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Nearest neighbor estimates of Kaniadakis entropy

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Abstract

The aim of this paper is to develop new nonparametric estimators of entropy based on the k^{th} nearest neighbor distances that are considered between n sample points, $k \leq (n-1)$ being a positive integer, fixed. The Method consists in using the new estimators which were useful in order to evaluate the entropies for random vectors. As results, using the Kaniadakis entropy measure, the asymptotic unbiasedness and consistency of the estimators are proven.

Keywords:*Kaniadakis entropy, estimator, k^{th}- nearest neighbor, variance, distribution.*

1 INTRODUCTION

k-nearest neighbor (k-NN) has become a popular and powerful idea that can be used as a technique for nonparametric density information, which takes as a function of the training data, the region volume.
This method has been firstly developed by Evelyn Fix and Joseph Hodges back in 1951. It is used both for classification and regression. The input embodies the k-closest training samples in the data set. In the k-NN classification, the function is approximated locally.
When k=1, we see the nearest neighbor due to the fact that the considered class is the closest training sample.

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Information measures and their variants can be applied in numerous data science domains such as causal inference, computational biology, sociology even. Mutual information is a well-established concept in information theory, one can quantify the mutual dependence between two variables by using it.
Regarding more pratical reasons, Beirlant et al. (2001) gave in [4] and overview of several methods used for the nonparametric estimation for the differential entropy of a random variable which is continuous. The properties of these methods were compared and application were given, among which we can mention the goodness-of fit test, the parameter estimation, spectral estimation and also the quantization theory. In [23], Singh et al. (2003), investigated for standard distributions the performance of the proposed estimators in the case of finite samples using simulations of type Monte-Carlo. The estimators were applied in order to estimate the entropy of the internal rotation in a methanol molecule, which was characterized by a one-dimensional random vector and also, for diethyl ether, which was described by a four-dimensional random vector. In [11] Lefvre et al. used the Monte-Carlo control technique to reduce training over-fitting and to improve robustness to semnatic noise in the user input. This uses a database of belief vector prototypes in order to choose the optimal system action.
A locally weighted k-nearest neighbor scheme is instituted to smooth the decision process by interpolating the value function, which resulted in higher user simulation performance.
The recent papers concerning the Markov models and different classes of distributions could be of use for our research, such as: Barbu [2] that defined in a discrete-time semi-Markov model and proposed a computation procedure for solving the corresponding Markov renewal equation, which is necessary for all the reliability measurements. Then, the reliability and its related measures were computed, and the results were applied to a three-state system. In 2014 Preda et al. constructed in [18] the minimal entropy martingale for semi-Markov regime switching interest rate models using some general entropy measures.
They proved that, for the one-period model, the minimal entropy martingale for semi-Markov processes in the case of the Tsallis and Kaniadakis entropies are the same as in the case of Shannon entropy.
Panait [13] introduced a weighted entropic copula from preliminary knowledge of dependence. It has been considered a copula with common distribution, based on which was then formulated the weighted entropy dependence model (WMEC). An approximator for the copula function for the studied problem and asymptotical properties regarding the unknown parameters of the model were determined.
In [3] Barbu, Karagrigoriu and Makrides were interested in a general class of distributions for independent (yet not necessarily identically distributed) random variables, closed under minima, in which the main parameter involved has been assumed to be time varying with several possible modeling options. It concerns the reliability and survival analysis for describing the time to event or failure. The maximum likelihood estimation of the parameters was addressed and the asymptotic properties of the estimators were discussed.
In 2009, Sricharan, Raich and Hero [26] analyzed a kth nearest neighbor class of plug

in estimators for estimating the Shannon and Rnyi entropy and derived explicit rates for the bias and variance of these plug-in estimators based on sample size and the underlying probability distribution. They also established a central limit theorem for these plug-in estimators that allow the specification of the confidence intervals on the entropy functionals. They used the created theory in an application regarding anomalies detection problems to specify thresholds in order to achieve desired false alarm rates. In the same year, Debreuve [7] solved various image and video processing tasks by handling the features is the same way, for both low and high-dimensions with the help of kth nearest neighbor estimators.

In 2011, Li et al. proposed in [12] a consistent entropy estimator for hyperspherical data, based on the kth nearest neighbor approach. Simulation studies were conducted to evaluate the performance of the estimators for models such as uniform distribution of von Mises-Fisher distributions.

In simulations, the k-NN entropy estimator was compared with the moment based counterpart and the results showed that the two methods were indeed, comparable.

Zamanzade et al. [29] introduced two new estimators that were used for the entropy estimation in the case of absolutely continuous random variables. They compared them with the first existing entropy estimators, such as the ones proposed by Dimitriev and Tarasenko [On the estimation functions of the probability density and its derivatives, Theory Probab. Appl. 18 (1973), pp. 628-633] and after, proposed goodness-of-fit tests on the newly introduced entropy estimators for normality and also compared their powers with the ones of the other for normality entropy based tests. Their simulation results performed well in the estimations of entropy and normality testing.

The entropy field is unlimited. And so are the types of estimators that could be given in the future. To mention just a couple of papers of interest and domains of feasibility in 2015, Sheraz, Dedu and Preda [18] applied the concept of entropy for underlying financial markets to make a comparison between volatile markets.

They considered as a first step Shannon entropy with different estimators, Tsallis entropy for different values of its parameter, Rnyi entropy and finally the approximate entropy and provided computational results for these entropies for weekly and monthly data in the case of four different stock indices.

In [14] Popescu et al. established new inequalities for JeffreysTsallis and Jensen-Shannon

Tsallis divergences. Their results refined and generalized recent results in Tsallis theory. In [17] Preda and Dedu derived new distribution families for modeling the income distribution by using the entropy maximization principle with Tsallis entropy. New classes of Lorenz curves were obtained by applying the entropy maximization principle with Tsallis entropy, under mean and Gini index equality and inequality constraints.In [14] Popescu et al. established new inequalities for JeffreysTsallis and JensenShannonTsallis divergences. Their results refined and generalized recent results in Tsallis theory. In [17] Preda and Dedu derived new distribution families for modeling the income distribution by using the entropy maximization principle with Tsallis entropy. New classes of Lorenz curves were obtained by applying the entropy maximization principle with Tsallis entropy, under mean and Gini index equality and

inequality constraints.
In 2016, Sfetcu [19] considered a sequence of generalized Jacobi polynomials. It has been defined a discrete probability distribution and considered the Tsallis (resp. Rnyi) divergence for it. It has been observed the asymptotic behavior of Tsallis (resp. Rnyi) divergence defined above and for the quadratic Tsallis (resp. Rnyi) divergence, an explicit formula has been given. In 2021, a new generalization of AwadShannon entropy, more precisely AwadVarma entropy has been studied in [20], a stochastic order on AwadVarma residual entropy was introduced and its properties of this order were studied, amongst which we can mention: closure, reversed closure and preservation in some stochastic models. Later, in [21] a stochastic order for Varma residual entropy was given, together with several of its properties, such as: closure, reversed closure and preservation.
In the past, Keller et al. [9] used mutual information to estimate raw data streams in an immediate period of time; also, Sorjamaa et al. [25] were using mutual information and k^{th} nearest neighbor approximator for time series prediction.
Considering to analyze, just like the Kozachenko-Leonenko fixed k^{th} nearest neighbor estimator for differential entropy, in 2016 Singh et al. [24] offered an analysis of k-nearest neighbour distances with applications to entropy estimation.
More recently, in 2018 Zhao and Lai [30] used the KSG mutual information estimator, which is based on the distances of each sample to its k^{th} nearest neighbor, to estimate mutual information between two continuous random variables.
In 2018 Jiao, Gao and Han [8] obtained the first uniform upper bound on its performance for a fixed k over Hlder balls, on a torus, without assuming any conditions on how close the density could be from zero.
Shannon entropy estimators can be calculated using the logarithm of the determinant that estimates the variance-covariance matrix S containing independent observations. For continuous distributions we can distinguish three types of methods that are proposed for estimation. The first one seeks to convert the continuous distribution to a discrete one, by using bin method. The second, tries to learn the underlying distribution first and after to calculate the entropy and the mutual information.
The last type of method directly estimates the entropy and mutual information based on k^{th} nearest neighbor, or the so-called k-NN distances. This is the one that we will approach throughout this paper.
Let $X_1, X_2, \ldots, X_n$ be n copies of a p-dimensional random variable X, all independent, having an absolutely continuous distribution function, F(x) and the probability density function f(x) where $f(x) = f(x_1, \ldots, x_n), dx = dx_1, \ldots, dx_{pi}$, p the torsional angles of molecules for evaluating the entropy.
We assume that the differential entropy $HK_a(f)$ as an integral estimate of entropy, under the following form, which is well-defined and finite:

$$HK_a(f) = E\left[-ln_a^K f(X)\right] = -\int_{-\infty}^{\infty} I_S(x) f(x) ln_a^K f(X)\, dx_1, \ldots, dx_n$$

E being the expectation taken relative to X . Here, S is the region where the pdf $f(.)$ is positive and $I_S(.)$ in the indicator function for S.
In the following, well make the notations starting with the ones of Dudewicz and

van der Meulen, 1981, Vasicek, 1976 in a one-dimensional space, so for $p = 1$, an estimator based on spacing has the form:

$$\widehat{V}_m^{(n)} = \frac{1}{n-m}\sum_{i=1}^{n-m} ln\left(\frac{n}{m}\left(X_{i+m:n} - X_{i:n}\right)\right) - \Psi\left(m\right) + ln\left(m\right)$$

where $m \leq (n-1)$ is a positive integer, fixed; $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$ are the order statistic of $X_1, X_2, \ldots, X_n$ and $\Psi\left(m\right) = \frac{\Gamma(m)}{\Gamma(m)}$ the digamma function. This estimator will be called the m-spacing estimator.

For p considered in a general case, Kozachenko and Leonenko [10] proposed a non-parametric estimate of entropy, to be based on the nearest neighbor distances between the sample points.

Thus, let $S_{r;x}$ be the sphere of radius $r > 0$, centered at $z \in R^p$, the p-dimensional Euclidean space.

Then, the volume of $S_{r;x}$ is:

$$V_r = \frac{\pi^{p/2} r^p}{\Gamma\left(\frac{p}{2}+1\right)}\ where\ \Gamma_a = \int_0^\infty x^{a-1}e^{-x}dx, a > 0$$

Let ρ_i be min $\{\|X_i - X_j\|, j \in \{1, 2, \ldots, n\} - \{i\}\}$

Based on the first nearest neighbor distances, for the π_i values, Kozachenko and Leonenko proposed in 1987 [10] the estimator H_n of $H(f)$ the Shannon entropy of f :

$$H_n = \frac{p}{n}\sum_{i=1}^{n} ln\rho_i + ln\left[\frac{\pi^{\frac{p}{2}}}{\Gamma\left(\frac{p}{2}+1\right)}\right] + \gamma + ln\left(n-1\right)$$

The differential entropy has important extremal properties, such as:

- for a concentrated density f on the unit interval [0,1], the differential entropy is maximal if f is uniform on [0, 1] and $H\left(f\right) = 0$.

- for a concentrated density on the positive half line and for a fixed expectation, the differential entropy takes the maximum for the exponential distribution.

- for a density with fixed variance, the differential entropy will be maximized by the Gaussian density.

Where $\gamma = 0.5772\ldots$ is the Eulers constant and by which formula they proved the asymptotic unbiasedness and consistency of H_n. For $p = 1$, Tsybakov and van der Meulen [27] have established the mean square root n consistenct for a truncated version of H_n. They proved the mean square $\sqrt{n}$-consistency of the estimator for a class of densities with unbounded support, including the Gaussian density.

Since the random variable S is absolutely continuous, its nearest neighbour distances (the values ρ_i) are expected to be small positive numbers. Due to the presence of n ln factors in the expression of H_n, small fluctuations in the small ρ_i values will result in higher fluctuations in the values of H_n.

In Section 2 of the present paper, is defined an entropy estimator based on the j-th nearest neighbour distances. We use the Kaniadakis entropy measure, that represents the basis of our theoretical developments. We will therefore be working with the Kaniadakis logarithm, which is represented under the following form:

$$f(x) = \begin{cases} \dfrac{f(x)^{\alpha} - f(x)^{-\alpha}}{2\alpha} & , \quad \alpha \neq 0 \\ lnf(x) & , \quad \alpha = 0 \end{cases}$$

where $f(x) > 0$, depending if the random variable X is continuous, f being the probability density function (pdf) of X. We suppose that the differential entropy $HK_{\alpha}(X)$ to exist and be finite.

In function approximation, system classification and prediction task, the goal is to find the best possible model and parameters to have a good performance.

This is the reason why diverse characterizations, models and properties are constantly and intensively studied for the entropy establishment. In order to give some examples, in [1], Bancescu proposed a new method for the construction of those statistical models that may be interpreted as the lifetime distributions of series-parallel/parallel-series systems and which are used in characterizing coherent systems. In [16], Preda and Bancescu used the Speed-gradient (SG) principle for the non-stationary process that obeys the group entropy maximization principle or the relative entropy group minimization principle and obtains its dynamic equations. In [5] for the entropy estimation, Botha et. al used the Dirichlet prior, which is considered a valuable choice in the Bayesian framework. Wang and Gui [28] used the maximum likelihood and Bayesian methods to obtain the estimators of the entropy for a two-parameter Burr type XII distribution under progressive type-II censored data. Popkov [15] approached the problem of randomized maximum entropy estimation for the probability density function of random model parameters with real data and measurement noises.This estimation procedure maximizes an information entropy functional, taken from a set of integral equalities. The technique of the Gteaux derivatives was developed to solve the problem under an analytical form.

Information theory quantities, otherwise called information measures, such as entropy, mutual information and Kullback-Leibler divergence quantify the amount of information among random variables and have a large set of applications in statistics. Bulinski et al. established in [6] the asymptotic unbiasedness and L^2-consistency under mild conditions, for the estimates of the Kullback-Leibler divergence between two probability measures in $\Re^d$ absolutely continuous with respect to (w.r.t.) the Lebesgue measure. The estimates were based on certain k-nearest neighbor statistics for pair of independent identically distributed (i.i.d.) due vector samples. The novelty of results is also in treating mixture models. In particular, they cover mixtures of nondegenerate Gaussian measures. The mentioned asymptotic properties of related estimators for the Shannon entropy and cross-entropy are strengthened. Some applications are indicated.

In the framework of this paper the random variables shall be restricted to non-negative ones. The asymptotic unbiasedness and consistency of the estimator will be proven.

This paper is organized as follows: Firstly, in section 2 well construct the k^{th}- NN Nearest Neighbor Estimator of Kaniadakis Entropy. Afterwards, in the third section well arrive at the asymptotic mean of the estimator $\widehat{GK}_{\alpha}^{(n)}(f)$ and we shall see the estimator $\widehat{HK}_\alpha(f)$ in the 4th section. Section 5 present the conditioned asymptotic variance of $TK_1^{(n)}$ and in Section 6 - the asymptotic variance of the estimator $\widehat{HK}_\alpha^{(n)}(f)$ is studied. Straight conclusions will be drawn at the end of the paper.

2 Construction of the k^{th} - NN Nearest Neighbor Estimator of Kaniadakis Entropy

Given a random sample $X_1, X_2, \ldots, X_n$, from the distribution with pdf $f(x)$, the purpose is to estimate the entropy$HK_\alpha(f)$.
For $\hat{f}(.)$ a suitable estimator of the pdf $f(.)$, a reasonable estimator of entropy $HK_\alpha(f)$ could be one of the following form:

$$\widehat{HK}_\alpha(f) = -\frac{1}{n}\sum_{i=1}^{n} ln_\alpha^K\left[\hat{f}(X_i)\right]$$

Let $1 \le k \le n$ be a positive integer, and for $i = 1, 2, \ldots, n$, let $R_{1,k,n}$ be the Euclidean distance from X_i to its k^{th} closest neighbor.
Therefore, a reasonable estimate defined as $\hat{f}(X_i)$ of $f(X_i)$ is given by:

$$\hat{f}(X_i)\frac{\pi^{\frac{p}{2}}R_{i,k,n}^p}{\Gamma\left(\frac{p}{2}+1\right)} = \frac{k}{n}$$

where $\frac{\pi^{\frac{p}{2}}R_{i,k,n}^p}{\Gamma\left(\frac{p}{2}+1\right)}$ is the volume of the sphere, having radius $R_{i,k,n}$. The previous equation gives

$$\hat{f}(X_i) = \frac{k\Gamma\left(\frac{p}{2}+1\right)}{n\pi^{\frac{p}{2}}R_{i,k,n}^p}, \qquad i = 1, 2, \ldots .n,$$

Therefore, a reasonable estimate of $HK_\alpha(f)$ is:

$$\widehat{GK}_\alpha^{(n)}(f) = -\frac{1}{n}\sum_{i=1}^{n} ln_\alpha^K\left[\hat{f}(X_i)\right] = \frac{1}{n}\sum_{i=1}^{n} TK_i^{(n)}$$

where $TK_i^{(n)} = ln_\alpha^K\left[\frac{n\pi^{\frac{p}{2}}R_{i,k,n}^p}{k\Gamma\left(\frac{p}{2}+1\right)}\right], \qquad i = 1, 2, \ldots .n$

$TK_i^{(n)} = ln_\alpha^K\left[\frac{n\pi^{\frac{p}{2}}R_{i,k,n}^p}{k\Gamma\left(\frac{p}{2}+1\right)}\right] > r, \qquad \frac{n\pi^{\frac{p}{2}}R_{i,k,n}^p}{k\Gamma\left(\frac{p}{2}+1\right)} > e_\alpha^K(r)$, where e_α^K is the inverse function of the ln_α^K function.
Therefore,

$$\widehat{GK}_\alpha^{(n)}(f) = -\frac{1}{n}\sum_{i=1}^{n} ln_\alpha^K\left[\frac{k\Gamma\left(\frac{p}{2}+1\right)}{n\pi^{\frac{p}{2}}R_{i,k,n}^p}\right], \quad i = 1, 2, \ldots .n$$

The theorem that follows gives the asymptotic mean of the estimator $TK_1^{(n)}$ conditioned of $X_1 = x$.

3 The asymptotic mean of the estimator $\widehat{GK}_\alpha^{(n)}(f)$

To arrive at the asymptotic mean of the estimator $\hat{GK}_\alpha^{(n)}(f)$ we must at first give a result for the asymptotic mean of the estimator $TK_1^{(n)}$ conditioned by $X_1 = x$.

Proposition 3.1

Let $E_x\left(f(X)^{-2\alpha}\right) < \infty, \quad 0 \neq |\alpha| < 1, k > 2|\alpha|$

The asymptotic mean of the estimator $TK_1^{(n)}$ conditioned of $X_1 = x$ is given by:

$$\lim_{n\to\infty} E\left[TK_1^{(n)}|X_1 = x\right] =$$

$$\frac{1}{4\alpha \cdot k! \cdot f(x)}\left[\frac{1}{(kf(x))^{2\alpha-1}}\Gamma(2\alpha+k) + \frac{1}{(kf(x))^{-2\alpha-1}}\Gamma(k-2\alpha)\right] k > 2|\alpha|$$

Proof

For r real number, we have:

$$P\left[TK_1^{(n)} > r|X_1 = x\right] = P\left[R_{1,k,n} > \rho_{r,n}^k|X_1 = x\right]$$

where

$$\rho_{r,n}^K = \left[\frac{k\Gamma\left(\frac{p}{2}+1\right)e_\alpha^K(r)}{n\pi^{\frac{p}{2}}}\right]^{\frac{1}{p}}$$

Thus,

$$P\left[TK_1^{(n)} > r|X_1 = x\right] = \sum_{i=0}^{k-1}\binom{n-1}{i}\left[P\left(S_{\rho_{r,n}^K:x}\right)\right]^i\left[1 - P\left(S_{\rho_{r,n}^K:x}\right)\right]^{n-1-i}$$

where

$$P\left(S_{\rho_{r,n}^K:x}\right) = \int_{S_{\rho_{r,n}^K:x}} f(t)\,dt$$

and $S_{\rho_{r,n}^K:x}$ is a general sphere of radius $r > 0$, on the p-dimensional Euclidean space. It was initially proposed for a general p, used for the non-parametric estimate of entropy, based on the nearest neighbor distances between the sample points

Since $\rho_{r,n}^K \to 0$ as $n \to \infty$ we have

$$lim_{n\to\infty}\left[nP\left(S_{\rho_{r,n}^K:x}\right)\right] = ke_\alpha^K(r)\,lim_{n\to\infty}\frac{P\left(S_{\rho_{r,n}^K:x}\right)}{V_{\rho_{r,n}^K}} = ke_\alpha^K(r)\,f(x)$$

Using the Poisson approximation for the binomial distribution, we have:

$$\lim_{n\to\infty} P\left[TK_1^{(n)} > r|X_1 = x\right] = \sum_{i=0}^{k-1}\frac{\left[kf(x)\,e_\alpha^K(r)\right]^i}{i!}e^{-kf(x)}e_\alpha^K(r) =$$

$$P\left[TK_x > r\right], -\infty < TK_x < \infty$$

where the random variable TK_x for a given x has the pdf

$$h_{TK_x}(y) = C\left(k, \alpha, f(x)\right) \frac{\left[kf(x)\, e_\alpha^K(y)\right]^k}{(k-1)!} e^{-kf(x)e_\alpha^K(y)}, \quad -\infty < y < \infty$$

C being the normalization constant.
Therefore,

$$lim_{n\to\infty} E\left[TK_1^{(n)}|X_1 = x\right] = E\left[TK_x\right] = \int_{-\infty}^{\infty} y \frac{\left[kf(x)\, e_\alpha^K(y)\right]^k}{(k-1)!} e^{-kf(x)e_\alpha^K(y)} dy$$

By making the variable change $z = kf(x)\, e_\alpha^K(y)\,, \quad z > 0.$
The consistent calculus for $E\left[TK_1^{(n)}|X_1 = x\right]$ is the following:
We denote

$$A_0 = \frac{1}{4\alpha \cdot k! \cdot f(x)}$$

$$\lim_{n\to\infty} E\left[TK_1^{(n)}|X_1 = x\right]$$

$$= \frac{1}{2kf(x)} E_z \left\{ \frac{Z^k}{(k-1)!} \left[\left(\frac{Z}{kf(x)}\right)^{\alpha-1} + \left(\frac{Z}{kf(x)}\right)^{-\alpha-1} \right] ln_\alpha^K \left(\frac{Z}{kf(x)}\right) \right\}$$

$$= \frac{1}{4\alpha \cdot k! \cdot f(x)} E_z \left\{ Z^k \left[\left(\frac{Z}{kf(x)}\right)^{\alpha-1} - \left(\frac{Z}{kf(x)}\right)^{-\alpha-1} \right] \left[\frac{z^\alpha}{(kf(x))^\alpha} - \frac{z^{-\alpha}}{(kf(x))^{-\alpha}} \right] \right\}$$

$$= A_0 E_Z \left\{ Z^k \left[\left(\frac{Z}{kf(x)}\right)^{\alpha-1} \right] \right\} - A_0 E_Z \left\{ Z^k \left[\left(\frac{Z}{kf(x)}\right)^{\alpha-1} \right] \left[\frac{z^\alpha}{(kf(x))^\alpha} - \frac{z^{-\alpha}}{(kf(x))^{-\alpha}} \right] \right\}$$

$$= A_0 \frac{1}{(kf(x))^{2\alpha-1}} E_Z\left[Z^{2\alpha-1+k}\right] - A_0 \frac{1}{(kf(x))^{-2\alpha-1}} E_Z\left[Z^{-2\alpha-1+k}\right]$$

$$= A_0 \frac{1}{(kf(x))^{2\alpha-1}} \Gamma(2\alpha + k) - A_0 \frac{1}{(kf(x))^{-2\alpha-1}} \Gamma(k - 2\alpha)$$

where $Z \sim exp(1)$.
Further we are about to use the Gamma function, described earlier in the introduction.
By letting $k \geq 2|\alpha|$, we have that and by denoting:

$$B_0 = \frac{1}{(kf(x))^{2\alpha-1}} \Gamma(2\alpha + k) - \frac{1}{(kf(x))^{-2\alpha-1}} \Gamma(k - 2\alpha)$$

We arrive at the conclusion that

$$lim_{n\to\infty} E\left[TK_1^{(n)}|X_1 = x\right] = A_0 \cdot B_0$$

Proposition 3.2
Let $E_X\left(f(X)^{-2\alpha}\right) < \infty, 0 \neq |\alpha| < 1, k > 2|\alpha|$

$$\lim_{n\to\infty} E\left[TK_1^{(n)}\right]$$
$$= \frac{1}{4\alpha\cdot(k-1)!\cdot f(x)}\left[k^{-2\alpha}\cdot\Gamma(2\alpha+k) - k^{2\alpha}\cdot\Gamma(k-2\alpha)\right]\cdot E_X\left[f(X)^{-2\alpha}\right]$$
$$+\frac{\Gamma(2\alpha+k)\cdot k^{\alpha}}{(k-1)!}\cdot HK_{2\alpha}(f)$$

Proof

By using Proposition 1,
$lim_{n\to\infty} E\left[TK_1^{(n)}\right] = \int_0^\infty E\left[TK_1^{(n)}|X_1 = x\right]\cdot f(x)\,dx$
$= \frac{1}{4\alpha\cdot k!}\left[k^{1-2\alpha}\cdot\Gamma(2\alpha+k)\cdot E\left[f(X)^{-2\alpha}\right] - k^{1+2\alpha}\cdot\Gamma(k-2\alpha)\cdot E\left[f(X)^{2\alpha}\right]\right]$
$= \frac{1}{4\alpha\cdot k!}\left\{k^{1-2\alpha}\cdot\Gamma(2\alpha+k)\cdot E\left[f(X)^{-2\alpha}\right] - k^{1+2\alpha} - \Gamma(k-2\alpha)\cdot 4\alpha\right.$

$$\left.\cdot E\left[\frac{f(X)^{2\alpha} - f(X)^{-2\alpha} + f(X)^{-2\alpha}}{4\alpha}\right]\right\}$$

$= \frac{1}{4\alpha\cdot k!}\left\{k^{1-2\alpha}\cdot\Gamma(2\alpha+k)\cdot E\left[f(X)^{-2\alpha}\right] - k^{1+2\alpha} - \Gamma(k-2\alpha)\cdot 4\alpha\cdot\left(E\left[\frac{f(X)^{2\alpha}-f(X)^{-2\alpha}}{4\alpha}\right]\right.\right.$

$$\left.\left.+E\left[\frac{f(X)^{-2\alpha}}{4\alpha}\right]\right)\right\}$$

$= \frac{1}{4\alpha\cdot k!}\left\{\left[k^{1-2\alpha}\cdot\Gamma(2\alpha+k) - k^{1+2\alpha}\cdot\Gamma(k-2\alpha)\right]\cdot E_X\left[f(X)^{-2\alpha}\right] - 4\alpha\cdot k^{1+2\alpha}\cdot\Gamma(k-2\alpha)\right.$

$$\left.\cdot HK_{2\alpha}(f)\right\}$$

$= \frac{1}{4\alpha\cdot(k-1)!}\left[k^{-2\alpha}\cdot\Gamma(2\alpha+k) - k^{2\alpha}\cdot\Gamma(k-2\alpha)\right]\cdot E_X\left[f(X)^{-2\alpha}\right] - \frac{\Gamma(2\alpha+k)\cdot k^{\alpha}}{(k-1)!}\cdot HK_{2\alpha}(f)$

Theorem 3.3

Let $E_X\left(f(X)^{-2\alpha}\right) < \infty, 0 \neq |\alpha| < 1, k > |2|\alpha|$
$lim_{n\to\infty} E\left[\widehat{GK}_{\alpha}^{(n)}(f)\right] = \frac{1}{(k-1)!\cdot 4\alpha}\cdot\left[k^{-2\alpha}\cdot\Gamma(2\alpha+k) + k^{2\alpha}\cdot\Gamma(k-2\alpha)\right]$

$$\cdot E_X\left[f(X)^{-2\alpha}\right] - \frac{\Gamma(2\alpha+k)\cdot k^{\alpha}}{(k-1)!}\cdot HK_{2\alpha}(f)$$

Proof

$TK_1^{(n)}, TK_2^{(n)}, \ldots, TK_n^{(n)}$ The random variables $TK_1^{(n)}, TK_2^{(n)}, \ldots, TK_n^{(n)}$ are identically distributed, therefore

$$E\left[\widehat{GK}_\alpha^{(n)}(f)\right] = E\left[TK_1^{(n)}\right]$$

By using Proposition 2,

$$lim_{n\to\infty} E\left[\widehat{GK}_\alpha^{(n)}(f)\right] = lim_{n\to\infty} E\left[TK_1^{(n)}|X_1 = x\right]$$

$$= \frac{1}{4\alpha\cdot k!\cdot f(x)}\cdot\left[k^{1-2\alpha} f(x)^{-2\alpha}\cdot\Gamma(2\alpha+k) + k^{1+2\alpha} f(x)^{2\alpha}\cdot\Gamma(k-2\alpha)\right]$$

This concludes our proof.
We have that

$$-2\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} I_S(x)\,\frac{f(x)^{1+2\alpha} - f(x)^{1-2\alpha}}{4\alpha} = HK_{2\alpha}(f)$$

Then,

$$lim_{n\to\infty} E\left[TK_1^{(n)}\right] = \tfrac{1}{(k-1)!\cdot 4\alpha}\left[k^{-2\alpha}\cdot\Gamma(2\alpha+k) - k^{2\alpha}\cdot\Gamma(k-2\alpha)\right]\cdot E\left[f(X)^{-2\alpha}\right]$$
$$-\frac{\Gamma(2\alpha+k)\cdot k^{\alpha}}{(k-1)!}\cdot\widehat{HK}_{2\alpha}(f)$$

From here on we can begin to calculate $\widehat{HK}_{2\alpha}(f)$ in order to get to $\widehat{HK}_\alpha(f)$.

4 The estimator $\widehat{HK}_\alpha(f)$.

Theorem 4.1

Let $E_X\left(f(X)^{-2\alpha}\right) < \infty, 0 \neq |\alpha| < \frac{1}{2}, k > |2|\alpha|$

$$\widehat{HK}_\alpha(f) = \tfrac{(k-1)!}{k^{\frac{\alpha}{2}}\Gamma(k-\alpha)}\left\{-\tfrac{1}{n} ln_{\frac{\alpha}{2}}^{K}\left[\tfrac{k\Gamma\left(\frac{p}{2}+1\right)}{n\pi^{\frac{p}{2}}}\right]\cdot\textstyle\sum_{i=1}^{n} R_{i,n,k}^{-\frac{\alpha p}{2}} + \tfrac{n^{\frac{\alpha}{2}-1}\pi^{\frac{p\alpha}{4}}p}{k^{\frac{\alpha}{2}}\left[\Gamma\left(\frac{p}{2}+1\right)\right]^{\frac{\alpha}{2}}}\textstyle\sum_{i=1}^{n} ln_{\frac{\alpha p}{2}}^{K} R_{i,n,k}\right\}$$
$$-\left(\frac{k^{\frac{3\alpha}{2}}}{2\alpha}\cdot\frac{\Gamma(\alpha+k)}{\Gamma(k-\alpha)} + \frac{k^{\frac{\alpha}{2}}}{2\alpha}\right) E_X\left[f(X)^{-\alpha}\right]$$

Proof

$$\widehat{HK}_\alpha(f) = \tfrac{-(k-1)!}{k^{\alpha}\Gamma(k-2\alpha)}\widehat{GK}_\alpha^{(n)}(f)$$

$$+\frac{(k-1)!}{(k-1)!\cdot 4\alpha\cdot k^{\alpha}\cdot\Gamma(k-2\alpha)}\left(k^{-2\alpha}\cdot\Gamma(2\alpha+k) - k^{2\alpha}\cdot\Gamma(k-2\alpha)\right)\cdot E_X\left[f(X)^{-2\alpha}\right]$$

$$= \frac{-(k-1)!}{k^{\alpha}\Gamma(k-2\alpha)}\widehat{GK}_{\alpha}^{(n)}(f) + \left(\frac{k^{-3\alpha}}{4\alpha}\cdot\frac{\Gamma(2\alpha+k)}{\Gamma(k-2\alpha)} + \frac{k^{\alpha}}{4\alpha}\right)E_X\left[f(X)^{-2\alpha}\right]$$

We can convert α in $\frac{\alpha}{2}$ and see that an asymptotically unbiased estimator for $HK_{\alpha}(f)$ is

$$\widehat{HK}_{\alpha}(f) = \frac{(k-1)!}{k^{\frac{\alpha}{2}}\Gamma(k-\alpha)}\widehat{GK}_{\frac{\alpha}{2}}^{(n)}(f) - \left(\frac{k^{\frac{3\alpha}{2}}}{2\alpha}\cdot\frac{\Gamma(\alpha+k)}{\Gamma(k-\alpha)} + \frac{k^{\frac{\alpha}{2}}}{2\alpha}\right)E_X\left[f(X)^{-\alpha}\right]$$

$$\widehat{GK}_{\frac{\alpha}{2}}^{(n)}(f) = -\frac{1}{n}\sum_{R_{i,n,k}^{p,\alpha}}^{1} ln_{\alpha}^{k}\left(\frac{k\Gamma\left(\frac{p}{2}+1\right)}{n\pi^{\frac{p}{2}}}\right) - \frac{1}{n}\frac{k^{-\alpha}\left(\Gamma\left(\frac{p}{2}+1\right)\right)^{-\alpha}}{n^{-\alpha}\pi^{\frac{-\alpha p}{4}}}ln_{\alpha}^{K}\left(\frac{1}{R_{I,n,k}^{p}}\right)^{-\alpha}$$

But $\widehat{GK}_{\alpha}^{(n)}$ can be also written as: $\widehat{GK}_{\alpha}^{(n)} = -\frac{1}{n}\sum_{i=1}^{n} ln_{\alpha}^{K}\left[\frac{k\Gamma\left(\frac{p}{2}+1\right)}{n\pi^{\frac{p}{2}}R_{i,n,k}^{p}}\right], \quad i = 1,2.\dots,n$
and so

$$\widehat{GK}_{\alpha}^{(n)}(f) = -\frac{1}{n}\sum_{i=1}^{n}\frac{1}{R_{i,n,k}^{\alpha p}}ln_{\alpha}^{K}\left[\frac{k\Gamma\left(\frac{p}{2}+1\right)}{n\pi^{\frac{p}{2}}}\right] - \frac{n^{\alpha}\pi^{\frac{p\alpha}{2}}}{k^{\alpha}\left[\Gamma\left(\frac{p}{2}+1\right)\right]^{\alpha}}\cdot ln_{\alpha}^{K}R_{i,n,k}^{p}$$

$$= -\frac{1}{n}\sum_{i=1}^{n}\frac{1}{R_{i,n,k}^{\alpha p}}ln_{\alpha}^{K}\left[\frac{k\Gamma\left(\frac{p}{2}+1\right)}{n\pi^{\frac{p}{2}}}\right] + \frac{1}{n}\sum_{i=1}^{n}\frac{n^{\alpha}\pi^{\frac{p\alpha}{2}}}{k^{\alpha}\left[\Gamma\left(\frac{p}{2}+1\right)\right]^{\alpha}}\cdot p\cdot ln_{\alpha p}^{K}R_{i,n,k}$$

$$= -\frac{1}{n}ln_{\alpha}^{K}\left[\frac{k\Gamma\left(\frac{p}{2}+1\right)}{n\pi^{\frac{p}{2}}}\right]\cdot\sum_{i=1}^{n}R_{i,n,k}^{-\alpha p} + \frac{n^{\alpha-1}\pi^{\frac{p\alpha}{2}}p}{k^{\alpha}\left[\Gamma\left(\frac{p}{2}+1\right)\right]^{\alpha}}\sum_{i=1}^{n}ln_{\alpha p}^{K}R_{i,n,k}$$

5 The conditioned asymptotic variance of $TK_1^{(n)}$

Proposition 5.1

Let $E_X\left(f(X)^{-2\alpha}\right) < \infty, 0 \neq |\alpha| < 1, k > |3|\alpha|$.
Let k be a positive integer and for a fixed x let $A_1 = \frac{1}{8\alpha^2\cdot k!\cdot f(x)}$.
Then,

$$\lim_{n\to\infty} E\left[\left(TK_1^{(n)}\right)^2 | X_1 = x\right]$$

$$= A_1\left[\frac{1}{(kf(x))^{3\alpha-1}}\cdot\Gamma(3\alpha+k)+\frac{1}{(kf(x))^{\alpha-1}}\cdot\Gamma(\alpha+k)+\frac{1}{(kf(x))^{-\alpha-1}}\cdot\Gamma(k-\alpha)\right.$$
$$\left.+\frac{1}{(kf(x))^{-3\alpha-1}}\cdot\Gamma(k-3\alpha)-\frac{2}{(kf(x))^{\alpha-1}}\cdot\Gamma(\alpha+k)-\frac{2}{(kf(x))^{-\alpha-1}}\cdot\Gamma(k-\alpha)\right]$$

Proof

$$\lim_{n\to\infty} E\left[\left(TK_1^{(n)}\right)^2|X_1=x\right]=E\left[TK_x^2|X_1=x\right]$$
$$=E_Z\left\{\frac{Z^k}{(k-1)!}\cdot\frac{1}{2kf(x)}\left[\left(\frac{Z}{kf(x)}\right)^{\alpha-1}+\left(\frac{Z}{kf(x)}\right)^{-\alpha-1}\right]\left[ln_\alpha^K\left(\frac{Z}{kf(x)}\right)\right]^2\right\}$$

$$\lim_{n\to\infty} E\left[\left(TK_1^{(n)}\right)^2|X_1=x\right]$$
$$=A_1\left\{\tfrac{1}{(kf(x))^{3\alpha-1}}E_Z\left[Z^{3\alpha+k}\right]+\tfrac{1}{(kf(x))^{\alpha-1}}E_Z\left[Z^{\alpha+k}\right]+\tfrac{1}{(kf(x))^{-\alpha-1}}E_Z\left[Z^{-\alpha+k}\right]\right.$$
$$\left.+\frac{1}{(kf(x))^{-3\alpha-1}}E_Z\left[Z^{-3\alpha+k}\right]-\frac{2}{(kf(x))^{\alpha-1}}E_Z\left[Z^{\alpha+k}\right]-\frac{2}{(kf(x))^{-\alpha-1}}E_Z\left[Z^{-\alpha+k}\right]\right\}$$
$$=A_1\left[\tfrac{1}{(kf(x))^{3\alpha-1}}\Gamma(3\alpha+k)+\tfrac{1}{(kf(x))^{\alpha-1}}\Gamma(\alpha+k)+\tfrac{1}{(kf(x))^{-\alpha-1}}\Gamma(k-\alpha)\right.$$
$$\left.+\frac{1}{(kf(x))^{-3\alpha-1}}\Gamma(k-3\alpha)-\frac{2}{(kf(x))^{\alpha-1}}\Gamma(\alpha+k)-\frac{2}{(kf(x))^{-\alpha-1}}\Gamma(k-\alpha)\right]$$

Proposition 5.2

Let $E_X\left(f(X)^{-2\alpha}\right)<\infty, 0\neq|\alpha|<1, k>|2|\alpha|$

$$\left\{\lim_{n\to\infty} E\left[\left(TK_1^{(n)}\right)^2|X_1=x\right]\right\}^2$$
$$=\tfrac{1}{16\alpha^2\cdot((k-1))^2}\left[k^{-4\alpha}f(x)^{-4\alpha}\Gamma^2(2\alpha+k)-k^{4\alpha}f(x)^{4\alpha}\Gamma^2(k-2\alpha)-2\Gamma(2\alpha+k)\Gamma(k-2\alpha)\right]$$

Proof

Using Proposition 3.1, we can easily make the calculations and come to the above result.

Thus, by calculating the square of

$$lim_{n\to\infty}E\left[\left(TK_1^{(n)}\right)|X_1=x\right]=\tfrac{1}{4\alpha\cdot k!\cdot f(x)}\cdot\left[\tfrac{1}{(kf(x))^{2\alpha-1}}\Gamma(2\alpha+k)-\tfrac{1}{(kf(x))^{-2\alpha-1}}\Gamma(k-2\alpha)\right]$$

we get:

$$lim_{n\to\infty}E\left[\left(TK_1^{(n)}\right)|X_1=x\right]^2$$
$$=\tfrac{1}{16\alpha^2\cdot(k!)^2\cdot f^2(x)}\left[\tfrac{1}{(kf(x))^{4\alpha-2}}\Gamma^2(2\alpha+k)+\tfrac{1}{(kf(x))^{-4\alpha-2}}\Gamma^2(k-2\alpha)\right.$$
$$\left.-2\frac{1}{(kf(x))^2}\Gamma(2\alpha+k)\Gamma(k-2\alpha)\right]$$

$$= \frac{1}{16\alpha^2 \cdot (k!)^2 \cdot f^2(x)} \left[\frac{1}{k^{4\alpha-2} \cdot f(x)^{4\alpha}} \Gamma^2(2\alpha+k) + \frac{1}{k^{-4\alpha-2} f(x)^{-4\alpha}} \Gamma^2(k-2\alpha) \right.$$
$$\left. - \frac{2}{k^{-2}} \Gamma(2\alpha+k) \Gamma(k-2\alpha) \right]$$

$$= \frac{k^2}{16\alpha^2 \cdot ((k-1)!)^2 \cdot k^2} \left[k^{-4\alpha} f(x)^{-4\alpha} \Gamma^2(2\alpha+k) + k^{4\alpha} f(x)^{4\alpha} \Gamma^2(k-2\alpha) \right.$$
$$\left. -2\Gamma(2\alpha+k) \Gamma(k-2\alpha) \right]$$

$$= \frac{1}{16\alpha^2 \cdot ((k-1)!)^2} \left[k^{-4\alpha} f(x)^{-4\alpha} \Gamma^2(2\alpha+k) + k^{-4\alpha} f(x)^{-4\alpha} \Gamma^2(k-2\alpha) \right.$$
$$\left. -2\Gamma(2\alpha+k) \Gamma(k-2\alpha) \right]$$

Theorem 5.3

Let $E_X \left(f(X)^{-2\alpha} \right) < \infty, 0 \neq |\alpha| < 1, k > |3|\alpha|$

$\lim_{n\to\infty} Var \left[\left(TK_1^{(n)} | X = x \right) \right]$

$$= A_1 \left[\frac{1}{(kf(x))^{3\alpha-1}} \cdot \Gamma(3\alpha+k) + \frac{1}{(kf(x))^{\alpha-1}} \cdot \Gamma(\alpha+k) + \frac{1}{(kf(x))^{-\alpha-1}} \cdot \Gamma(k-\alpha) \right.$$
$$\left. + \frac{1}{(kf(x))^{-3\alpha-1}} \cdot \Gamma(k-3\alpha) - \frac{2}{(kf(x))^{\alpha-1}} \cdot \Gamma(\alpha+k) - \frac{2}{(kf(x))^{-\alpha-1}} \cdot \Gamma(k-\alpha) \right]$$
$$- \frac{1}{16\alpha^2 \cdot ((k-1)!)^2} \left[k^{-4\alpha} f(x)^{-4\alpha} \Gamma^2(2\alpha+k) + k^{4\alpha} f(x)^{4\alpha} \Gamma^2(k-2\alpha) \right.$$
$$\left. -2\Gamma(2\alpha+k) \Gamma(k-2\alpha) \right]$$

Proof

$$\lim_{n\to\infty} Var \left[\left(TK_1^{(n)} | X = x \right) \right]$$
$$= \lim_{n\to\infty} E \left[\left(\left(TK_1^{(n)} \right)^2 | X = x \right) \right] \left\{ \lim_{n\to\infty} E \left[TK_1^{(n)} | X_1 = x \right] \right\}^2$$

Therefore, the proof is direct from Prop. 3 and 4.

6 The asymptotic variance of the estimator $\widehat{HK}_\alpha^{(n)}(f)$

Proposition 6.1

$$\lim_{n\to\infty} Cov\left(TK_1^{(n)}, TK_2^{(n)}\right) = 0$$

Proof

$$\lim_{n\to\infty} P\left[T_1^{(n)} > r, T_2^{(n)} > s | X_1 = x, X_2 = y\right]$$

$$= \sum_{i=0}^{k-1}\sum_{j=0}^{k-1} \frac{\left[kf\left(x\right)e^{r}\right]^{i}}{i!}\frac{\left[kf\left(y\right)e^{s}\right]^{j}}{j!} e^{\left[kf(x)e^{r}+kf(y)e^{s}\right]}$$

$$= P\left[T_x > r\right] P\left[T_y > s\right]$$

For finding the limit of the covariance between $TK_1^{(n)}$ and $TK_2^{(n)}$we shall consider r and s, both finite.

$$P\left[TK_1^{(n)} > r, TK_2^{(n)} > s | X_1 = x, X_2 = y\right]$$
$$= P\left[R_{1,k,n} > \rho_{r,n}^{K}, R_{2,k,n} > \rho_{s,n}^{K} | X_1 = x, X_2 = y\right]$$

By having $x \neq y$, since $\rho_{r,n}^{K}$ and $\rho_{s,n}^{K}$ tend to 0, when $n \to \infty$ we assume that, for n large $S_{\rho_{r,n}^{K}:x} \bigcap S_{\rho_{s,n}^{K}:y} = \phi$, an empty set.

$$P\left[TK_1^{(n)} > r, TK_2^{(n)} > s | X_1 = x, X_2 = y\right]$$
$$= P\left[\begin{array}{ll} at & most\,(k-1)\,of\,X_{3,} \ldots, X_n \in S_{\rho_{r,n}^{K}:x} and \\ at & most\,(k-1)\,of\,X_{3,} \ldots, X_n \in S_{\rho_{s,n}^{K}:y} \end{array}\right]$$
$$= \sum_{0\le i,m\le j, i+m\le n-2} \frac{(n-2)!}{i!m!\,(n-2-i-m)} P\left[\left(S_{\rho_{r,n}^{K}:x}\right)\right]^{i} P\left[\left(S_{\rho_{s,n}^{K}:y}\right)\right]^{m}$$
$$\cdot\left[1 - P\left(S_{\rho_{r,n}^{K}:x}\right) - P\left(S_{\rho_{s,n}^{K}:y}\right)\right]^{n-2-i-m}$$

We shall denote $\left[\frac{P\left(S_{\rho_{r,n}^{K}:x}\right)}{V_{S_{\rho_{r,n}^{K}}}}\right]^{i}\left[\frac{P\left(S_{\rho_{s,n}^{K}:y}\right)}{V_{S_{\rho_{s,n}^{K}}}}\right]^{m}$ be $A_{i,m}^{k}\left(x,y\right)$

and

$$\left\{1 - \tfrac{1}{n}\left[ke_{\alpha}^{K}\left(r\right)\cdot\frac{P\left(S_{\rho_{r,n}^{K}:x}\right)}{V_{S_{\rho_{r,n}^{K}}}} + ke_{\alpha}^{K}\left(x\right)\cdot\frac{P\left(S_{\rho_{s,n}^{K}:y}\right)}{V_{S_{\rho_{s,n}^{K}}}}\right]^{n-2-i-m}\right\} \text{ be } B_{i,m}^{k}\left(x,y\right)$$

$$= \sum_{0\le i,m\le j, i+m\le n-2} \frac{(n-2)!}{i!m!\,(n-2-i-m)} \frac{\left(ke_{\alpha}^{k}\left(r\right)\right)^{i}}{n^{i}}\cdot\frac{\left(ke_{\alpha}^{k}\left(s\right)\right)^{m}}{n^{m}}\cdot A_{i,m}^{k}\left(x,y\right)\cdot B_{i,m}^{k}\left(x,y\right)$$

Theorem 6.2

$$\lim_{n\to\infty} Var\left[\widehat{HK}_{\alpha}^{(n)}(f)\right] = 0$$

Proof

The distribution of a random vector $\left(TK_1^{(n)}, TK_2^{(n)}, \ldots, TK_n^{(n)}\right)$ remains the same regarding any permutation that we may consider of it, and then we have

$$Var\left[\widehat{GK}_{\frac{\alpha}{2}}^{(n)}(f)\right] = \frac{Var\left(TK_1^{(n)}\right)}{n} + \frac{n(n-1)}{n^2} Cov\left(K_1^{(n)}, TK_2^{(n)}\right)$$

We have

$$\lim_{n\to\infty} E\left[TK_1^{(n)}, TK_2^{(n)}\right] = \lim_{n\to\infty}\left[E\left[TK_1^{(n)}\right] E\left[TK_2^{(n)}\right]\right],$$

which inferrs that

$$\lim_{n\to\infty} Cov\left[TK_1^{(n)}, TK_2^{(n)}\right] = E\left[TK_1^{(n)}, TK_2^{(n)}\right] - E\left[TK_1^{(n)}\right] E\left[TK_2^{(n)}\right]$$

$$\lim_{n\to\infty} Cov\left[TK_1^{(n)}, TK_2^{(n)}\right] = 0$$

Hence, from the recent calculus for $Var\left[TK_1^{(n)}\right]$ and $\lim_{n\to\infty} Cov\left[TK_1^{(n)}, TK_2^{(n)}\right]$ we have that

$$\lim_{n\to\infty} Var\left[\widehat{GK}_{\frac{\alpha}{2}}^{(n)}(f)\right] = 0$$

resulting that

$$\lim_{n\to\infty} Var\left[\widehat{HK}_{\alpha}^{(n)}(f)\right] = 0$$

Asymptotic efficiency is another property worth consideration in the evaluation of estimators. Though there are many definitions, we can see the asymptotic variance as how far the set of numbers is spread out, of the limit distribution of the estimator. In the above results we evidence the properties for the Kaniadakis-based estimations in this direction.

7 Conclusions

We constructed the k^{th} - NN Nearest Neighbor Estimator of the Kaniadakis Entropy and found the asymptotic mean of the estimator $\widehat{GK}_{\alpha}^{(n)}(f)$. Also, a theorem for the asymptotic mean of the estimator $TK_1^{(n)}$ conditioned by $X_1 = x$ has been given.

Then, we calculated the estimator $\widehat{HK}_{\alpha}^{(n)}(f)$ from which we derived the conditioned asymptotic variance of $TK_1^{(n)}$ and further, the asymptotic variance of the estimator $\widehat{HK}_{\alpha}^{(n)}(f)$.

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