



Non-autonomous weighted elliptic equations with double exponential growth

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Abstract

We consider the existence of solutions of the following weighted problem:

$$\begin{cases} L := -\operatorname{div}(\rho(x)|\nabla u|^{N-2}\nabla u) + \xi(x)|u|^{N-2}u = f(x, u) & \text{in } B \\ u > 0 & \text{in } B \\ u = 0 & \text{on } \partial B, \end{cases}$$

where B is the unit ball of \mathbb{R}^N , $N > 2$, $\rho(x) = \left(\log \frac{e}{|x|}\right)^{N-1}$ the singular logarithm weight with the limiting exponent $N - 1$ in the Trudinger-Moser embedding, and $\xi(x)$ is a positif continuous potential. The nonlinearities are critical or subcritical growth in view of Trudinger-Moser inequalities of double exponential type. We prove the existence of positive solution by using Mountain Pass theorem. In the critical case, the function of Euler Lagrange does not fulfil the requirements of Palais-Smale conditions at all levels. We dodge this problem by using adapted test functions to identify this level of compactness.

1 Introduction

In this paper we study the following weighted problem

$$\begin{cases} L := -\operatorname{div}(\rho(x)|\nabla u|^{N-2}\nabla u) + \xi(x)|u|^{N-2}u = f(x, u) & \text{in } B \\ u > 0 & \text{in } B \\ u = 0 & \text{on } \partial B, \end{cases} \quad (1)$$

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where B is the unit ball of \mathbb{R}^N , $N > 2$, $f(x, t)$ is continuous in $B \times \mathbb{R}$ and behaves like $\exp\{e^{\alpha t} \frac{N}{N-1}\}$ as $t \rightarrow +\infty$, for some $\alpha > 0$. $\xi : B \rightarrow \mathbb{R}$ is a positive continuous function satisfying some conditions. The weight $\rho(x)$ is given by

$$\rho(x) = \left(\log \frac{e}{|x|}\right)^{N-1}. \tag{2}$$

Since 1970, when Moser gave the famous result on the Trudinger-Moser inequality, many applications have taken place such as in the theory of conformal deformation on collectors, the study of the prescribed Gauss curvature and the mean field equations. After that, a logarithmic Trudinger-Moser inequality was used in crucial way in [31] to study the Liouville equation of the form

$$\begin{cases} -\Delta u &= \lambda \frac{e^u}{\int_{\Omega} e^u} & \text{in } \Omega \\ u &= 0 & \text{on } \partial\Omega, \end{cases} \tag{3}$$

where Ω is an open domain of \mathbb{R}^N , $N \geq 2$ and λ a positive parameter. The equation (3) has a long history and has been derived in the study of multiple condensate solution in the Chern-Simons-Higgs theory [36, 37] and also, it appeared in the study of Euler Flow [10, 11, 17, 27]. Later, the Trudinger-Moser inequality was improved to weighted inequalities [2, 12, 13, 16, 21, 30]. The influence of the weight in the Sobolev norm was studied as the compact embedding in [29].

When the weight is of logarithmic type, Calanchi and Ruf [14] extend the Trudinger-Moser inequality and give some applications when $N = 2$ and for prescribed nonlinearities. After that, Calanchi et al. [15] consider more general nonlinearities and prove the existence of radial solutions.

In this paper, we focus on the case $N > 2$ and use the Trudinger-Moser inequality to study and prove the existence of solutions to the problem (1).

We note that the semi-linear problem of Schrödinger

$$\begin{cases} -\Delta u + V(x)u &= g(x, u) & \text{in } \mathbb{R}^N \\ u &\in W^{1,N}(\mathbb{R}^N), \end{cases} \tag{4}$$

where $N \geq 3$ and $|g(x, u)| \leq c(|u| + |u|^{q-1})$, $1 < q \leq 2^* = \frac{2N}{N-2}$ was studied by many authors, we refer the readers to Kryszewski and Szulkin [28], Alama and Li [3], Ding and Ni [19] and Jeanjean [26].

For $N = 2$, with the same operator as in the problem (4) and where the nonlinearity is acting like $\exp(\alpha t^2)$ as $t \rightarrow +\infty$, for some $\alpha > 0$, many works have been processed, see [5, 22, 25].

In the case $N \geq 2$, a lot of works (see [24] and their references) are treated with

the N-laplacian operator associated to a potential i.e. the following problem

$$\begin{cases} -\operatorname{div}(|\nabla u|^{N-2}\nabla u) + a(x)|u|^{N-2}u = h(x, u) & \text{in } \mathbb{R}^N \\ u \in W^{1,N}(\mathbb{R}^N), \end{cases}$$

where the nonlinearity h has subcritical or critical growth in view of Trudinger-Moser inequalities and the potential a verifies some conditions which guarantee some compact embedding.

Problems when $\xi(x) = 0$ and in the non weighted case i.e. $\rho(x) = 1$, have been extensive studies by many authors, we refer the reader to [1, 4, 23, 33] and their references. We also mention that the following problem

$$\begin{cases} -\operatorname{div}(|\nabla u|^{N-2}\nabla u) = f(x, u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth domain of \mathbb{R}^N , $N \geq 2$ and the nonlinearity f behaves like $\exp\{t^{\frac{N}{N-1}}\}$ as $t \rightarrow +\infty$ was studied by Adimurthi [1] and Ruf et al. [22, 23].

We point out that the following problem

$$\begin{cases} L_w := -\operatorname{div}(w(x)\nabla u) = f(x, u) & \text{in } B_1 \\ u > 0 & \text{in } B_1 \\ u = 0 & \text{on } \partial B_1, \end{cases}$$

where B_1 is the unit disk of \mathbb{R}^2 , $w(x) = \log \frac{e}{|x|}$ and the nonlinearities f are of double exponential growth, was studied in [15]. Recently, Deng, Hu and Tang [18] studied the following problem

$$\begin{cases} -\operatorname{div}(\rho(x)|\nabla u|^{N-2}\nabla u) = f(x, u) & \text{in } B \\ u = 0 & \text{on } \partial B, \end{cases}$$

where $N \geq 2$, the function $f(x, t)$ is continuous in $B \times \mathbb{R}$ and behaves like $\exp\{e^{\alpha t^{\frac{N}{N-1}}}\}$ as $t \rightarrow +\infty$, for some $\alpha > 0$. The authors proved that there is a non-trivial solution to this problem using Mountain Pass theorem. They circumvented the loss of compactness of the associated energy function by an asymptotic condition on the nonlinearity and using appropriate Moser sequences. Also, they followed the method of Buccardo and Murat to show the convergence almost everywhere of the gradient. A similar result is proved in [40].

In literature, more attention has been accorded to the subspace of radial functions

$$W_{0,rad}^1(B, \rho) = cl\{u \in C_{0,rad}^\infty(B); \int_B \rho(x)|\nabla u|^N dx < \infty\},$$

endowed with the norm

$$\|\nabla u\|_{N,\rho} = \left(\int_B \rho(x) |\nabla u|^N dx \right)^{\frac{1}{N}}$$

and we are motivated by the following double exponential inequality proved in [13], which is an improvement of the Trudinger-Moser inequality in a weighted Sobolev space.

Theorem 1.1 [13] Let ρ given by (3), then

$$\int_B \exp\{e^{|u|^{\frac{N}{N-1}}}\} dx < +\infty, \quad \forall u \in W_{0,rad}^1(B, \rho) \quad (5)$$

and

$$\sup_{\substack{u \in W_{0,rad}^1(B, \rho) \\ \|\nabla u\|_{N,\rho} \leq 1}} \int_B \exp\{\beta e^{\omega_{N-1}^{\frac{1}{N-1}} |u|^{\frac{N}{N-1}}}\} dx < +\infty \quad \Leftrightarrow \quad \beta \leq N, \quad (6)$$

where ω_{N-1} is the area of the unit sphere S^{N-1} in \mathbb{R}^N .

It seems that the Trudinger-Moser inequality (6), can be considered as a borderline case of the famous Sobolev inequality.

The solution to the problem of the form (1) is important in several applications such as the study of classical and quantum mechanics, the evolution equations appearing in non-Newtonian fluids, reaction diffusion problem, turbulent flows in porous media and image treatment [7, 8, 34, 38]. We report that in recent years, PDE of divergence form have many applications in digital image restoration.

Let us now state our results. For this paper, we hypothesize that the nonlinearity $f(x, t)$ verifies the following assumptions

(A₁) $f : B \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous, radial in x , and $f(x, t) = 0$ for $t \leq 0$.

(A₂) There exists $t_0 > 0, M > 0$ such that

$$0 < F(x, t) = \int_0^t f(x, s) ds \leq M |f(x, t)|, \forall t > t_0, \forall x \in B.$$

(A₃) $0 < F(x, t) \leq \frac{1}{N} f(x, t)t, \forall t > 0, \forall x \in B$

and the potential ξ is continuous on \bar{B} and verifies

(ξ_1) $\xi(x) \geq \xi_0 > 0$ in B for some $\xi_0 > 0$.

(ξ_2) The function $\frac{1}{\xi}$ belongs to $L^{\frac{1}{N-1}}(B)$.

In view of (4) and (5) we say that f has subcritical growth at $+\infty$ if

$$\lim_{s \rightarrow +\infty} \frac{|f(x, s)|}{\exp\{Ne^{\alpha s^{N'}}\}} = 0, \quad \text{for all } \alpha > 0 \quad (7)$$

and f has critical growth at $+\infty$ if there exists some $\alpha_0 > 0$ such that

$$\lim_{s \rightarrow +\infty} \frac{|f(x, s)|}{\exp\{Ne^{\alpha s^{N'}}\}} = 0, \quad \forall \alpha > \alpha_0 \quad \text{and} \quad \lim_{s \rightarrow +\infty} \frac{|f(x, s)|}{\exp\{Ne^{\alpha s^{N'}}\}} = +\infty, \quad \forall \alpha < \alpha_0. \quad (8)$$

To study the solvability of the problem (1), consider the space

$$\mathfrak{W} = \left\{ u \in W_{0,rad}^1(B, \rho) / \int_B \xi(x)|u|^N dx < +\infty \right\},$$

endowed with the norm

$$\|u\| = \left(\int_B \rho(x)|\nabla u|^N dx + \int_B \xi(x)|u|^N dx \right)^{\frac{1}{N}}. \quad (9)$$

We say that u is a solution to the problem (1), if u is a weak solution in the following sense.

Definition 1.1 A function u is called a solution to (1) if $u \in \mathfrak{W}$ and

$$\int_B (\rho(x)|\nabla u|^{N-2} \nabla u \nabla \varphi + \xi|u|^{N-2} u \varphi) dx = \int_B f(x, u) \varphi dx, \quad \text{for all } \varphi \in \mathfrak{W}. \quad (10)$$

It is clear that finding weak solutions of the problem (1) is equivalent to finding nonzero critical points of the following functional on \mathfrak{W} :

$$\mathcal{E}(u) = \frac{1}{N} \int_B \rho(x)|\nabla u|^N dx + \frac{1}{N} \int_B \xi(x)|u|^N dx - \int_B F(x, u) dx, \quad (11)$$

where $F(x, u) = \int_0^u f(x, t) dt$.

In order to find critical points of the functional \mathcal{E} associated with (1), one generally applies the mountain pass given by Ambrosotti and Robinowitz, see [6].

Before announcing our first result, we denote

$$\lambda_1 = \inf_{u \neq 0, u \in \mathfrak{W}} \frac{\|u\|^N}{\int_B |u|^N dx}, \quad (12)$$

the first eigenvalue of the operator with Dirichlet boundary condition. This eigenvalue λ_1 exists and the corresponding normalized eigen function ϕ_1 is positive and belongs to $L^\infty(B)$ [20].

We start by the first result, in the subcritical double exponential growth, we have the following result.

Theorem 1.2 Assume that ξ is continuous and verifies (ξ_1) , (ξ_2) . Let f a function that has a subcritical growth at $+\infty$ and satisfy (A_1) , (A_2) and (A_3) . If in addition f verifies the condition

$$(A_4) \quad \limsup_{t \rightarrow 0} \frac{NF(x, t)}{t^N} < \lambda_1, \text{ uniformly in } x \in B,$$

where λ_1 is defined by (12), then problem (1) has a non trivial radial solution.

In the case of the critical double exponential growth, the study of problem (1) becomes more difficult than in the case of subcritical exponential growth. Our EulerLagrange functional does not satisfy the PalaisSmale condition at all level anymore. To overcome the verification of compactness of Euler Lagrange functional at some suitable level, we choose testing functions, which are extremal to the Trudinger-Moser inequality (6). Our result is as follows.

Theorem 1.3 Assume that ξ is continuous and verifies (ξ_1) , (ξ_2) . Assume that the function f has critical growth at $+\infty$ and satisfies the conditions (A_1) , (A_2) , (A_3) and (A_4) . If in addition f verifies the asymptotic condition

$$(A_5) \quad \lim_{s \rightarrow \infty} \frac{s|f(x, s)|}{\exp(Ne^{\alpha_0 s^{\frac{N}{N-1}}})} \geq \beta_0 \text{ uniformly in } x, \text{ with}$$

$$\beta_0 > \frac{N}{\alpha_0^{N-1} e^N (N-1 + e^{-\frac{\mathfrak{C}(m, N)}{N-1}})}$$

where $m = \max_{x \in \overline{B}} \xi(x)$,

$$\mathfrak{C}(m, N) = m(\mathfrak{S}(N) + \frac{(N+1)(N-1)!}{N^N} + 1)$$

and

$$\mathfrak{S}(N) = \frac{2}{N} + \frac{N-1}{N^2} + \frac{(N-1)(N-2)}{N^3} + \dots + \frac{(N-1)(N-2)\dots 3}{N^{N-2}},$$

then problem (1) has a non trivial radial solution.

The main reason for this study is that, to our knowledge, there are few research taking into account both this type of non-linearity and the potential $\xi \neq 0$ for a non-linear elliptic equation in the framework of Sobolev weighted spaces.

This paper is organized as follows. In Section 2, we present some necessary preliminary knowledge about working space, and we give some useful lemmas for the compactness analysis. In section 3, we prove that the energy \mathcal{E} satisfied the two geometric properties, and the compactness condition but under a given level for the critical nonlinearity case. Finally, we fulfil the proof of the main results in section 4. In this work, the constant C may change from line to another and sometimes we index the constants in order to show how they change.

2 Sobolev Spaces setting and compactness analysis

2.1 Weighted Lebesgue and Sobolev Spaces setting

Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a bounded domain in \mathbb{R}^N and let $\rho \in L^1(\Omega)$ be a nonnegative function. Following Drabek et al. and Kufner in [20, 29], the weighted Lebesgue space $L^p(\Omega, \rho)$ is defined as follows:

$$L^p(\Omega, \rho) = \{u : \Omega \rightarrow \mathbb{R} \text{ measurable; } \int_{\Omega} \rho(x)|u|^p dx < \infty\},$$

for any real number $1 \leq p < \infty$.

This is a normed vector space equipped with the norm

$$\|u\|_{p,\rho} = \left(\int_{\Omega} \rho(x)|u|^p dx \right)^{\frac{1}{p}}$$

and for $\rho(x) = 1$, we find the standard Lebesgue space $L^p(\Omega)$ and its norm

$$\|u\|_p = \left(\int_{\Omega} |u|^p dx \right)^{\frac{1}{p}}.$$

In [20], the corresponding weighted Sobolev space was defined as

$$W^{1,p}(\Omega, \rho) = \{u \in L^p(\Omega); \nabla u \in L^p(\Omega, \rho)\}$$

and equipped with the norm defined on $W^{1,p}(\Omega)$ by

$$\|u\|_{W^{1,p}(\Omega,\rho)} = (\|u\|_p^p + \|\nabla u\|_{p,\rho}^p)^{\frac{1}{p}}. \quad (13)$$

$L^p(\Omega, \rho)$ and $W^{1,p}(\Omega, \rho)$ are separable, reflexive Banach spaces provided that $\rho(x)^{\frac{1}{p-1}} \in L^1_{loc}(\Omega)$.

Furthermore, if $\rho(x) \in L^1_{loc}(\Omega)$, then $C_0^\infty(\Omega)$ is a subset of $W^{1,p}(\Omega, \rho)$ and we can introduce the space $W_0^{1,p}(\Omega, \rho)$ as the closure of $C_0^\infty(\Omega)$ in $W^{1,p}(\Omega, \rho)$. The space $W_0^{1,p}(\Omega, \rho)$ is equipped with the following norm,

$$\|u\|_{W_0^{1,p}(\Omega,\rho)} = \left(\int_{\Omega} \rho(x) |\nabla u|^p dx \right)^{\frac{1}{p}}, \quad (14)$$

which is equivalent to the one given by (13).

Also, we will use the space $W_0^{1,N}(\Omega, \rho)$, which is the closure of $C_0^\infty(\Omega)$ in $W^{1,N}(\Omega, \rho)$, equipped with the norm

$$\|u\|_{W_0^1(\Omega,\rho)} = \left(\int_{\Omega} \rho(x) |\nabla u|^N dx \right)^{\frac{1}{N}}.$$

Let s the real such that

$$s \in (1, +\infty) \quad \text{and} \quad \rho^{-s} \in L^1(\Omega). \quad (15)$$

The last condition gives important embedding of the space $W^{1,N}(\Omega, \rho)$ into usual Lebesgues spaces without weight. More precisely, following [20] we have

$$W^{1,N}(\Omega, \rho) \hookrightarrow L^N(\Omega) \quad \text{with compact injection} \quad (16)$$

and

$$W^{1,N}(\Omega, \rho) \hookrightarrow L^{N+\eta}(\Omega) \quad \text{with compact injection} \quad \text{for } 0 \leq \eta < N(s-1),$$

provided

$$\rho^{-s} \in L^1(\Omega) \quad \text{with } s \in (1, +\infty).$$

Let the subspace

$$W_{0,rad}^1(B, \rho) = cl\{u \in C_{0,rad}^\infty(B); \int_B \rho(x) |\nabla u|^N dx < \infty\},$$

with $\rho(x) = \left(\log \frac{\epsilon}{|x|}\right)^{N-1}$.

Then the space $\mathfrak{W} = \{u \in W_{0,rad}^1(B, \rho) / \int_B \xi(x)|u|^N dx < +\infty\}$ is a Banach and reflexive space provided (ξ_1) is satisfied. \mathfrak{W} is endowed with the norm

$$\|u\| = \left(\int_B \rho(x)|\nabla u|^N dx + \int_B \xi(x)|u|^N dx \right)^{\frac{1}{N}},$$

which is equivalent to the following norm

$$\|u\|_{W_{0,rad}^1(B, \rho)} = \left(\int_B \rho(x)|\nabla u|^N dx \right)^{\frac{1}{N}}.$$

2.2 Compactness analysis step one

In this section, we will present a number of technical Lemmas for our future use. We begin by the following Lemma.

Lemma 1 The following embedding is continuous

$$\mathfrak{W} \hookrightarrow L^q(B) \quad \text{for all } q \geq 1.$$

Moreover, this embedding is also compact for all $q \geq 1$.

Proof. Since $\rho(x) \geq 1$ for all $x \in B$, the following embedding

$$\mathfrak{W} \hookrightarrow W_{0,rad}^1(B) \hookrightarrow L^q(B),$$

are continuous for all $q \geq 1$. To show that it is also compact, take a sequence of function $u_k \subset \mathfrak{W}$ such that $\|u_k\| \leq C$ for all k . Then $\|u_k\|_{W_{0,rad}^1} \leq C$ for all k . On the other hand, we have the following compact embedding[20] $W_{0,rad}^1 \hookrightarrow L^q$ for all q such that $1 \leq q < Ns$, with $s > 1$, then up to a subsequence, there exists some $u \in W_{0,rad}^1$, such that u_k convergent to u strongly in $L^q(B)$ for all q such that $1 \leq q < Ns$. Without loss of generality, we may assume that

$$\begin{cases} u_k & \rightharpoonup u & \text{weakly in } \mathfrak{W} \\ u_k & \rightarrow u & \text{strongly in } L^1(B) \\ u_k(x) & \rightarrow u(x) & \text{almost everywhere in } B. \end{cases} \quad (17)$$

For $q > 1$, it follows from (17) and the continuous embedding $\mathfrak{W} \hookrightarrow L^p(B)$ ($p \geq 1$) that

$$\begin{aligned} \int_B |u_k - u|^q dx &= \int_B |u_k - u|^{\frac{1}{2}} |u_k - u|^{q-\frac{1}{2}} dx \\ &\leq \left(\int_B |u_k - u| dx \right)^{\frac{1}{2}} \left(\int_B |u_k - u|^{2q-1} dx \right)^{\frac{1}{2}} \\ &\leq C \left(\int_B |u_k - u| dx \right)^{\frac{1}{2}} \rightarrow 0. \end{aligned}$$

This concludes the Lemma. \square

A second important Lemma.

Lemma 2 [23] Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and $f : \overline{\Omega} \times \mathbb{R}$ a continuous function. Let $\{u_n\}_n$ be a sequence in $L^1(\Omega)$ converging to u in $L^1(\Omega)$. Assume that $f(x, u_n)$ and $f(x, u)$ are also in $L^1(\Omega)$. If

$$\int_{\Omega} |f(x, u_n)u_n| dx \leq C,$$

where C is a positive constant, then

$$f(x, u_n) \rightarrow f(x, u) \text{ in } L^1(\Omega).$$

In an attempt to prove a compactness condition for the energy \mathcal{E} , we need a lions type result [32] about an improved TM-inequality when we deal with weakly convergent sequences and double exponential case.

Lemma 3 Let $\{u_k\}_k$ be a sequence in \mathfrak{W} . Suppose that $\|u_k\| = 1$, $u_n \rightharpoonup u$ weakly in \mathfrak{W} , $u_n(x) \rightarrow u(x)$ and $\nabla u_n(x) \rightarrow \nabla u(x)$ almost everywhere in B . Then

$$\sup_k \int_B \exp(Ne^{p\omega \frac{1}{N-1}} |u_k|^{N'}) dx < +\infty,$$

for all $1 < p < \mathcal{P}$, where

$$\mathcal{P} := \begin{cases} (1 - \|u\|^N)^{\frac{-1}{N-1}} & \text{if } \|u\| < 1 \\ +\infty & \text{if } \|u\| = 1. \end{cases}$$

Proof. By young inequality we have

$$\exp(Ne^{a+b}) \leq \frac{1}{q} \exp(Ne^{qa}) + \frac{1}{q'} \exp(Ne^{q'b}), \quad \forall a, b \in \mathbb{R}, \quad q > 1,$$

with $\frac{1}{q} + \frac{1}{q'} = 1$. And also we can estimate $|u_k|^{N'}$ using the following inequality

$$(1+a)^q \leq (1+\varepsilon)a^q + \left(1 - \frac{1}{(1+\varepsilon)^{\frac{1}{q-1}}}\right)^{1-q}, \quad \forall a \geq 0, \quad \forall \varepsilon > 0 \quad \forall q > 1.$$

So, we get

$$|u_k|^{N'} \leq (1+\varepsilon)|u_k - u|^{N'} + \left(1 - \frac{1}{(1+\varepsilon)^{N-1}}\right)^{\frac{-1}{N-1}} |u|^{N'}.$$

Therefore for any $p > 1$, using the above inequalities we obtain

$$\begin{aligned} \int_B \exp(Ne^{p\omega \frac{1}{N-1} |u_k|^{N'}}) dx &\leq \frac{1}{q} \int_B \exp(Ne^{pq\omega \frac{1}{N-1} (1+\varepsilon) |u_k - u|^{N'}}) dx \\ &+ \frac{1}{q'} \int_B \exp(Ne^{pq'\omega \frac{1}{N-1} (1 - \frac{1}{(1+\varepsilon)^{N-1}})^{\frac{-1}{N-1}} |u|^{N'}}) dx. \end{aligned}$$

From (4) the last integral is finite and to complete the proof we should prove that for every p such that $1 < p < \mathcal{P}$, we have

$$\sup_k \int_B \exp(Ne^{pq\omega \frac{1}{N-1} (1+\varepsilon) |u_k - u|^{N'}}) dx < +\infty,$$

for some $\varepsilon > 0$ and $q > 1$.

By Brezis-Lieb's lemma we have

$$\|u_n - u\|^N = \|u_n\|^N - \|u\|^N + o_n(1) \quad \text{where } o_n(1) \rightarrow 0 \text{ as } n \rightarrow +\infty.$$

Then

$$\|u_n - u\|^N = 1 - \|u\|^N + o_n(1) \quad \text{where } o_n(1) \rightarrow 0 \text{ as } n \rightarrow +\infty.$$

We may assume that $\|u\| < 1$. The proof in the case $\|u\| = 1$ is similar. If $\|u\| < 1$ then for

$$p < \frac{1}{(1 - \|u\|^N)^{\frac{1}{N-1}}},$$

there exists $\nu > 0$ such that

$$p(1 - \|u\|^N)^{\frac{1}{N-1}} (1 + \nu) < 1.$$

On the other hand,

$$\lim_{k \rightarrow +\infty} \|u_k - u\|^N = 1 - \|u\|^N$$

and so

$$\lim_{k \rightarrow +\infty} \|u_k - u\|^{N'} = (1 - \|u\|^N)^{\frac{1}{N-1}}.$$

Therefore, for every $\varepsilon > 0$, there exists $k_\varepsilon \geq 1$ such that

$$\|u_k - u\|^{N'} \leq (1 + \varepsilon)(1 - \|u\|^N)^{\frac{1}{N-1}}, \quad \forall k \geq k_\varepsilon.$$

Then, for $q = 1 + \varepsilon$ with ε such that $\varepsilon = \sqrt[3]{1 + \nu} - 1$ and for every $k \geq k_\varepsilon$, we get

$$pq(1 + \varepsilon)\|u_k - u\|^{N'} \leq 1.$$

From (5), this leads to

$$\begin{aligned}
 & \int_B \exp(Ne^{pq\omega \frac{1}{N-1} (1+\varepsilon) |u_k - u|^{N'}}) dx \\
 \leq & \int_B \exp(Ne^{(1+\varepsilon)pq\omega \frac{1}{N-1} (\frac{|u_k - u|}{\|u_k - u\|})^{N'}} \|u_k - u\|^{N'}) dx \\
 \leq & \int_B \exp(Ne^{\omega \frac{1}{N-1} (\frac{|u_k - u|}{\|u_k - u\|})^{N'}}) dx \\
 \leq & \sup_{\|u\| \leq 1} \int_B \exp(Ne^{\omega \frac{1}{N-1} |u|^{N'}}) dx < +\infty
 \end{aligned}$$

and the proof is complete. \square

3 The variational formulation

As the reaction term f is critical or sub-critical growth, there are positive constants b and c such as

$$|f(x, t)| \leq b \exp\{e^{ct^{N'}}\}, \quad \forall x \in B, \quad \forall t \in \mathbb{R}. \quad (18)$$

3.1 The geometrical properties of the energy \mathcal{E}

As we mentioned in the introduction, problems (1) have variational structure. In the sequel, we prove that the functional \mathcal{E} has a mountain pass geometry. We begin by the first.

Lemma 4 Assume that the hypothesis $(A_1), (A_2), (A_3), (A_4), (A_5), (\xi_1)$ and (ξ_2) hold. Then there exist $a > 0$ and $\sigma > 0$ such that

$$\mathcal{E}(u) \geq a \quad \forall u : \|u\| = \sigma.$$

Proof. By (A_4) , there exist $\varepsilon_0 \in (0, 1)$ and $\delta_0 > 0$ such that for all $(x, t) \in B \times \mathbb{R}$

$$F(x, t) \leq \frac{1}{N} \lambda_1 (1 - \varepsilon_0) t^N, \quad \text{for } |t| \leq \delta_0.$$

Indeed, from (A_4) we have

$$\limsup_{t \rightarrow 0} \frac{NF(x, t)}{t^N} < \lambda_1,$$

or

$$\inf_{\beta > 0} \sup\left\{ \frac{NF(x, t)}{t^N}, \quad 0 < t < \beta \right\} < \lambda_1.$$

This inequality is strict, then there exists $\varepsilon_0 > 0$ such that

$$\inf_{\beta > 0} \sup\left\{\frac{NF(x, t)}{t^N}, 0 < t < \beta\right\} < \lambda_1 - \varepsilon_0,$$

hence, there exists $\delta_0 > 0$ such that

$$\sup\left\{\frac{NF(x, t)}{t^N}, 0 < t < \delta_0\right\} < \lambda_1 - \varepsilon_0$$

and consequently

$$\forall |t| < \delta_0 F(x, t) \leq \frac{1}{N} \lambda_1 (1 - \varepsilon_0) t^N.$$

From (18), we deduce that for $q > N$ there exist constant $c_0 > 0$ and $c_1 > 0$ such that

$$F(x, t) \leq c_1 |t|^q \exp(e^{c_0 t^{\frac{N}{N-1}}}), \quad \forall |t| \geq \delta_0$$

and hence we get

$$F(x, t) \leq c_1 |t|^q \exp(e^{c_0 t^{\frac{N}{N-1}}}) + \frac{1}{N} \lambda_1 (1 - \varepsilon_0) t^N, \quad \forall t \in \mathbb{R}.$$

Then, using the fact that $\lambda_1 \int_B |u|^N dx \leq \|u\|^N$ and the Hölder inequality we get

$$\mathcal{E}(u) \geq \frac{1}{N} \varepsilon_0 \|u\|^N - c_1 \left(\int_B \exp(N e^{c_0 |u|^{\frac{N}{N-1}}}) dx \right)^{\frac{1}{N}} \left(\int_B |u|^{\frac{N-1}{N} q} dx \right)^{\frac{N-1}{N}}.$$

We choose $\varrho > 0$ such that $c_0 \varrho^{N'} \leq \omega_{\frac{1}{N-1}}$, then we get

$$\int_B \exp(N e^{c_0 |u|^{N'}}) dx = \int_B \exp(N e^{c_0 \|u\|^{N'} \left(\frac{|u|}{\|u\|}\right)^{N'}}) dx \leq c_2, \quad \forall u \in \mathfrak{W} \text{ with } \|u\| = \varrho$$

and this follows from (5). On the other hand, by lemma 1, there exist a constant $c_4 > 0$, such that $\|u\|_{N', q} \leq c_4 \|u\|$, so we deduce that there exists c_5 such that

$$\mathcal{E}(u) \geq \frac{1}{N} \varepsilon_0 \|u\|^N - c_5 \|u\|^q \quad \forall u \in \mathfrak{W} \quad \|u\| = \varrho,$$

provided $\varrho > 0$ and $c_0 \varrho^{N'} \leq \omega_{\frac{1}{N-1}}$. Finally, we choose $\sigma > 0$ as the maximum point of the function $g(\varrho) = \frac{\varepsilon_0}{N} \varrho^N - c_5 \varrho^q$ on the interval $[0, \frac{\omega_{\frac{1}{N-1}}}{c_0^{\frac{1}{N'}}}]$ and

let $a = \mathcal{E}(\sigma)$ then the Lemma follows. \square

Lemma 5 Suppose that (A_1) , (A_2) , (ξ_1) and (ξ_2) hold. Let ϕ_1 be a normalized eigenfunction associated to λ_1 in \mathfrak{W} . Then, $\mathcal{E}(t\phi_1) \rightarrow -\infty$, as $t \rightarrow +\infty$.

Proof. Let $\phi_1 \in E \cap L^\infty(B)$ be the normalized eigen function associated to the eigen-value defined by (12) ie such that $\|\phi_1\| = 1$. We have

$$\mathcal{E}(t\phi_1) = \frac{t^N}{N} \|\phi_1\|^N - \int_B F(x, t\phi_1) dx.$$

Then using (A_1) and (A_2) and integrating, we get the existence of a constant $C > 0$ such that

$$F(x, t) \geq C e^{\frac{1}{M}t}, \quad \forall |t| \geq t_0.$$

Consequently, there exist $\gamma > \lambda_1$ and $C > 0$ such that $F(x, t) \geq \frac{\gamma}{N} t^N + C$ for all $t > 0$.

$$\mathcal{E}(t\phi_1) \leq \frac{t^N}{N} \|\phi_1\|^N - \frac{\gamma}{N} t^N \|\phi_1\|_N^N - C|B|,$$

where $|B| = \text{mes}(B) = \text{Vol}(B)$. Then, from the definition of λ_1 , we get

$$\mathcal{E}(t\phi_1) \leq t^N \frac{\lambda_1 - \gamma}{N} \|\phi_1\|_N^N < 0 \quad \forall t > 0.$$

This achieves the proof.

3.2 Compactness analysis step two : the compactness level of the energy \mathcal{E}

The main difficulty in the approach to the critical problem of growth is the lack of compactness. Precisely the overall condition of Palais-Smale does not hold except for a certain level of energy. In the following proposition, we identify the first level of compactness.

Proposition 3.1 Let \mathcal{E} be the energy associated to the problem (1) defined by (11), then

- (i) In the subcritical case the functional \mathcal{E} satisfies the Palais-Smale condition $(PS)_d$ at all level $d \in \mathbb{R}$.
- (ii) In the critical case the functional \mathcal{E} satisfies the Palais-Smale condition $(PS)_d$ only for level d such that

$$d < \frac{\omega_{N-1}}{N\alpha_0^{N-1}}.$$

Proof. (ii) We begin by the critical case. Let $d \in \mathbb{R}$ and $\{u_n\}_n$ in \mathfrak{W} be a $(PS)_d$ sequence, that is

$$\mathcal{E}(u_n) = \frac{1}{N} \|u_n\|^N - \int_B F(x, u_n) dx \rightarrow d, \quad n \rightarrow +\infty \quad (19)$$

and

$$\begin{aligned} |\mathcal{E}'(u_n)v| &= \left| \int_B \rho(x) |\nabla u_n|^{N-2} \nabla u_n \cdot \nabla v dx \right. \\ &\quad \left. + \int_B \xi |u_n|^{N-2} u_n v dx - \int_B f(x, u_n) v dx \right| \\ &\leq \varepsilon_n \|v\|, \quad \forall v \in \mathfrak{W}, \end{aligned} \quad (20)$$

where $\varepsilon_n \rightarrow 0$ as $n \rightarrow +\infty$.

By (A_2) , we get for any $\varepsilon > 0$, a real $t_\varepsilon > 0$ such that

$$F(x, t) \leq \varepsilon t f(x, t), \quad \forall |t| > t_\varepsilon, \quad \text{uniformly in } x \in B. \quad (21)$$

Hence, for any $\varepsilon > 0$, we have

$$\begin{aligned} \frac{1}{N} \|u_n\|^N &\leq C + \int_B F(x, u_n) dx \\ &\leq C + \int_{|u_n| \leq t_\varepsilon} F(x, u_n) dx + \varepsilon \int_B f(x, u_n) u_n dx \\ &\leq C_\varepsilon + \varepsilon \varepsilon_n \|u_n\| + \varepsilon \|u_n\|^N. \end{aligned}$$

Therefore

$$\left(\frac{1}{N} - \varepsilon\right) \|u_n\|^N \leq C + \varepsilon \varepsilon_n \|u_n\|$$

and so (u_n) is bounded in \mathfrak{W} , then there exists $u \in \mathfrak{W}$ such that, up to a subsequence

$$\begin{aligned} u_n &\rightharpoonup u \quad \text{in } \mathfrak{W} \\ u_n &\rightarrow u \quad \text{in } L^q(B) \quad \forall q \geq 1 \\ u_n &\rightarrow u \quad \text{a.e in } B. \end{aligned}$$

We follow the schema of [1] to show the convergence almost everywhere of the gradient $\nabla u_n(x) \rightarrow \nabla u(x)$ a.e $x \in B$ and $|\nabla u_n|^{N-2} \nabla u_n \rightharpoonup |\nabla u|^{N-2} \nabla u$ weakly in $(L^{\frac{N}{N-1}}(B, w))^N$. Now, from (18) and (19), we have

$$\left| \int_B f(x, u_n) u_n dx \right| \leq \varepsilon_n \|u_n\| + \|u_n\|^N \leq C.$$

By Lemma 2, we obtain

$$f(x, u_n) \rightarrow f(x, u) \quad \text{in } L^1(B) \quad \text{as } n \rightarrow +\infty. \quad (22)$$

From (21), we have

$$\left| \int_B F(x, u_n) dx \right| \leq C + \varepsilon \left| \int_B f(x, u_n) u_n dx \right| \leq C.$$

Using the condition (A_2) and the generalized Lebesgue dominated convergence theorem, we get

$$F(x, u_n) \rightarrow F(x, u) \text{ in } L^1(B) \text{ as } n \rightarrow +\infty. \quad (23)$$

It follows from (19) that

$$\lim_{n \rightarrow +\infty} \frac{1}{N} \|u_n\|^N = d + \int_B F(x, u) dx. \quad (24)$$

Then by (A_3) and (20), we have

$$\lim_{n \rightarrow +\infty} N \int_B F(x, u_n) dx \leq \lim_{n \rightarrow +\infty} \int_B f(x, u_n) u_n dx = N(d + \int_B F(x, u) dx). \quad (25)$$

So, $d \geq 0$. Moreover, from (19), (20) and passing to the limit we get

$$\int_B \rho(x) |\nabla u|^{N-2} \nabla u \cdot \nabla v dx + \int_B \xi |u|^{N-2} u v dx = \int_B f(x, u) v dx, \quad \forall v \in \mathfrak{W}. \quad (26)$$

Therefore u is solution of the problem (1).

Taking $u = v$, we get

$$\int_B \rho(x) |\nabla u|^N dx + \int_B \xi |u|^N dx = \int_B f(x, u) u dx \geq N \int_B F(x, u) dx,$$

which implies that $\mathcal{E}(u) \geq 0$.

Next, we will distinguish three cases:

- (1) $d = 0$
- (2) $d > 0, u = 0$
- (3) $d > 0, u > 0$.

Case (1): $d = 0$. We have

$$0 \leq \mathcal{E}(u) \leq \liminf_{n \rightarrow +\infty} \mathcal{E}(u_n) = 0$$

and then

$$\frac{1}{N} \|u\|^N = \int_B F(x, u) dx.$$

From (23), we deduce

$$\|u_n\| \rightarrow \|u\| \quad \text{as } n \rightarrow +\infty.$$

Therefore, up to subsequence,

$$u_n \rightarrow u \quad \text{strongly in } \mathfrak{W}.$$

Case (2): $d > 0$, $u = 0$. Then, from (18) and (19), we get

$$\lim_{n \rightarrow +\infty} \|u_n\|^N = Nd \quad \text{and} \quad \lim_{n \rightarrow +\infty} \int_B f(x, u_n) u_n dx = Nd.$$

We will prove this is impossible and the *Case (2)* can not occur.

Claim: There exists $q > 1$ such that

$$\int_B |f(x, u_n)|^q dx \leq C, \quad (27)$$

for some constant C .

Let α_0 be the real that appear in the definition of critical or subcritical non-linearity f . For every $\varepsilon > 0$ and $q > 1$ there exists $t_\varepsilon > 0$ and $C = C(q, \varepsilon) > 0$ such that

$$|f(x, t)|^q \leq C_{\varepsilon, q} \left(\exp\{N e^{\alpha_0(\varepsilon+1)t^{\frac{N}{N-1}}}\} \right), \quad \forall |t| \geq t_\varepsilon \quad \text{and uniformly in } x \in B. \quad (28)$$

Therefore

$$\begin{aligned} \int_B |f(x, u_n)|^q dx &= \int_{\{|u_n| \leq t_\varepsilon\}} |f(x, u_n)|^q dx + \int_{\{|u_n| > t_\varepsilon\}} |f(x, u_n)|^q dx \\ &\leq \omega_{N-1} \max_{B \times [-t_\varepsilon, t_\varepsilon]} |f(x, t)|^q + C_{\varepsilon, q} \int_B \exp(N e^{\alpha_0(\varepsilon+1)|u_n|^{\frac{N}{N-1}}}) dx. \end{aligned}$$

The last integral is finite. Indeed, since $Nd < \frac{\omega_{N-1}}{\alpha_0^{\frac{N-1}{N}}}$, there exists $\eta \in (0, \frac{1}{N})$ such that $Nd = (1 - N\eta) \frac{\omega_{N-1}}{\alpha_0^{\frac{N-1}{N}}}$. On the other hand, $\|u_n\|^{N'} \rightarrow (Nd)^{\frac{1}{N-1}}$, so

there exists $n_\eta > 0$ such that for all $n \geq n_\eta$, we get $\|u_n\|^{N'} \leq (1 - \eta) \frac{\omega_{N-1}^{\frac{1}{N-1}}}{\alpha_0}$. Therefore,

$$\alpha_0(1 + \varepsilon) \left(\frac{|u_n|}{\|u_n\|} \right)^{N'} \|u_n\|^{N'} \leq (1 + \varepsilon)(1 - \eta) \omega_{N-1}^{\frac{1}{N-1}}.$$

We choose $\varepsilon > 0$ small enough to get

$$\alpha_0(1 + \varepsilon) \|u_n\|^{N'} \leq \omega_{N-1}^{\frac{1}{N-1}},$$

therefore the second integral is uniformly bounded in view of (5) and the claim follows.

Now using (20) when $v = u_n$ and for $q > 1$, we get

$$\left| \|u_n\|^N - \int_B f(x, u_n) u_n dx \right| \leq C\varepsilon_n$$

and so

$$\|u_n\|^N \leq C\varepsilon_n + \left(\int_B |f(x, u_n)|^q dx \right)^{\frac{1}{q}} \left(\int_B |u_n(x)|^{q'} dx \right)^{\frac{1}{q'}}, \quad (29)$$

where q and q' are conjugate.

Since $u_n \rightarrow 0$ in $L^{q'}(B)$, the inequality (29) gives

$$\|u_n\| \rightarrow 0,$$

then $d = 0$ which is impossible since we supposed that $d > 0$ and so this case can not occur.

Case (3): $d > 0$ and $u > 0$.

We prove that $\mathcal{E}(u) = d$.

We have

$$\begin{aligned} \mathcal{E}(u) &\leq \liminf_{n \rightarrow +\infty} \mathcal{E}(u_n) \\ &\leq \frac{1}{N} \liminf_{n \rightarrow +\infty} \left(\|u_n\|^N - \int_B F(x, u) dx \right) \\ &\leq d. \end{aligned}$$

Now suppose that $\mathcal{E}(u) < d$. Then

$$\|u\|^{N'} < \left(N \left(d + \int_B F(x, u) dx \right) \right)^{\frac{1}{N-1}}. \quad (30)$$

Let

$$v_n = \frac{u_n}{\|u_n\|}$$

and

$$v = \frac{u}{\left(N \left(d + \int_B F(x, u) dx \right) \right)^{\frac{1}{N}}}.$$

It's clear that $v_n \rightharpoonup v$ weakly in \mathfrak{W} , v a non zero function $\|v_n\| = 1$ and $\|v\| < 1$.

Applying the Lions-type Lemma 3, we get

$$\sup_n \int_B \exp \left(N e^{p\omega \frac{1}{N-1}} |v_n|^{N'} \right) dx < \infty,$$

for some $1 < p < (1 - \|v\|^N)^{\frac{1}{N-1}}$.

As in the case(2), we are going to estimate $\int_B |f(x, u_n)|^q dx$.

For any $\varepsilon > 0$, we have

$$\begin{aligned} \int_B |f(x, u_n)|^q dx &= \int_{\{|u_n| \leq t_\varepsilon\}} |f(x, u_n)|^q dx + \int_{\{|u_n| > t_\varepsilon\}} |f(x, u_n)|^q dx \\ &\leq \omega_{N-1} \max_{B \times [-t_\varepsilon, t_\varepsilon]} |f(x, t)|^q + C_{\varepsilon, q} \int_B \exp(N e^{\alpha_0(1+\varepsilon)} |u_n|^{N'}) dx \\ &\leq C_\varepsilon + C_{\varepsilon, q} \int_B \exp(N e^{\alpha_0(1+\varepsilon)} \|u_n\|^{N'} |v_n|^{N'}) dx \leq C, \end{aligned}$$

provided $\alpha_0(1+\varepsilon)\|u_n\|^{N'} \leq p \omega_{N-1}^{\frac{1}{N-1}}$, for some $1 < p < (1 - \|v\|^N)^{\frac{1}{N-1}}$.
Indeed, we have

$$(1 - \|v\|^N)^{\frac{1}{N-1}} = \left(\frac{N(d + \int_B F(x, u) dx)}{N(d + \int_B F(x, u) dx) - \|u\|^N} \right)^{\frac{1}{N-1}} = \left(\frac{d + \int_B F(x, u) dx}{d - \mathcal{E}(u)} \right)^{\frac{1}{N-1}}.$$

Since

$$\lim_{n \rightarrow +\infty} \|u_n\|^{N'} = \left(N(d + \int_B F(x, u) dx) \right)^{\frac{1}{N-1}},$$

then,

$$\alpha_0(1+\varepsilon)\|u_n\|^{N'} \leq \alpha_0(1+2\varepsilon) \left(N(d + \int_B F(x, u) dx) \right)^{\frac{1}{N-1}}$$

and to get the desired estimate it's enough to show that we can choose $\varepsilon > 0$ small enough such that

$$\frac{\alpha_0}{\omega_{N-1}^{\frac{1}{N-1}}} (1+2\varepsilon) < \left(\frac{1}{N(d - \mathcal{E}(u))} \right)^{\frac{1}{N-1}},$$

that is

$$(1+2\varepsilon)(d - \mathcal{E}(u))^{\frac{1}{N-1}} < \frac{\omega_{N-1}^{\frac{1}{N-1}}}{N^{\frac{1}{N-1}} \alpha_0} \quad (31)$$

and the last inequality holds since $\mathcal{E}(u) \geq 0$ and $d < \frac{\omega_{N-1}}{N \alpha_0^{\frac{1}{N-1}}}$.

From (20) with $v = u_n - u$, we get

$$\begin{aligned} \int_B \rho(x) |\nabla u_n|^{N-2} \nabla u_n \cdot (\nabla u_n - \nabla u) dx + \int_B \xi(x) |u_n|^{N-2} u_n (u_n - u) dx \\ - \int_B f(x, u_n) (u_n - u) dx = o_n(1). \end{aligned} \quad (32)$$

On the other hand, since $u_n \rightharpoonup u$ weakly in \mathfrak{W} then

$$\int_B \rho(x) |\nabla u|^{N-2} \nabla u \cdot (\nabla u_n - \nabla u) dx + \int_B \xi(x) |u|^{N-2} u (u_n - u) dx = o_n(1). \quad (33)$$

Combining (32) and (33), we obtain

$$\begin{aligned} & \int_B \rho(x) (|\nabla u_n|^{N-2} \nabla u_n - |\nabla u|^{N-2} \nabla u) \cdot (\nabla u_n - \nabla u) dx + \\ & \int_B \xi(x) (|u_n|^{N-2} u_n - |u|^{N-2} u) (u_n - u) dx - \int_B f(x, u_n) (u_n - u) dx = o_n(1). \end{aligned}$$

Using the well known inequality

$$(|x|^{N-2} x - |y|^{N-2} y) \cdot (x - y) \geq 2^{2-N} |x - y|^N, \quad \forall x, y \in \mathbb{R}^N \text{ and } N \geq 2, \quad (34)$$

we obtain

$$\begin{aligned} 0 & \leq 2^{2-N} \left(\int_B \rho(x) |\nabla u_n - \nabla u|^N dx + \int_B \xi(x) |u_n - u|^N dx \right) \\ & \leq \int_B f(x, u_n) (u_n - u) dx + o_n(1). \end{aligned} \quad (35)$$

By the Hölder inequality, we obtain

$$\begin{aligned} 2^{2-N} \|u_n - u\|^N & \leq \int_B f(x, u_n) (u_n - u) dx + o_n(1) \\ & \leq \left(\int_B |f(x, u_n)|^q \right)^{\frac{1}{q}} \left(\int_B |u_n - u|^{q'} \right)^{\frac{1}{q'}} dx + o_n(1). \end{aligned} \quad (36)$$

So,

$$\|u_n - u\| \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

By Brezis-Lieb's lemma, up to subsequence, we get

$$\lim_{n \rightarrow +\infty} \|u_n\|^N = N(d + \int_B F(x, u) dx) = \|u\|^N,$$

which contradicts (49).

(i) In the subcritical case, the Palais-Smale condition is satisfied for all level $d \in \mathbb{R}$. Indeed, up to subsequences, we can assume that

$$\begin{cases} \|u_n\| & \leq M & \text{in } \mathfrak{W} \\ u_n & \rightharpoonup u & \text{weakly in } \mathfrak{W} \\ u_n & \rightarrow u & \text{strongly in } L^q(B) \quad \forall q \geq 1 \\ u_n(x) & \rightarrow u(x) & \text{almost everywhere in } B. \end{cases}$$

Since f is subcritical at $+\infty$, there exists a constant $C_M > 0$ such that

$$f(x, s) \leq C_M \exp\left\{e^{M \frac{1}{N-1}} s^{\frac{N}{N-1}}\right\}, \forall (x, s) \in B \times (0, +\infty).$$

Using the Hölder inequality

$$\begin{aligned} \left| \int_B f(x, u_n)(u_n - u) dx \right| &\leq \int_B |f(x, u_n)(u_n - u)| dx \\ &\leq \left(\int_B |f(x, u_n)|^2 dx \right)^{\frac{1}{2}} \left(\int_B |u_n - u|^2 dx \right)^{\frac{1}{2}} \\ &\leq C \left(\int_B \exp\left\{2e^{M \frac{1}{N-1}} u_n^{\frac{N}{N-1}}\right\} dx \right)^{\frac{1}{2}} \|u_n - u\|_2 \\ &\leq C \left(\int_B \exp\left\{2e^{M \frac{1}{N-1}} \|u_n\|^{\frac{N}{N-1}} \frac{|u_n|^{\frac{N}{N-1}}}{\|u_n\|^{\frac{N}{N-1}}}\right\} dx \right)^{\frac{1}{2}} \|u_n - u\|_2 \\ &\leq C \|u_n - u\|_2 \rightarrow 0 \text{ as } n \rightarrow +\infty. \end{aligned}$$

Proceeding as in the case (3), with $v = u_n - u$ in (20), we get

$$2^{2-N} \|u_n - u\|^N \leq \left| \int_B f(x, u_n)(u_n - u) dx \right| + o_n(1) \rightarrow 0 \text{ as } n \rightarrow +\infty.$$

This completes the proof of the Proposition 3.2. \square

4 Proof of the main results

Proof of Theorem 1.2

Since $f(x, t)$ satisfies the condition (7) for all $\alpha_0 > 0$, then by Proposition 3.2, the functional \mathcal{E} satisfies the (PS) condition (at each possible level d). So, by Lemma 4 and Lemma 5, we deduce that the functional \mathcal{E} has a nonzero critical point u in \mathfrak{W} . From maximum principle, the solution u of the problem (1) is positive. \square

Proof of Theorem 1.3 We are going to estimate the minmax value of the functional \mathcal{E} . The idea is to set up a sequence of functions $v_n \in \mathfrak{W}$, and estimate $\max\{\mathcal{E}(tv_n) : t \geq 0\}$. For this purpose, let consider the following Moser function

$$w_n(x) = \frac{1}{\omega_{N-1}^{\frac{1}{N}}} \begin{cases} \frac{\log(\log(\frac{e}{|x|}))}{\log^{\frac{1}{N}}(1+n)} & \text{if } e^{-n} \leq |x| \leq 1 \\ \log^{\frac{N-1}{N}}(1+n) & \text{if } 0 \leq |x| \leq e^{-n}. \end{cases}$$

Let $v_n(x) = \frac{w_n(x)}{\|w_n\|}$. Then $v_n \in \mathfrak{W}$ and $\|v_n\| = 1$.

4.1 Helpful Lemmas

We need two technical Lemmas who will help us to reach our aims and objectives.

Lemma 6 Assume $\xi(x)$ is continuous and (ξ_1) is satisfied. Then there holds

(i)

$$\|w_n\|^N \leq 1 + \frac{m(\mathfrak{S}(N) + \frac{(N+1)(N-1)!}{N^N} + 1 + o_n(1))}{\log(1+n)} + o_n(1),$$

where $m = \max_{x \in \bar{B}} \xi(x)$, $o_n(1) \rightarrow 0$ as $n \rightarrow +\infty$, and

$$\mathfrak{S}(N) = \frac{2}{N} + \frac{N-1}{N^2} + \frac{(N-1)(N-2)}{N^3} + \dots + \frac{(N-1)(N-2)\dots 3}{N^{N-2}}.$$

(ii)

$$E(m, N, n) \leq \frac{1}{\|w_n\|^{N'}} \leq D(\xi_0, N, n),$$

where

$$E(m, N, n) = 1 - \frac{m(\mathfrak{S}(N) + \frac{(N+1)(N-1)!}{N^N} + 1 + o_n(1))}{(N-1)\log(1+n)} + o_n(1)$$

and

$$D(\xi_0, N, n) = 1 - \frac{\xi_0(\mathfrak{S}(N) + \frac{(N+1)(N-1)!}{N^N} + 1 + o_n(1))}{(N-1)\log(1+n)} + o_n(1).$$

Proof. (i) We have

$$\begin{aligned} & \frac{1}{\omega_{N-1}} \int_B \log^{N-1}\left(\frac{e}{|x|}\right) |\nabla w_n|^N dx \\ &= \frac{1}{\log(1+n)} \int_{e^{-n}}^1 r^{N-1} \left| \frac{1}{r \log \frac{e}{r}} \right|^N \log^{N-1} \frac{e}{r} dr = 1 \end{aligned}$$

and,

$$I = \int_{e^{-n} \leq |x| \leq 1} \log^N(\log(\frac{e}{|x|})) dx \leq \int_{e^{-n} \leq |x| \leq 1} \log^N(\frac{e}{|x|}) dx.$$

Making the change of variable, $|x| = e^{-t}$ and integrating by part, we get

$$\begin{aligned} I &= \int_{e^{-n} \leq |x| \leq 1} \log^N(\log(\frac{e}{|x|})) dx \leq \omega_{N-1} \int_0^n e^{-Nt} (1+t)^N dt \\ &= \omega_{N-1} \left(-(1+n)^N \frac{e^{-Nn}}{N} + \frac{1}{N} \right) + \int_0^n e^{-Nt} (1+t)^{N-1} dt \\ &= \omega_{N-1} \left(\frac{1}{N} - \frac{1}{N} e^{-nN} (1+n)^N + \frac{N-1}{N} \int_0^n e^{-Nt} (1+t)^{N-2} dt \right) \\ &= \omega_{N-1} \left(\mathfrak{S}(N) - \frac{e^{-nN}}{N} \mathfrak{B}(n, N) + \frac{(N-1)!}{N^{N-2}} \int_0^n e^{-Nt} (1+t) dt \right), \end{aligned}$$

where

$$\begin{aligned} \mathfrak{B}(n, N) &= (1+n)^N + (1+n)^{N-1} \\ &+ \sum_{j=2}^{j=N-2} \frac{(N-1)(N-2)(N-3)\dots(N-(j-1))}{N^j} (1+n)^{N-j} \end{aligned}$$

and

$$\mathfrak{S}(N) = \frac{2}{N} + \frac{N-1}{N^2} + \frac{(N-1)(N-2)}{N^3} + \dots + \frac{(N-1)(N-2)\dots 3}{N^{N-2}}.$$

Then

$$\begin{aligned} I &= \omega_{N-1} \left(\mathfrak{S}(N) + \frac{(N+1)(N-1)!}{N^N} - \frac{e^{-nN}}{N} \mathfrak{B}(n, N) \right. \\ &\left. + \frac{(N-1)!}{N^N} + \frac{(1+N^2)(N-1)!}{N^N} \right) \end{aligned}$$

and

$$I = \omega_{N-1} \left(\mathfrak{S}(N) + \frac{(N+1)(N-1)!}{N^N} + o_n(1) \right).$$

Hence

$$\begin{aligned}
 & \int_B |w_n(x)|^N dx \\
 \leq & \frac{1}{\omega_{N-1} \log(1+n)} \int_{e^{-n} \leq |x| \leq 1} \log^N \left(\frac{e}{|x|} \right) dx \\
 + & \frac{1}{\omega_{N-1}} \int_{0 \leq |x| \leq e^{-n}} \log^{N-1}(1+n) dx \\
 = & \frac{1}{\log(1+n)} \int_0^n e^{-Nt} (1+t)^N dt + \frac{1}{N} e^{-Nn} \log^{N-1}(1+n) \\
 \leq & \frac{1}{\log(1+n)} \left(\mathfrak{S}(N) + \frac{(N+1)(N-1)!}{N^N} + o_n(1) \right) + \frac{1}{\log(1+n)} \\
 \leq & \frac{1}{\log(1+n)} \left(\mathfrak{S}(N) + \frac{(N+1)(N-1)!}{N^N} + 1 + o_n(1) \right)
 \end{aligned}$$

and thus

$$\begin{aligned}
 \|w_n(x)\|^N &= \int_B \log^{N-1} \left(\frac{e}{|x|} \right) |\nabla w_n|^N dx + \int_B \xi(x) |w_n(x)|^N dx \\
 &\leq 1 + \frac{m \left(\mathfrak{S}(N) + \frac{(N+1)(N-1)!}{N^N} + 1 + o_n(1) \right)}{\log(1+n)}.
 \end{aligned}$$

Then,

$$\|w_n(x)\|^{N'} \leq 1 + \frac{m \left(\mathfrak{S}(N) + \frac{(N+1)(N-1)!}{N^N} + 1 + o_n(1) \right)}{(N-1) \log(1+n)}.$$

(ii) We make a development of order one, we obtain ,

$$\frac{1}{\|w_n(x)\|^{N'}} \geq 1 - \frac{m \left(\mathfrak{S}(N) + \frac{(N+1)(N-1)!}{N^N} + 1 + o_n(1) \right)}{(N-1) \log(1+n)}.$$

Using (ξ_1) and by proceeding in the same way, we get the inequality of the left.

□

Now, we present the second elementary Lemma.

Lemma 7

$$\begin{aligned}
 & \lim_{n \rightarrow +\infty} \int_{e^{-n} \leq |x| \leq 1} \exp \{ N e^{\omega_{N-1}^{-1} v_n^{N'}} \} dx \\
 = & \lim_{n \rightarrow +\infty} \omega_{N-1} \int_0^n \exp \left\{ N e^{\frac{\log^{N'}(1+t)}{\log^{N-1}(1+n) \|w_n\|^{N'}}} - Nt \right\} dt \geq \omega_{N-1} e^N.
 \end{aligned}$$

Proof. We make the changes of variable $s = 1 + t$, $j = n + 1$, and using lemma 6 (ii), we get,

$$\int_{e^{-n} \leq |x| \leq 1} \exp\{N e^{\omega \frac{1}{N-1} v_n^{N'}}\} dx = \int_0^n \exp\{N e^{\frac{\log^{N'}(1+t)}{\log^{N-1}(1+n)} \|w_n\|^{N'}} - Nt\} dt$$

and

$$\int_{e^{-n} \leq |x| \leq 1} \exp\{N e^{\omega \frac{1}{N-1} v_n^{N'}}\} dx \geq \int_0^n \exp\left(N e^{\frac{\log^{N'}(1+t)(1-E(m,N,n))}{\log^{N-1}(1+k)}} - Nt\right) dt.$$

Let $b := b(m; N; j) = 1 - E(m; N; n)$, then

$$\begin{aligned} \int_{e^{-n} \leq |x| \leq 1} \exp\{N e^{\omega \frac{1}{N-1} v_n^{N'}}\} dx &\geq \int_0^n \exp\left(N e^{\frac{\log^{N'}(1+t)(1-E(m,N,n))}{\log^{N-1}(1+k)}} - Nt\right) dt \\ &= \int_1^j \exp\left(N s^{\left(\frac{b^{N-1} \log s}{\log j}\right)^{\frac{1}{N-1}}} - N(s-1)\right) ds \\ &= e^N \int_1^j \exp\left(N s^{\left(\frac{b^{N-1} \log s}{\log j}\right)^{\frac{1}{N-1}}} - Ns\right) ds. \end{aligned}$$

We claim that

$$\lim_{j \rightarrow +\infty} \int_1^j \exp\left(N s^{\left(\frac{b^{N-1} \log s}{\log j}\right)^{\frac{1}{N-1}}} - Ns\right) ds = 1. \quad (37)$$

Indeed, using the fact that $b \leq 1$, and for any $j > 4$, we have

$$\psi_j(s) := N s^{\left(\frac{b^{N-1} \log s}{\log j}\right)^{\frac{1}{N-1}}} - Ns \leq N s^{\left(\frac{\log s}{\log j}\right)^{\frac{1}{N-1}}} - Ns, \quad \text{with } s \geq 1.$$

The interval $[1; j]$ is then divided as follows

$$[1, j] = [1, j^{\frac{1}{2(N-1)}}] \cup [j^{\frac{1}{2(N-1)}}, j - j^{\frac{1}{2(N-1)}}] \cup [j - j^{\frac{1}{2(N-1)}}, j].$$

First we consider the interval $[1, j^{\frac{1}{2(N-1)}}]$, and since

$$\chi_{[1, j^{\frac{1}{2(N-1)}}]}(s) e^{\psi_j(s)} \leq e^{N s^{\frac{1}{2(N-1)}} - Ns} \leq e^{N s^{\frac{1}{2}} - Ns} \in L^1([1, +\infty))$$

$$\chi_{[1, j^{\frac{1}{2(N-1)}}]}(s) e^{\psi_j(s)} \rightarrow e^{N - Ns} \quad \text{for a.e. } s \in [1, +\infty), \text{ as } j \rightarrow +\infty,$$

using the Lebesgue dominated convergence theorem, we get

$$\begin{aligned} & \lim_{j \rightarrow +\infty} \int_1^{j^{\frac{1}{2^{(N-1)}}}} \exp\left(Ns^{\left(\frac{b^{N-1} \log s}{\log j}\right)^{\frac{1}{N-1}}} - Ns\right) ds \\ &= \lim_{j \rightarrow +\infty} \int_1^j \chi_{[1, j^{\frac{1}{2^{(N-1)}}}]}(s) e^{\psi_j(s)} ds = \frac{1}{N}. \end{aligned}$$

Now we are going to study the limit of this integral on $[j^{\frac{1}{2^{(N-1)}}}, j - j^{\frac{1}{2^{(N-1)}}}]$ and $[j - j^{\frac{1}{2^{(N-1)}}}, j]$, so we compute

$$\psi_j(j^{\frac{1}{2^{(N-1)}}}) = -Nj^{\frac{1}{2^{(N-1)}}} \left(1 - j^{\frac{b-2}{2^N}}\right)$$

and

$$\psi_j(j^{\frac{1}{2^{(N-1)}}}) \leq -j^{\frac{1}{2^{(N-1)}}} \quad \text{for all } j \geq \left(\frac{N}{N-1}\right)^{2^N}, \quad (38)$$

we have also

$$\begin{aligned} & \psi_j(j - j^{\frac{1}{2^{(N-1)}}}) \\ &= N \exp\left(\frac{b}{\log^{\frac{1}{N-1}} j} \left[\log j + \log(1 - j^{\frac{1}{2^{(N-1)}} - 1})\right]^{N'}\right) - N(j - j^{\frac{1}{2^{(N-1)}}}) \\ &= N \exp\left(b \log j \left\{1 + \frac{\log(1 - j^{\frac{1}{2^{(N-1)}} - 1})}{\log j}\right\}^{N'}\right) - N(j - j^{\frac{1}{2^{(N-1)}}}) \\ &\leq N \left[\exp\left(\log j \left\{1 - N' \frac{j^{\frac{1}{2^{(N-1)}} - 1}}{\log j} + o\left(\frac{1}{\log j}\right) + o\left(\frac{1}{j}\right)\right\} - 1\right)\right] + Nj^{\frac{1}{2^{(N-1)}}} \\ &= Nj \left[\exp\left(-N' j^{\frac{1}{2^{(N-1)}} - 1} + o\left(\frac{1}{j}\right)\right) - 1\right] + Nj^{\frac{1}{2^{(N-1)}}}. \end{aligned}$$

Therefore, for every $\varepsilon \in (0, 1)$ there exists $j_\varepsilon \geq 1$ such that

$$\psi_j(j - j^{\frac{1}{2^{(N-1)}}}) \leq Nj^{\frac{1}{2^{(N-1)}}} (1 - (1 - \varepsilon)N') \quad \text{for every } j \geq j_\varepsilon. \quad (39)$$

Let j fixed and large enough. A qualitative study conducted on ψ_j in $[1, +\infty)$, shows that there exists a unique $s_j \in (1, j)$ such that $\psi'_j(s_j) = 0$ and consequently

$$\int_{j^{\frac{1}{2^{(N-1)}}}}^{j - j^{\frac{1}{2^{(N-1)}}}} e^{\psi_j(s)} ds \leq (j - 2j^{\frac{1}{2^{(N-1)}}}) e^{\max(\psi_j(j^{\frac{1}{2^{(N-1)}}}), \psi_j(j - j^{\frac{1}{2^{(N-1)}}}))}.$$

In addition, from (38) and (39) with $\varepsilon = \frac{1}{N^2}$, we obtain

$$\max[\psi_j(j^{\frac{1}{2^{(N-1)}}}), \psi_j(j - j^{\frac{1}{2^{(N-1)}}})] \leq -j^{\frac{1}{2^{(N-1)}}},$$

as condition that j is large enough. Hence, there exists $\bar{j} \geq 1$ such that

$$\int_{j \frac{1}{2^{(N-1)}}}^{j - j \frac{1}{2^{(N-1)}}} e^{\psi_j(s)} ds \leq (j - 2j \frac{1}{2^{(N-1)}}) e^{-j \frac{1}{2^{(N-1)}}} \quad \text{for all } j \geq \bar{j}.$$

Therefore

$$\lim_{j \rightarrow +\infty} \int_{j \frac{1}{2^{(N-1)}}}^{j - j \frac{1}{2^{(N-1)}}} \exp\left(N e^{s \left(\frac{b^{N-1} \log s}{\log j}\right)^{\frac{1}{N-1}}} - Ns\right) ds = 0. \quad (40)$$

Finally, we will study the limit on the interval $[j - j \frac{1}{2^{(N-1)}}, j]$. We mention that for a fixed $j \geq 1$ large enough, ψ_j is a convex function on $[j - j \frac{1}{2^{(N-1)}}, +\infty)$, and $\psi_j(j) = Nj^b - Nj \leq 0$, so we can get this estimate

$$\begin{aligned} \psi_j(s) &\leq \frac{j-s}{j \frac{1}{2^{(N-1)}}} \psi_j(j - j \frac{1}{2^{(N-1)}}) + \psi_j(j) \\ &\leq \frac{j-s}{j \frac{1}{2^{(N-1)}}} \psi_j(j - j \frac{1}{2^{(N-1)}}), \quad s \in [j - j \frac{1}{2^{(N-1)}}, j]. \end{aligned}$$

On the another hand, in view of (37) and (38), if $\varepsilon \in (0, \frac{1}{N^2})$ and $j \geq j_\varepsilon$ we have

$$\psi_j(s) \leq N(1 - (1 - \varepsilon)N')(j - s), \quad s \in [j - j \frac{1}{2^{(N-1)}}, j], \quad (41)$$

furthermore, using the fact that ψ_j is convex on $[j - j \frac{1}{2^{(N-1)}}, +\infty)$ and $\psi'_j(j) = NN'j^{b-1} - N$, we get

$$\begin{aligned} \psi_j(s) &\geq \psi_j(j) + \psi'_j(j)(s - j) \\ &= Nj^b - Nj + (NN'j^{b-1} - N)(s - j), \quad s \in [j - j \frac{1}{2^{(N-1)}}, j]. \end{aligned} \quad (42)$$

So,

$$\int_{j - j \frac{1}{2^{(N-1)}}}^j e^{\psi_j(s)} ds \geq \frac{e^{Nj^{b-Nj}}}{NN'j^{b-1} - N} (1 - e^{-j \frac{1}{2^{(N-1)}}}). \quad (43)$$

Then by bringing together (41), (42) and (43), we deduce

$$\begin{aligned} \lim_{j \rightarrow +\infty} \frac{e^{Nj^{b-Nj}}}{NN'j^{b-1} - N} (1 - e^{-j \frac{1}{2^{(N-1)}}}) &\leq \lim_{j \rightarrow +\infty} \int_{j - j \frac{1}{2^{(N-1)}}}^j e^{\varphi_j(s)} ds \\ &\leq \frac{-1}{N(1 - (1 - \varepsilon)N')} \end{aligned}$$

and since $\lim_{j \rightarrow +\infty} b = 1$, then

$$\frac{1}{N'} \leq \lim_{j \rightarrow +\infty} \int_{j^{-j} 2^{\frac{1}{N-1}}}^j e^{\varphi_j(s)} ds \leq \frac{-1}{N(1 - (1 - \varepsilon)N')}.$$

By tending ε to zero, we get

$$\lim_{j \rightarrow +\infty} \int_{j^{-j} 2^{\frac{1}{N-1}}}^{j^{-j} 2^{\frac{1}{N-1}}} \exp \left(N e^{s^{\left(\frac{\log s}{\log j}\right)^{\frac{1}{N-1}}} - Ns} \right) ds = \frac{1}{N'}.$$

So our claim (37) is proved, and the Lemma follows. \square

4.2 The minmax value of the energy \mathcal{E}

In this sub-section we will give an estimation of the min-max of the energy.

Lemma 8 Assume (ξ_1) , (A_2) , (A_3) , (A_4) and (A_5) holds. There exists some $n \in \mathbb{N}$ such that

$$\max\{\mathcal{E}(tv_n) : t \geq 0\} < \frac{\omega_{N-1}}{N\alpha_0^{N-1}} \text{ for some } n \geq 1. \quad (44)$$

Proof. Let us assume by contradiction that for all $n \in \mathbb{N}$,

$$\max\{\mathcal{E}(tv_n) : t \geq 0\} \geq \frac{\omega_{N-1}}{N\alpha_0^{N-1}}. \quad (45)$$

So, for every $n \in \mathbb{N}$, there are $t_n > 0$ such as

$$\mathcal{E}(t_n v_n) = \max_{t \geq 0} \mathcal{E}(tv_n).$$

Then

$$\frac{\omega_{N-1}}{N\alpha_0^{N-1}} \leq \max_{t \geq 0} \mathcal{E}(t_n v_n) = \frac{1}{N} t_n^N - \int_B F(x, t_n v_n) dx$$

and

$$0 = \frac{d}{dt} \mathcal{E}(tv_n) \Big|_{t=t_n} = t_n^{N-1} - \int_B f(x, t_n v_n) v_n dx. \quad (46)$$

By (A_5) , for any $\varepsilon > 0$, there exists $t_\varepsilon > 0$ such that

$$f(x, t) t \geq (\beta_0 - \varepsilon) \exp(N e^{\alpha_0 t^{N'}}) \quad \forall |t| \geq t_\varepsilon, \text{ uniformly in } x. \quad (47)$$

By Lemma 6, if $|x| \leq e^{-k}$ we have

$$\begin{aligned}
 v_n^{N'} &\geq \frac{1}{\omega_{N-1}^{\frac{1}{N-1}}} \frac{\log(1+n)}{1 + \frac{m(\mathfrak{S}(N) + \frac{(N+1)(N-1)!}{N^N} + 1) + o_n(1)}{\log(1+n)}} + o_n(1) \\
 &= \frac{1}{\omega_{N-1}^{\frac{1}{N-1}}} \log(1+n) - \frac{m(\mathfrak{S}(N) + \frac{(N+1)(N-1)!}{N^N} + 1)}{(N-1)\omega_{N-1}^{\frac{1}{N-1}}} + o_n(1) \\
 &= \frac{1}{\omega_{N-1}^{\frac{1}{N-1}}} \log(1+n) - \frac{\mathfrak{C}(N, M)}{(N-1)\omega_{N-1}^{\frac{1}{N-1}}} + o_n(1),
 \end{aligned} \tag{48}$$

where

$$\mathfrak{C}(M, N) = m(\mathfrak{S}(N) + \frac{(N+1)(N-1)!}{N^N} + 1).$$

Using (46) and (47), we get

$$\begin{aligned}
 t_n^N &\geq (\beta_0 - \varepsilon) \int_{0 \leq |x| \leq e^{-n}} \exp\{N e^{\alpha_0 t_n^{N'} v_n^{N'}}\} dx \\
 &\geq (\beta_0 - \varepsilon) \int_{0 \leq |x| \leq e^{-n}} \exp\{N e^{\frac{\alpha_0 t_n^{N'} (\frac{1}{\omega_{N-1}^{\frac{1}{N-1}}} \log(1+n) - \frac{\mathfrak{C}(M, N)}{(N-1)\omega_{N-1}^{\frac{1}{N-1}}} + o_n(1))}{\omega_{N-1}^{\frac{1}{N-1}}}}\} dx \\
 &= \omega_{N-1} (\beta_0 - \varepsilon) \exp\{N e^{\frac{\alpha_0 t_n^{N'} (\frac{1}{\omega_{N-1}^{\frac{1}{N-1}}} \log(1+n) - \frac{\mathfrak{C}(M, N)}{(N-1)\omega_{N-1}^{\frac{1}{N-1}}} + o(1))}{\omega_{N-1}^{\frac{1}{N-1}}}} - Nn\}.
 \end{aligned} \tag{49}$$

As a result, (t_n) is a bounded sequence. It should be noted that if

$$\lim_{n \rightarrow +\infty} t_n^N > \frac{\omega_{N-1}}{\alpha_0^{N-1}}, \tag{50}$$

one then obtains a contradiction with the boundedness of (t_n) . Indeed, otherwise there exists some $\delta > 0$ such that for n large enough,

$$t_n^N \geq (\delta + \frac{\omega_{N-1}^{\frac{1}{N-1}}}{\alpha_0})^{N-1}.$$

Thus

$$\frac{\alpha_0}{\omega_{N-1}^{\frac{1}{N-1}}} t_n^{N'} \geq \frac{\alpha_0}{\omega_{N-1}^{\frac{1}{N-1}}} \delta + 1$$

and hence, the right hand of (49) tends to infinity which contradicts the boundedness of (t_n) . Therefore (50) cannot hold and we get

$$\lim_{k \rightarrow +\infty} t_n^N = \frac{\omega_{N-1}}{\alpha_0^{N-1}}. \tag{51}$$

We claim that (51) leads to a contradiction with (A₅). For this purpose, the following sets should be used

$$\mathcal{A}_n = \{x \in B \mid t_n v_n \geq t_\varepsilon\} \quad \text{and} \quad \mathcal{C}_n = B \setminus \mathcal{A}_n,$$

where t_ε is given in (45). We have

$$\begin{aligned} t_n^N &= \int_B f(x, t_n v_n) t_n v_n dx = \int_{\mathcal{A}_n} f(x, t_n v_n) t_n v_n dx + \int_{\mathcal{C}_n} f(x, t_n v_n) t_n v_n \\ &\geq (\beta_0 - \varepsilon) \int_{\mathcal{A}_n} \exp\{N e^{\alpha_0 t_n^{N'} v_n^{N'}}\} dx + \int_{\mathcal{C}_n} f(x, t_n v_n) t_n v_n dx \\ &= (\beta_0 - \varepsilon) \int_B \exp\{N e^{\alpha_0 t_n^{N'} v_n^{N'}}\} dx - (\beta_0 - \varepsilon) \int_{\mathcal{C}_n} \exp\{N e^{\alpha_0 t_n^{N'} v_n^{N'}}\} dx \\ &\quad + \int_{\mathcal{C}_n} f(x, t_n v_n) t_n v_n dx. \end{aligned}$$

Since $v_k \rightarrow 0$ a.e in B , $\chi_{\mathcal{C}_n} \rightarrow 1$ a.e in B , therefore using the dominated convergence theorem, we get

$$\lim_{n \rightarrow +\infty} t_n^N = \frac{\omega_{N-1}}{\alpha_0^{N-1}} \geq (\beta_0 - \varepsilon) \lim_{n \rightarrow +\infty} \int_B \exp\{N e^{\alpha_0 t_n^{N'} v_n^{N'}}\} dx - (\beta_0 - \varepsilon) \frac{\omega_{N-1}}{N} e^N.$$

By using the fact that

$$t_n^N \geq \frac{\omega_{N-1}}{\alpha_0^{N-1}},$$

we get

$$\begin{aligned} \int_B \exp\{N e^{\alpha_0 t_n^{N'} v_n^{N'}}\} dx &\geq \int_{0 \leq |x| \leq e^{-n}} \exp\{N e^{N \omega_{N-1}^{\frac{1}{N-1}} v_n^{N'}}\} dx \\ &\quad + \int_{e^{-k} \leq |x| \leq 1} \exp\{N e^{\omega_{N-1}^{\frac{1}{N-1}} v_n^{N'}}\} dx. \end{aligned}$$

On one hand, we have by (48)

$$\begin{aligned} &\int_{0 \leq |x| \leq e^{-n}} \exp\{N e^{\omega_{N-1}^{\frac{1}{N-1}} v_n^{N'}}\} dx \\ &\geq \int_{0 \leq |x| \leq e^{-n}} \exp\left\{N e^{\log(1+n) - \frac{\mathfrak{C}(M, N)}{N-1} + o_n(1)}\right\} dx \\ &= \frac{\omega_{N-1}}{N} \exp\left\{N + Nn - \frac{\mathfrak{C}(M, N)}{N-1} + o_n(1)\right\} e^{-Nn} \\ &= \frac{\omega_{N-1}}{N} \exp\left\{N - \frac{\mathfrak{C}(M, N)}{N-1} + o_n(1)\right\} \rightarrow \frac{\omega_{N-1}}{N} e^{N - \frac{\mathfrak{C}(M, N)}{N-1}}. \end{aligned}$$

On other side, we have by (48), the definition of v_n and the result of Lemma 4.1,

$$\int_{e^{-n} \leq |x| \leq 1} \exp\{N e^{\omega_{N-1}^{\frac{1}{N-1}} v_n^{N'}}\} dx = \int_{e^{-n} \leq |x| \leq 1} \exp\{N e^{\frac{\log^{N'}(\log(\frac{e}{|x|}))}{\log^{N-1}(1+n)}}\} dx \geq \omega_{N-1} e^N.$$

Hence,

$$\lim_{n \rightarrow +\infty} t_n^N = \frac{\omega_{N-1}}{\alpha_0^{N-1}} \geq (\beta_0 - \varepsilon) \omega_{N-1} \frac{e^N}{N} (N - 1 + e^{-\frac{\mathfrak{C}(M,N)}{N-1}}).$$

Since $\varepsilon > 0$ is arbitrary, we have

$$\frac{N}{\alpha_0^{N-1} e^N (N - 1 + e^{-\frac{\mathfrak{C}(M,N)}{N-1}})} \geq \beta_0.$$

This contradicts (A_5) and establishes the proof.

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