mathfrak{r}(s), \mathfrak{p}(s))ds \\ &+ \frac{1}{\Gamma(\kappa_1)}\int_0^{\mathfrak{t}}(\mathfrak{t} - s)^{\kappa_1-1}\pounds(s, \mathfrak{r}(s), \mathfrak{p}(s))ds\Bigg)\Bigg| \\ &+ \Bigg(\frac{|\mathfrak{Z}_1|\mathfrak{t}^{\upsilon_1-1}}{\Gamma(\kappa_1)}\sum_{i=1}^m c_i\int_0^{\xi_i}(\xi_i - s)^{\kappa_1-1}|\pounds(s, \mathfrak{r}(s), \mathfrak{p}(s)) \\ &- \pounds(s, \mathfrak{r}'(s), \mathfrak{p}'(s))|ds + \frac{1}{\Gamma(\kappa_1)}\int_0^{\mathfrak{t}}(\mathfrak{t} - s)^{\kappa_1-1}|\pounds(s, \mathfrak{r}(s), \mathfrak{p}(s)) \end{aligned}$$

$$
\begin{aligned}
&- \mathcal{L}(s, \mathfrak{r}'(s), \mathfrak{p}'(s))|ds\Big) \\
&\leq \mathcal{M}_1\delta_1 + \frac{|\mathfrak{Z}_1|\mathfrak{t}^{\upsilon_1-1}}{\Gamma(\kappa_1)} \sum_{i=1}^{m} c_i \int_0^{\xi_i} (\xi_i - s)^{\kappa_1-1} p_1(s)\Big(|\mathfrak{r}(s) - \mathfrak{r}'(s)| \\
&+ |\mathfrak{p}(s) - \mathfrak{p}'(s)|\Big) ds + \frac{1}{\Gamma(\kappa_1)} \int_0^{\mathfrak{t}} (\mathfrak{t} - s)^{\kappa_1-1} p_1(s) \\
&\times \Big(|\mathfrak{r}(s) - \mathfrak{r}'(s)| + |\mathfrak{p}(s) - \mathfrak{p}'(s)|\Big) ds.
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
\|\mathfrak{r} - \mathfrak{r}'\|_{\mathcal{C}_{\upsilon_1}} &\leq \mathcal{M}_1\delta_1 + \Bigg[\frac{p_1^*|\mathfrak{Z}_1|T^{\upsilon_1-1}}{\Gamma(\kappa_1)} \sum_{i=1}^{m} c_i \xi_i^{\kappa_1+\upsilon_1-1} B(\upsilon_1, \kappa_1) \\
&\quad + \frac{p_1^* T^{\kappa_1+\upsilon_1-1}}{\Gamma(\kappa_1)} B(\upsilon_1, \kappa_1)\Bigg][\|\mathfrak{r} - \mathfrak{r}'\|_{\mathcal{C}_{\upsilon_1}} + \|\mathfrak{p} - \mathfrak{p}'\|_{\mathcal{C}_{\upsilon_2}}] \\
&\leq \mathcal{M}_1\delta_1 + \mathcal{Q}_{\mathfrak{r}}[\|\mathfrak{r} - \mathfrak{r}'\|_{\mathcal{C}_{\upsilon_1}} + \|\mathfrak{p} - \mathfrak{p}'\|_{\mathcal{C}_{\upsilon_2}}] \\
(1 - \mathcal{Q}_{\mathfrak{r}})\|\mathfrak{r} - \mathfrak{r}'\|_{\mathcal{C}_{\upsilon_1}} &\leq \mathcal{M}_1\delta_1 + \mathcal{Q}_{\mathfrak{r}}\|\mathfrak{p} - \mathfrak{p}'\|_{\mathcal{C}_{\upsilon_2}}. \qquad (6)
\end{aligned}
$$

Similarly, we get

$$
(1 - \mathcal{Q}_{\mathfrak{p}})\|\mathfrak{p} - \mathfrak{p}'\|_{\mathcal{C}_{\upsilon_2}} \leq \mathcal{M}_2\delta_2 + \mathcal{Q}_{\mathfrak{p}}\|\mathfrak{r} - \mathfrak{r}'\|_{\mathcal{C}_{\upsilon_1}}, \qquad (7)
$$

where

$$
\mathcal{Q}_{\mathfrak{r}} = \frac{p_1^*|\mathfrak{Z}_1|T^{\upsilon_1-1}}{\Gamma(\kappa_1)} \sum_{i=1}^{m} c_i \xi_i^{\kappa_1+\upsilon_1-1} B(\upsilon_1, \kappa_1) + \frac{p_1^* T^{\kappa_1+\upsilon_1-1}}{\Gamma(\kappa_1)} B(\upsilon_1, \kappa_1)
$$

and

$$
\mathcal{Q}_{\mathfrak{p}} = \frac{p_2^*|\mathfrak{Z}_2|T^{\upsilon_2-1}}{\Gamma(\kappa_2)} \sum_{i=1}^{m} d_i \zeta_i^{\kappa_2+\upsilon_2-1} B(\upsilon_2, \kappa_2) + \frac{p_2^* T^{\kappa_2+\upsilon_2-1}}{\Gamma(\kappa_2)} B(\upsilon_2, \kappa_2).
$$

Inequalities (6) and (7) also written as

$$
\begin{cases}
(1 - \mathcal{Q}_{\mathfrak{r}})\|\mathfrak{r} - \mathfrak{r}'\|_{\mathcal{C}_{\upsilon_1}} - \mathcal{Q}_{\mathfrak{r}}\|\mathfrak{p} - \mathfrak{p}'\|_{\mathcal{C}_{\upsilon_2}} \leq \mathcal{M}_1\delta_1, \\
-\mathcal{Q}_{\mathfrak{p}}\|\mathfrak{r} - \mathfrak{r}'\|_{\mathcal{C}_{\upsilon_1}} + (1 - \mathcal{Q}_{\mathfrak{p}})\|\mathfrak{p} - \mathfrak{p}'\|_{\mathcal{C}_{\upsilon_2}} \leq \mathcal{M}_2\delta_2.
\end{cases} \qquad (8)
$$

The matrix form of equation (8) is

$$
\begin{bmatrix} (1 - \mathcal{Q}_{\mathfrak{r}}) & -\mathcal{Q}_{\mathfrak{r}} \\ -\mathcal{Q}_{\mathfrak{p}} & (1 - \mathcal{Q}_{\mathfrak{p}}) \end{bmatrix} \begin{bmatrix} \|\mathfrak{r} - \mathfrak{r}'\|_{\mathcal{C}_{\upsilon_1}} \\ \|\mathfrak{p} - \mathfrak{p}'\|_{\mathcal{C}_{\upsilon_2}} \end{bmatrix} \leq \begin{bmatrix} \mathcal{M}_1\delta_1 \\ \mathcal{M}_2\delta_2 \end{bmatrix}
$$

which implies that

$$\begin{bmatrix} \|\mathfrak{r}-\mathfrak{r}'\|_{\mathcal{C}_{\upsilon_1}} \\ \|\mathfrak{p}-\mathfrak{p}'\|_{\mathcal{C}_{\upsilon_2}} \end{bmatrix} \leq \frac{1}{\mathcal{A}} \begin{bmatrix} (1-\mathcal{Q}_{\mathfrak{p}}) & \mathcal{Q}_{\mathfrak{r}} \\ \mathcal{Q}_{\mathfrak{p}} & (1-\mathcal{Q}_{\mathfrak{r}}) \end{bmatrix} \begin{bmatrix} \mathcal{M}_1\delta_1 \\ \mathcal{M}_2\delta_2 \end{bmatrix}$$

where $\mathcal{A} = 1-\mathcal{Q}_{\mathfrak{p}}-\mathcal{Q}_{\mathfrak{r}}$. Further implies

$$\|\mathfrak{r}-\mathfrak{r}'\|_{\mathcal{C}_{\upsilon_1}} \leq \frac{(1-\mathcal{Q}_{\mathfrak{p}})}{\mathcal{A}}\mathcal{M}_1\delta_1 + \frac{\mathcal{Q}_{\mathfrak{r}}}{\mathcal{A}}\mathcal{M}_2\delta_2 \tag{9}$$

and

$$\|\mathfrak{p}-\mathfrak{p}'\|_{\mathcal{C}_{\upsilon_2}} \leq \frac{\mathcal{Q}_{\mathfrak{p}}}{\mathcal{A}}\mathcal{M}_1\delta_1 + \frac{(1-\mathcal{Q}_{\mathfrak{r}})}{\mathcal{A}}\mathcal{M}_2\delta_2. \tag{10}$$

Hence, from equations (9) and (10), we obtain

$$\begin{aligned} \|(\mathfrak{r},\mathfrak{p})-(\mathfrak{r}',\mathfrak{p}')\|_{\mathfrak{C}} &\leq \|\mathfrak{r}-\mathfrak{r}'\|_{\mathcal{C}_{\upsilon_1}} + \|\mathfrak{p}-\mathfrak{p}'\|_{\mathcal{C}_{\upsilon_2}} \\ &\leq \frac{(1-\mathcal{Q}_{\mathfrak{p}})}{\mathcal{A}}\mathcal{M}_1\delta_1 + \frac{\mathcal{Q}_{\mathfrak{r}}}{\mathcal{A}}\mathcal{M}_2\delta_2 + \frac{\mathcal{Q}_{\mathfrak{p}}}{\mathcal{A}}\mathcal{M}_1\delta_1 \\ &+ \frac{(1-\mathcal{Q}_{\mathfrak{r}})}{\mathcal{A}}\mathcal{M}_2\delta_2 \\ &\leq \mathcal{Q}_{\mathcal{M}_{1,2},\mathcal{A}}\delta. \end{aligned} \tag{11}$$

where $\mathcal{M}_{1,2} = max\{\mathcal{M}_1,\ \mathcal{M}_2\} > 0$, $\delta = max\{\delta_1, \delta_2\}$ and $\mathcal{Q}_{\mathcal{M}_{1,2},\mathcal{A}} = \frac{2}{\mathcal{A}}\mathcal{M}_{1,2}$. Hence the system (1)-(2) is Hyers-Ulam stable. □

**Lemma 4.6.** *If* $(\mathfrak{r},\mathfrak{p}) \in \mathcal{C}\times\mathcal{C}$ *satisfies the inequalities* (5) *and there are* $\mathcal{N}_1,\ \mathcal{N}_2 > 0$*, such that*

*(A4)*

$$I_{0+}^{\kappa_1}\varphi_1(\mathfrak{t}) \leq \mathcal{N}_1\varphi_1(\mathfrak{t}),\ I_{0+}^{\kappa_2}\varphi_2(\mathfrak{t}) \leq \mathcal{N}_2\varphi_2(\mathfrak{t}),$$

*then* $(\mathfrak{r},\mathfrak{p})$ *is the solution of the inequalities*

$$\begin{aligned} \Bigg|\mathfrak{r}(\mathfrak{t}) - \Bigg( &\frac{\mathfrak{Z}_1\mathfrak{t}^{\upsilon_1-1}}{\Gamma(\kappa_1)}\sum_{i=1}^{m} c_i\int_0^{\xi_i}(\xi_i-s)^{\kappa_1-1}\mathcal{L}(s,\mathfrak{r}(s),\mathfrak{p}(s))ds \\ &+ \frac{1}{\Gamma(\kappa_1)}\int_0^{\mathfrak{t}}(\mathfrak{t}-s)^{\kappa_1-1}\mathcal{L}(s,\mathfrak{r}(s),\mathfrak{p}(s))ds\Bigg)\Bigg| \leq \mathcal{N}_1\mathcal{M}_1\delta_1\varphi_1(\mathfrak{t}) \end{aligned}$$

*and*

$$\begin{aligned} \Bigg|\mathfrak{p}(\mathfrak{t}) - \Bigg( &\frac{\mathfrak{Z}_2\mathfrak{t}^{\upsilon_2-1}}{\Gamma(\kappa_2)}\sum_{i=1}^{m} d_i\int_0^{\zeta_i}(\zeta_i-s)^{\kappa_2-1}\mathfrak{N}(s,\mathfrak{r}(s),\mathfrak{p}(s))ds \\ &+ \frac{1}{\Gamma(\kappa_2)}\int_0^{\mathfrak{t}}(\mathfrak{t}-s)^{\kappa_2-1}\mathfrak{N}(s,\mathfrak{r}(s),\mathfrak{p}(s))ds\Bigg)\Bigg| \leq \mathcal{N}_2\mathcal{M}_2\delta_2\varphi_2(\mathfrak{t}). \end{aligned}$$

*Proof.* The proof of this lemma is directly derived from Lemma 4.4 and Remark 4.3. □

**Theorem 4.7.** *If the assumptions (A2), (A3), (A4) hold along with the condition $\mathcal{Q}_{\mathfrak{r}} + \mathcal{Q}_{\mathfrak{p}} < 1$, then the solution of system (1)-(2) is Hyers-Ulam-Rassias stable.*

*Proof.* It is easy to see that all the assumptions of the Theorem (3.2) are satisfied. Also let $(\mathfrak{r}, \mathfrak{p}) \in \mathcal{C} \times \mathcal{C}$ be an approximate solution of inequalities (5) and $(\mathfrak{r}', \mathfrak{p}') \in \mathcal{C} \times \mathcal{C}$ be the unique solution of the couple system.

$$\begin{cases} D_{0^+}^{\kappa_1,\eta_1}\mathfrak{r}'(\mathfrak{t}) = \pounds(\mathfrak{t}, \mathfrak{r}'(\mathfrak{t}), \mathfrak{p}'(\mathfrak{t})), \\ D_{0^+}^{\kappa_2,\eta_2}\mathfrak{p}'(\mathfrak{t}) = \mathfrak{N}(\mathfrak{t}, \mathfrak{r}'(\mathfrak{t}), \mathfrak{p}'(\mathfrak{t})), \\ I^{1-\upsilon_1}\mathfrak{r}'(0) = \sum_{i=1}^m c_i\mathfrak{r}'(\xi_i), \ \xi_i \in [0, T], \\ I^{1-\upsilon_2}\mathfrak{p}'(0) = \sum_{i=1}^m d_i\mathfrak{p}'(\zeta_i), \ \zeta_i \in [0, T]. \end{cases}$$

In view of the Lemma (2.6), we have

$$\begin{aligned} \mathfrak{r}'(\mathfrak{t}) = {} & \frac{\mathfrak{Z}_1\mathfrak{t}^{\upsilon_1-1}}{\Gamma(\kappa_1)}\sum_{i=1}^m c_i\int_0^{\xi_i}(\xi_i - s)^{\kappa_1-1}\pounds(s, \mathfrak{r}'(s), \mathfrak{p}'(s))ds \\ & + \frac{1}{\Gamma(\kappa_1)}\int_0^{\mathfrak{t}}(\mathfrak{t} - s)^{\kappa_1-1}\pounds(s, \mathfrak{r}'(s), \mathfrak{p}'(s))ds \end{aligned}$$

and

$$\begin{aligned} \mathfrak{p}'(\mathfrak{t}) = {} & \frac{\mathfrak{Z}_2\mathfrak{t}^{\upsilon_2-1}}{\Gamma(\kappa_2)}\sum_{i=1}^m d_i\int_0^{\zeta_i}(\zeta_i - s)^{\kappa_2-1}\mathfrak{N}(s, \mathfrak{r}'(s), \mathfrak{p}'(s))ds \\ & + \frac{1}{\Gamma(\kappa_2)}\int_0^{\mathfrak{t}}(\mathfrak{t} - s)^{\kappa_2-1}\mathfrak{N}(s, \mathfrak{r}'(s), \mathfrak{p}'(s))ds. \end{aligned}$$

Using Lemma (4.6), we have

$$\begin{aligned} |\mathfrak{r}(\mathfrak{t}) - \mathfrak{r}'(\mathfrak{t})| = {} & \Bigg|\mathfrak{r}(\mathfrak{t}) - \frac{\mathfrak{Z}_1\mathfrak{t}^{\upsilon_1-1}}{\Gamma(\kappa_1)}\sum_{i=1}^m c_i\int_0^{\xi_i}(\xi_i - s)^{\kappa_1-1}\pounds(s, \mathfrak{r}'(s), \mathfrak{p}'(s))ds \\ & - \frac{1}{\Gamma(\kappa_1)}\int_0^{\mathfrak{t}}(\mathfrak{t} - s)^{\kappa_1-1}\pounds(s, \mathfrak{r}'(s), \mathfrak{p}'(s))ds\Bigg| \\ \le {} & \Bigg|\mathfrak{r}(\mathfrak{t}) - \Bigg(\frac{\mathfrak{Z}_1\mathfrak{t}^{\upsilon_1-1}}{\Gamma(\kappa_1)}\sum_{i=1}^m c_i\int_0^{\xi_i}(\xi_i - s)^{\kappa_1-1}\pounds(s, \mathfrak{r}(s), \mathfrak{p}(s))ds \\ & + \frac{1}{\Gamma(\kappa_1)}\int_0^{\mathfrak{t}}(\mathfrak{t} - s)^{\kappa_1-1}\pounds(s, \mathfrak{r}(s), \mathfrak{p}(s))ds\Bigg)\Bigg| \end{aligned}$$

$$
\begin{aligned}
&+\Bigg(\frac{|\mathfrak{Z}_1|\mathfrak{t}^{\upsilon_1-1}}{\Gamma(\kappa_1)}\sum_{i=1}^{m}c_i\int_0^{\xi_i}(\xi_i-s)^{\kappa_1-1}|\mathcal{L}(s,\mathfrak{r}(s),\mathfrak{p}(s))\\
&-\mathcal{L}(s,\mathfrak{r}'(s),\mathfrak{p}'(s))|ds+\frac{1}{\Gamma(\kappa_1)}\int_0^{\mathfrak{t}}(\mathfrak{t}-s)^{\kappa_1-1}|\mathcal{L}(s,\mathfrak{r}(s),\mathfrak{p}(s))\\
&-\mathcal{L}(s,\mathfrak{r}'(s),\mathfrak{p}'(s))|ds\Bigg)\\
&\le \mathcal{N}_1\mathcal{M}_1\delta_1\varphi_1(\mathfrak{t})+\frac{|\mathfrak{Z}_1|\mathfrak{t}^{\upsilon_1-1}}{\Gamma(\kappa_1)}\sum_{i=1}^{m}c_i\int_0^{\xi_i}(\xi_i-s)^{\kappa_1-1}p_1(s)\\
&\times\Bigg(|\mathfrak{r}(s)-\mathfrak{r}'(s)|+|\mathfrak{p}(s)-\mathfrak{p}'(s)|\Bigg)ds\\
&+\frac{1}{\Gamma(\kappa_1)}\int_0^{\mathfrak{t}}(\mathfrak{t}-s)^{\kappa_1-1}p_1(s)\Bigg(|\mathfrak{r}(s)-\mathfrak{r}'(s)|+|\mathfrak{p}(s)-\mathfrak{p}'(s)|\Bigg)ds.
\end{aligned}
$$

Therefore, we get

$$
\begin{aligned}
\|\mathfrak{r}-\mathfrak{r}'\|_{\mathcal{C}_{\upsilon_1}} &\le \mathcal{N}_1\mathcal{M}_1\delta_1\varphi_1(\mathfrak{t})+\Bigg[\frac{p_1^*|\mathfrak{Z}_1|T^{\upsilon_1-1}}{\Gamma(\kappa_1)}\sum_{i=1}^{m}c_i\xi_i^{\kappa_1+\upsilon_1-1}B(\upsilon_1,\kappa_1)\\
&\quad+\frac{p_1^*T^{\kappa_1+\upsilon_1-1}}{\Gamma(\kappa_1)}B(\upsilon_1,\kappa_1)\Bigg][\|\mathfrak{r}-\mathfrak{r}'\|_{\mathcal{C}_{\upsilon_1}}+\|\mathfrak{p}-\mathfrak{p}'\|_{\mathcal{C}_{\upsilon_2}}]\\
&\le \mathcal{N}_1\mathcal{M}_1\delta_1\varphi_1(\mathfrak{t})+\mathcal{Q}_{\mathfrak{r}}[\|\mathfrak{r}-\mathfrak{r}'\|_{\mathcal{C}_{\upsilon_1}}+\|\mathfrak{p}-\mathfrak{p}'\|_{\mathcal{C}_{\upsilon_2}}].
\end{aligned}
$$

This implies that

$$
(1-\mathcal{Q}_{\mathfrak{r}})\|\mathfrak{r}-\mathfrak{r}'\|_{\mathcal{C}_{\upsilon_1}}\le \mathcal{N}_1\mathcal{M}_1\delta_1\varphi_1(\mathfrak{t})+\mathcal{Q}_{\mathfrak{r}}\|\mathfrak{p}-\mathfrak{p}'\|_{\mathcal{C}_{\upsilon_2}}. \tag{12}
$$

Simillarly, we get

$$
(1-\mathcal{Q}_{\mathfrak{p}})\|\mathfrak{p}-\mathfrak{p}'\|_{\mathcal{C}_{\upsilon_2}}\le \mathcal{N}_2\mathcal{M}_2\delta_2\varphi_2(\mathfrak{t})+\mathcal{Q}_{\mathfrak{p}}\|\mathfrak{r}-\mathfrak{r}'\|_{\mathcal{C}_{\upsilon_1}}, \tag{13}
$$

where

$$
\mathcal{Q}_{\mathfrak{r}}=\frac{p_1^*|\mathfrak{Z}_1|T^{\upsilon_1-1}}{\Gamma(\kappa_1)}\sum_{i=1}^{m}c_i\xi_i^{\kappa_1+\upsilon_1-1}B(\upsilon_1,\kappa_1)+\frac{p_1^*T^{\kappa_1+\upsilon_1-1}}{\Gamma(\kappa_1)}B(\upsilon_1,\kappa_1)
$$

and

$$
\mathcal{Q}_{\mathfrak{p}}=\frac{p_2^*|\mathfrak{Z}_2|T^{\upsilon_2-1}}{\Gamma(\kappa_2)}\sum_{i=1}^{m}d_i\zeta_i^{\kappa_2+\upsilon_2-1}B(\upsilon_2,\kappa_2)+\frac{p_2^*T^{\kappa_2+\upsilon_2-1}}{\Gamma(\kappa_2)}B(\upsilon_2,\kappa_2).
$$

Inequalities (12) and (13) also written as

$$
\begin{cases}
(1-\mathcal{Q}_{\mathfrak{r}})\|\mathfrak{r}-\mathfrak{r}'\|_{\mathcal{C}_{\upsilon_1}}-\mathcal{Q}_{\mathfrak{r}}\|\mathfrak{p}-\mathfrak{p}'\|_{\mathcal{C}_{\upsilon_2}}\le \mathcal{N}_1\mathcal{M}_1\delta_1\varphi_1(\mathfrak{t}),\\
-\mathcal{Q}_{\mathfrak{p}}\|\mathfrak{r}-\mathfrak{r}'\|_{\mathcal{C}_{\upsilon_1}}+(1-\mathcal{Q}_{\mathfrak{p}})\|\mathfrak{p}-\mathfrak{p}'\|_{\mathcal{C}_{\upsilon_2}}\le \mathcal{N}_2\mathcal{M}_2\delta_2\varphi_2(\mathfrak{t}).
\end{cases} \tag{14}
$$

The matrix form of equation (14) is

$$\begin{bmatrix}(1-\mathcal{Q}_{\mathfrak{r}}) & -\mathcal{Q}_{\mathfrak{r}}\\ -\mathcal{Q}_{\mathfrak{p}} & (1-\mathcal{Q}_{\mathfrak{p}})\end{bmatrix}\begin{bmatrix}\|\mathfrak{r}-\mathfrak{r}'\|_{\mathcal{C}_{\upsilon_1}}\\ \|\mathfrak{p}-\mathfrak{p}'\|_{\mathcal{C}_{\upsilon_2}}\end{bmatrix}\leq\begin{bmatrix}\mathcal{N}_1\mathcal{M}_1\delta_1\varphi_1(\mathfrak{t})\\ \mathcal{N}_2\mathcal{M}_2\delta_2\varphi_2(\mathfrak{t})\end{bmatrix}.$$

This implies that

$$\begin{bmatrix}\|\mathfrak{r}-\mathfrak{r}'\|_{\mathcal{C}_{\upsilon_1}}\\ \|\mathfrak{p}-\mathfrak{p}'\|_{\mathcal{C}_{\upsilon_2}}\end{bmatrix}\leq\frac{1}{\mathcal{A}}\begin{bmatrix}(1-\mathcal{Q}_{\mathfrak{p}}) & \mathcal{Q}_{\mathfrak{r}}\\ \mathcal{Q}_{\mathfrak{p}} & (1-\mathcal{Q}_{\mathfrak{r}})\end{bmatrix}\begin{bmatrix}\mathcal{N}_1\mathcal{M}_1\delta_1\varphi_1(\mathfrak{t})\\ \mathcal{N}_2\mathcal{M}_2\delta_2\varphi_2(\mathfrak{t})\end{bmatrix}.$$

where $\mathcal{A}=1-\mathcal{Q}_{\mathfrak{p}}-\mathcal{Q}_{\mathfrak{r}}$. Further implies

$$\|\mathfrak{r}-\mathfrak{r}'\|_{\mathcal{C}_{\upsilon_1}}\leq\frac{(1-\mathcal{Q}_{\mathfrak{p}})}{\mathcal{A}}\mathcal{N}_1\mathcal{M}_1\delta_1\varphi_1(\mathfrak{t})+\frac{\mathcal{Q}_{\mathfrak{r}}}{\mathcal{A}}\mathcal{N}_2\mathcal{M}_2\delta_2\varphi_2(\mathfrak{t}) \tag{15}$$

and

$$\|\mathfrak{p}-\mathfrak{p}'\|_{\mathcal{C}_{\upsilon_2}}\leq\frac{\mathcal{Q}_{\mathfrak{p}}}{\mathcal{A}}\mathcal{N}_1\mathcal{M}_1\delta_1\varphi_1(\mathfrak{t})+\frac{(1-\mathcal{Q}_{\mathfrak{r}})}{\mathcal{A}}\mathcal{N}_2\mathcal{M}_2\delta_2\varphi_2(\mathfrak{t}). \tag{16}$$

Hence, from equations (15) and (16), we have

$$\begin{aligned}\|(\mathfrak{r},\mathfrak{p})-(\mathfrak{r}',\mathfrak{p}')\|_{\mathfrak{C}}&\leq\|\mathfrak{r}-\mathfrak{r}'\|_{\mathcal{C}_{\upsilon_1}}+\|\mathfrak{p}-\mathfrak{p}'\|_{\mathcal{C}_{\upsilon_2}}\\ &\leq\frac{(1-\mathcal{Q}_{\mathfrak{p}})}{\mathcal{A}}\mathcal{N}_1\mathcal{M}_1\delta_1\varphi_1(\mathfrak{t})+\frac{\mathcal{Q}_{\mathfrak{r}}}{\mathcal{A}}\mathcal{N}_2\mathcal{M}_2\delta_2\varphi_2(\mathfrak{t})\\ &+\frac{\mathcal{Q}_{\mathfrak{p}}}{\mathcal{A}}\mathcal{N}_1\mathcal{M}_1\delta_1\varphi_1(\mathfrak{t})+\frac{(1-\mathcal{Q}_{\mathfrak{r}})}{\mathcal{A}}\mathcal{N}_2\mathcal{M}_2\delta_2\varphi_2(\mathfrak{t})\\ &\leq\mathcal{Q}_{\mathcal{N}_{1,2},\mathcal{M}_{1,2},\mathcal{A}}\delta\varphi(\mathfrak{t}).\end{aligned} \tag{17}$$

where $\mathcal{N}_{1,2}=max\{\mathcal{N}_1,\ \mathcal{N}_2\}>0$, $\mathcal{M}_{1,2}=max\{\mathcal{M}_1,\ \mathcal{M}_2\}>0$, $\delta=max\{\delta_1,\delta_2\}$, $\varphi(\mathfrak{t})=max\{\varphi_1(\mathfrak{t}),\ \varphi_2(\mathfrak{t})\}$ and $\mathcal{Q}_{\mathcal{N}_{1,2},\mathcal{M}_{1,2},\mathcal{A}}=\frac{2}{\mathcal{A}}\mathcal{N}_{1,2}\mathcal{M}_{1,2}$. Hence the system (1)-(2) is Hyers-Ulam-Rassias stable. □

## 5 Applications

In this section, we validate our results with the help of examples.

**Example 5.1.** *Consider fractional differential equation*

$$\begin{cases} D_{0+}^{\frac{1}{2},\frac{1}{4}}\mathfrak{r}(s) &= \frac{e^{-s}}{10+s^2}+\frac{\mathfrak{r}(s)}{40(1+\mathfrak{p}^2(s))}+\frac{1}{30\sqrt{4+s^2}}\cos\mathfrak{p}(s),\ \ s\in\mathcal{J}=(0,1],\\ D_{0+}^{\frac{1}{3},\frac{1}{5}}\mathfrak{p}(s) &= \frac{1}{\sqrt{25+s^2}}\cos(s)+\frac{1}{15}(\mathfrak{r}(s))+\frac{1}{100}e^{-2s}\sin\mathfrak{p}(s),\end{cases} \tag{18}$$

*with nonlocal conditions*

$$\begin{cases} I_{0+}^{\frac{3}{8}}\mathfrak{r}(0) &= 3\mathfrak{r}(\frac{1}{6}),\\ I_{0+}^{\frac{8}{15}}\mathfrak{p}(0) &= 4\mathfrak{p}(\frac{1}{7}).\end{cases} \tag{19}$$

*Here* $\kappa_1 = \frac{1}{2}$, $\eta_1 = \frac{1}{4}$, $\kappa_2 = \frac{1}{3}$, $\eta_2 = \frac{1}{5}$, $\upsilon_1 = \frac{5}{8}$ *and* $\upsilon_2 = \frac{7}{15}$. *Take*

$$\pounds(s, \mathfrak{r}, \mathfrak{p}) = \frac{e^{-s}}{10 + s^2} + \frac{\mathfrak{r}(s)}{40(1 + \mathfrak{p}^2(s))} + \frac{1}{30\sqrt{4 + s^2}} \cos \mathfrak{p}(s)$$

*and*

$$\mathfrak{N}(s, \mathfrak{r}, \mathfrak{p}) = \frac{1}{\sqrt{25 + s^2}} \cos(s) + \frac{1}{15}(\mathfrak{r}(s)) + \frac{1}{100} e^{-2s} \sin \mathfrak{p}(s),$$

$s \in \mathfrak{J}$ *and* $\mathfrak{r},\ \mathfrak{p} \in [0, \infty)$. *Clearly,* $\pounds$ *and* $\mathfrak{N}$ *are continuous and*

$$|\pounds(s, \mathfrak{r}, \mathfrak{p})| \leq \frac{1}{10} + \frac{1}{40}|\mathfrak{r}| + \frac{1}{60}|\mathfrak{p}|$$

*and*

$$|\mathfrak{N}(s, \mathfrak{r}, \mathfrak{p})| \leq \frac{1}{5} + \frac{1}{15}|\mathfrak{r}| + \frac{1}{100}|\mathfrak{p}| \text{ for any } \mathfrak{r}, \mathfrak{p} \in [0, \infty) \text{ and } s \in \mathfrak{J}.$$

*Therefore, the assumption (A1) is satisfied. Also* $\mathfrak{U} = \mathfrak{B}_0 + \mathfrak{B}_1 = 0.0587 + 0.1770 = 0.2357$, $\mathfrak{T} = \mathfrak{D}_0 + \mathfrak{D}_1 = 0.0391 + 0.0265 = 0.0656$ *and* $\max\{\mathfrak{U}, \mathfrak{T}\} = 0.2357 < 1$. *Hence all the hypothesis of Theorem* (3.1) *are satisfied and therefore the system* (18)-(19) *has at least one solution.*

**Example 5.2.** *Consider fractional differential equation of the form*

$$\begin{cases} D_{2^+}^{\frac{1}{2},\frac{1}{3}}\mathfrak{r}(s) &= \frac{6+|\mathfrak{r}(s)|+|\mathfrak{p}(s)|}{5e^{s+4}(1+|\mathfrak{r}(s)|+|\mathfrak{p}(s)|)}, \quad s \in \mathfrak{J} = (2, 3], \\ D_{2^+}^{\frac{1}{5},\frac{1}{4}}\mathfrak{p}(s) &= \frac{1}{e^{2s}}[\sin(\mathfrak{r}(s)) + \sin(\mathfrak{p}(s))], \end{cases} \tag{20}$$

*with nonlocal conditions as*

$$\begin{cases} I_{2^+}^{\frac{1}{3}}\mathfrak{r}(2) &= 2\mathfrak{r}(\frac{11}{5}) + 3\mathfrak{r}(\frac{13}{5}), \\ I_{2^+}^{\frac{3}{5}}\mathfrak{p}(2) &= 4\mathfrak{p}(\frac{12}{5}) + 5\mathfrak{p}(\frac{14}{5}). \end{cases} \tag{21}$$

*Here* $\kappa_1 = \frac{1}{2}$, $\eta_1 = \frac{1}{3}$, $\kappa_2 = \frac{1}{5}$, $\eta_2 = \frac{1}{4}$, $\upsilon_1 = \frac{2}{3}$ *and* $\upsilon_2 = \frac{2}{5}$. *Take*

$$\pounds(s, \mathfrak{r}, \mathfrak{p}) = \frac{6 + |\mathfrak{r}(s)| + |\mathfrak{p}(s)|}{5e^{s+4}(1 + |\mathfrak{r}(s)| + |\mathfrak{p}(s)|)}, \ s \in \mathfrak{J} \text{ and } \mathfrak{r},\ \mathfrak{p} \in [0, \infty)$$

*and*

$$\mathfrak{N}(s, \mathfrak{r}, \mathfrak{p}) = \frac{1}{e^{2s}}[\sin(\mathfrak{r}(s)) + \sin(\mathfrak{p}(s))],\ s \in \mathfrak{J} \text{ and } \mathfrak{r},\ \mathfrak{p} \in [0, \infty).$$

*Clearly* $\pounds$ *and* $\mathfrak{N}$ *is continuous and*

$$|\pounds(s, \mathfrak{r}, \mathfrak{p}) - \pounds(s, \bar{\mathfrak{r}}, \bar{\mathfrak{p}})| \leq \frac{1}{5e^6}(|\mathfrak{r} - \bar{\mathfrak{r}}| + |\mathfrak{p} - \bar{\mathfrak{p}}|)$$

*and*

$$|\mathfrak{N}(s,\mathfrak{r},\mathfrak{p})-\mathfrak{N}(s,\bar{\mathfrak{r}},\bar{\mathfrak{p}})| \leq \frac{1}{e^4}(|\mathfrak{r}-\bar{\mathfrak{r}}|+|\mathfrak{p}-\bar{\mathfrak{p}}|) \text{ for any } \mathfrak{r},\mathfrak{p},\bar{\mathfrak{r}},\bar{\mathfrak{p}} \in [0,\infty) \text{ and } s \in \mathfrak{J}.$$

*Therefore the assumption (A2) is satisfied. Since* $p_1^* = \frac{1}{5e^6}$ *and* $p_2^* = \frac{1}{e^4}$ *then we have*

$$\begin{aligned}\mathfrak{G} = &\left[\frac{p_1^*|\mathfrak{Z}_1|}{\Gamma(\kappa_1)}\sum_{i=1}^{m} c_i\xi_i^{\kappa_1+\upsilon_1-1}B(\upsilon_1,\kappa_1)+\frac{p_1^*T^{\kappa_1}}{\Gamma(\kappa_1)}B(\upsilon_1,\kappa_1)\right.\\ &\left.+\frac{p_2^*|\mathfrak{Z}_2|}{\Gamma(\kappa_2)}\sum_{i=1}^{m} d_i\zeta_i^{\kappa_2+\upsilon_2-1}B(\upsilon_2,\kappa_2)+\frac{p_2^*T^{\kappa_2}}{\Gamma(\kappa_2)}B(\upsilon_2,\kappa_2)\right]\\ &\leq 0.0103\\ &< 1.\end{aligned}$$

*As assumption (A3) is satisfied. It follows from Theorem (3.2) that the problem (20) has unique solution. Also the condition*

$$\begin{aligned}\mathfrak{Q}_{\mathfrak{r}}+\mathfrak{Q}_{\mathfrak{p}} = &\frac{p_1^*|\mathfrak{Z}_1|T^{\upsilon_1-1}}{\Gamma(\kappa_1)}\sum_{i=1}^{m} c_i\xi_i^{\kappa_1+\upsilon_1-1}B(\upsilon_1,\kappa_1)+\frac{p_1^*T^{\kappa_1+\upsilon_1-1}}{\Gamma(\kappa_1)}B(\upsilon_1,\kappa_1)\\ &+\frac{p_2^*|\mathfrak{Z}_2|T^{\upsilon_2-1}}{\Gamma(\kappa_2)}\sum_{i=1}^{m} d_i\zeta_i^{\kappa_2+\upsilon_2-1}B(\upsilon_2,\kappa_2)+\frac{p_2^*T^{\kappa_2+\upsilon_2-1}}{\Gamma(\kappa_2)}B(\upsilon_2,\kappa_2)\\ &= 0.0059\\ &< 1.\end{aligned}$$

*Hence we conclude from Theorem (4.5) that the problem (20) is Ulam-Hyers stable.*

*If we take* $\varphi_1(s) = s$ *and* $\varphi_2(s) = e^{4s}$ *for* $s \in \mathfrak{J}$, *then we have*

$$\begin{aligned}I_{2+}^{\frac{1}{2}}\varphi_1(s) = &\frac{1}{\Gamma(\frac{1}{2})}\int_2^s (s-\mathfrak{t})^{-\frac{1}{2}}\mathfrak{t}\,d\mathfrak{t}\\ \leq &\frac{s}{\Gamma(\frac{1}{2})}\int_2^s (s-\mathfrak{t})^{-\frac{1}{2}}\,d\mathfrak{t}\\ \leq &\frac{2\varphi_1(s)}{\sqrt{\pi}}\end{aligned}$$

*and*

$$I_{2+}^{\frac{1}{5}}\varphi_2(s) = \frac{1}{\Gamma(\frac{1}{5})}\int_2^s (s-\mathfrak{t})^{-\frac{4}{5}}e^{4\mathfrak{t}}\,d\mathfrak{t}$$

$$
\begin{aligned}
&\leq \frac{e^{4s}}{\Gamma(\frac{1}{5})} \int_2^s (s-\mathfrak{t})^{-\frac{4}{5}}\, d\mathfrak{t} \\
&\leq \frac{5\varphi_2(s)}{\Gamma(\frac{1}{5})}.
\end{aligned}
$$

*Consequently hypothesis* $(A4)$ *is satisfied with* $\mathcal{N}_1 = \frac{2}{\sqrt{\pi}}$, $\mathcal{N}_2 = \frac{5}{\Gamma(\frac{1}{5})}$. *Hence all assumptions of Theorem* (4.7) *are satisfied then we say that system* (20) *is Ulam-Hyers-Rassias stable.*

## 6 Conclusion

In this manuscript, we have used the Arzelá-Ascoli theorem, Leray-Schauder's alternative and Banach contraction principle for establishing the existence and uniqueness results for the solution of the proposed couple Hilfer fractional differential system with nonlocal conditions. The nonlocal conditions which we have considered in this manuscript yields better results than the initial and boundary conditions. With certain assumptions, for the given problem, we have succeed to establish the stability results for the solution using Ulam-Hyers and Ulam-Hyers-Rassias stabilities. From the obtained results, we conclude that such an approach is very powerful, effective, and suitable for the solutions of systems of nonlinear Hilfer fractional differential equation with a nonlocal conditions. For future works, we would like to consider other fractional operators such as the $\psi$-Hilfer fractional operator, Katugampola fractional operator, etc. to establish the stability results.

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Kanika Dhawan
Department of Mathematics & Scientific Computing
National Institute of Technology Hamirpur.
HP-177005 India
Email: dhawankanika83@gmail.com

Ramesh Kumar Vats
Department of Mathematics & Scientific Computing
National Institute of Technology Hamirpur
HP-177005 India
Email: rkvatsnitham@gmail.com

Ravi P. Agarwal
Department of Mathematics
Texas A&M University-Kingsville 700 University Blvd., MSC 172 Kingsville, Texas.
Email: Ravi.Agarwal@tamuk.edu