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On the exponential Diophantine equation $m^x + (m+1)^y = (1+m+m^2)^z$

Murat Alan

Abstract

Let $m > 1$ be a positive integer. We show that the exponential Diophantine equation $m^x + (m+1)^y = (1+m+m^2)^z$ has only the positive integer solution $(x, y, z) = (2, 1, 1)$ when $m \geq 2$.

1 Introduction

Let a, b and c be relatively prime positive integers greater than one. Let us consider the simple looking exponential Diophantine equation

$$a^x + b^y = c^z \tag{1}$$

in positive integers x, y, z. Although some results such as finiteness of solutions of it goes back to 1933 [18], there are some conjectures still remain unproved related to uniqueness of the solutions (x, y, z) of this equation. One of the famous conjecture is due to Jeśmanowicz on Pythagorean triples, i.e., positive integers satisfying $a^2 + b^2 = c^2$. In 1955, Jeśmanowicz has conjectured that if a, b, c are any Pythagorean triples then the equation (1) has only the positive integer solution $(x, y, z) = (2, 2, 2)$ [11]. There exist many positive results on Jeśmanowicz' Conjecture with some conditions, see for example [15, 17, 20, 21, 28]. A similar conjecture is due to Terai which states that if a, b, c, p, q, r are fixed positive integers satisfying $a^p + b^q = c^r$ with $r \geq 2$ and $\gcd(a, b) = 1$,

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then the equation (1) has only the positive integer solution $(x, y, z) = (p, q, r)$ [25, 26]. Terai's conjecture has a few exceptional cases [30]. Although there exist many positive results in special cases, for example [1, 3, 6, 8, 7, 12, 13, 19, 22, 24, 27, 29, 31], this conjecture is also an unsolved problem in general yet. For more detailed information on these two conjectures we refer [16, 23]. In this paper we consider positive integer solutions of the exponential Diophantine equation

$$m^x + (m+1)^y = (1+m+m^2)^z \tag{2}$$

where $m > 1$ is a positive integer, and we prove the following theorem

Theorem 1.1. *Let $m > 1$ be a positive integer. Then the equation (2) has only the positive integer solution $(x, y, z) = (2, 1, 1)$.*

If $m = 1$ then the equation (2) turns into the equation $1 + 2^y = 3^z$ and it is easy to see that this equation has only two positive integer solutions $(y, z) = (1, 1)$ and $(y, z) = (3, 2)$. In Theorem 1.1 we exclude the case $m = 1$ just for preserving the exponential expression in (2).

The tools to solve this kind of exponential Diophantine equations can be vary according to specific equation. Although in some cases linear forms in logarithms can be very effective, see for example [2], in this paper to prove the above theorem we mainly rely on two results which one is known as classification method due to Lee [14] and the other one is famous primitive divisor theorem [4, 33].

2 Preliminaries

Let $h(-4D)$ be the class number of positive binary quadratic forms of discriminant $-4D$. There is a bound on the class number $h(-4D)$.

Lemma 2.1 ([10], Theorems 11.4.3, 12.10.1 and 12.14.3)**.**

$$h(-4D) < \frac{4}{\pi}\sqrt{D}\log(2e\sqrt{D})$$

Following theorem is simplified and combined version of two results from [14].

Theorem 2.2 ([14], Theorems 1 and 2)**.** *Let D, k be positive integers such that $2 \nmid k$ and $\gcd(D, k) = 1$. If $D > 1$ then every solution (X, Y, Z) of the equation*

$$X^2 + DY^2 = k^Z, \qquad X, Y, Z \in \mathbb{Z}, \qquad \gcd(X, Y) = 1, \qquad Z > 0 \tag{3}$$

can be expressed as

$$Z = Z_1 t, \quad t \in \mathbb{N},$$

$$X + Y\sqrt{-D} = \lambda_1 (X_1 + \lambda_2 Y_1 \sqrt{-D})^t, \qquad \lambda_1, \lambda_2 \in \{\pm 1\},$$

where (X_1, Y_1, Z_1) *is a positive integer solution of the above equation* (3) *such that* $Z_1 \mid h(-4D)$.

Let α, β be algebraic integers. A Lucas pair is a pair (α, β) such that $\alpha + \beta$ and $\alpha\beta$ are non-zero coprime rational integers and $\dfrac{\alpha}{\beta}$ is not a root of unity. Any two Lucas pairs (α_1, β_1) and (α_2, β_2) are called equivalent if $\dfrac{\alpha_1}{\alpha_2} = \dfrac{\beta_1}{\beta_2} = \mp 1$. For any Lucas pair (α, β) the corresponding sequences of Lucas numbers defined by

$$L_n(\alpha, \beta) = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \; n = 0, 1, 2, \ldots$$

An important notion related to Lucas sequences is existence of primitive divisors of $L_n(\alpha, \beta)$. A prime number is a primitive divisor of $L_n(\alpha, \beta)$ if $p \mid L_n(\alpha, \beta)$ and $p \nmid (\alpha - \beta)^2 L_1(\alpha, \beta) \ldots L_{n-1}(\alpha, \beta)$ $(n > 1)$. So for any equivalent Lucas pairs we have that $L_n(\alpha_1, \beta_1) = \mp L_n(\alpha_2, \beta_2)$. Thus equivalent Lucas pairs have common corresponding primitive divisors. If $L_n(\alpha, \beta)$ has no primitive divisors then the Lucas pair (α, β) is called $n-$defective Lucas pair.

Theorem 2.3 ([4, 33])**.** *Every* $n-th$ *term of any Lucas sequences* $L_n(\alpha, \beta)$ *has a primitive divisors if* $n > 30$. *If* $4 < n \leq 30$ *and* $n \neq 6$ *then, up to equivalence, all* $n-$*defective Lucas pairs* (α, β) *are of the form* $\left(\dfrac{a - \sqrt{b}}{2}, \dfrac{a + \sqrt{b}}{2}\right)$ *as follows*

$n = 5$, $(a, b) = (1, 5)$, $(1, -7)$, $(2, -40)$, $(1, -11)$, $(1, -15)$, $(12, -76)$, $(12, -1364)$*;* $n = 7$, $(a, b) = (1, -7), (1, -19)$*;* $n = 8$, $(a, b) = (1, -7), (2, -24)$*;* $n = 10$, $(a, b) = (2, -8)$, $(5, -3)$, $(5, -47)$*;* $n = 12$, $(a, b) = (1, -5)$, $(1, -7)$, $(1, -11)$, $(2, -56)$, $(1, -15)$, $(1, -19)$*;* $n = 13, 18, 30$, $(a, b) = (1, -7)$.

Following theorem is an early version of the primitive divisor theorem for integers which is known as Zsigmondy's theorem.

Theorem 2.4 ([36, 5], Zsigmondy's Theorem)**.** *Let* $a > b \geq 1$ *be relatively prime integers and let* $\{u_n\}_{n \geq 1}$ *be the sequence defined as*

$$u_n = a^n - b^b.$$

If $n > 1$ *then* u_n *has a primitive divisor, that is a prime* p *such that* $p \mid u_n$ *and* $p \nmid u_m$ *for* $1 \leq m \leq n-1$ *except for* $(a, b, n) = (2, 1, 6)$ *or* $n = 2$ *and* $a + b = 2^k$ *for some positive integer* k.

3 Proof of Theorem 1.1

Through this section (x, y, z) will be a positive integer solution of the equation (2).

Lemma 3.1. *If $m > 1$, then x is even and y is odd integer. Further if $x = 2X$ then $3 \mid X - y$.*

Proof. First we take (2) modulo $m+1$ so that we get $(-1)^x \equiv 1 \pmod{m+1}$. Since $m > 1$, we see that x is even. Let $x = 2X$. Again from (2) we get that $m^{2X} + (-m^2)^y \equiv 0 \pmod{(1+m+m^2)}$. So $m^{|2X-2y|} + (-1)^y \equiv 0 \pmod{(1+m+m^2)}$. Let $|2X - 2y| = 3q + r$ for some positive integers q and r with $0 \leq r \leq 2$. Since $m^3 \equiv 1 \pmod{(1+m+m^2)}$ we have that $m^r + (-1)^y \equiv 0 \pmod{(1+m+m^2)}$. If y is even then we have that $m^r + 1 \equiv 0 \pmod{(1+m+m^2)}$ which is a contradiction for $0 \leq r \leq 2$. So y must be an odd integer. Thus $m^r - 1 \equiv 0 \pmod{(1+m+m^2)}$. Since $1+m+m^2 \nmid m-1$ and $1+m+m^2 \nmid m^2-1$ we get that $r = 0$ and hence $3 \mid 2X - 2y$ and therefore $3 \mid X - y$. □

Lemma 3.2. *Let $m > 1$. If $z < m+1$ then the equation (2) has only the positive integer solution $(x, y, z) = (2, 1, 1)$.*

Proof. From Lemma 3.1 we know that $x \geq 2$. So from (2) we write $1 + my \equiv 1 + mz \pmod{m^2}$ and hence $y \equiv z \pmod{m}$. Thus $|z - y| = mk$ for some non negative integer k. If $y > z$ then $y \geq z + mk \geq z + m \geq 2z$ which is a contradiction since the inequality

$$(m+1)^y < (1+m+m^2)^z < (1+m)^{2z}$$

clearly implies that $y < 2z$. On the other hand the case $z > y$ also leads to a contradiction with the hypothesis $z < m+1$ since $z > y$ implies that $z = y + mk \geq m+1$. So we conclude that $k = 0$ and therefore $z = y$. Let $z = y = n$. Then from (2) we write

$$m^x = (1+m+m^2)^n - (m+1)^n. \tag{4}$$

Now we use Zsigmondy's theorem. Since $m > 1$, from (4), we see that the sequence $u_n = (1+m+m^2)^n - (m+1)^n$, $n \geq 1$ has no primitive divisors. Further if $y = z = n > 1$ from Zsigmondy's Theorem we have two possibilities: either $1+m+m^2 = 2$ and $1+m = 1$ or $m^2+2m+2 = 2^k$ for some positive integer k, but both of them clearly false. So we have that $n = y = z = 1$, and therefore $x = 2$. So $(x, y, z) = (2, 1, 1)$ is the only solution of (2) when $z < m+1$. □

Lemma 3.3. *Let* $3 \mid z$ *and* $3 \mid m+1$. *If* $y > 1$ *then the equation* (2) *has no positive integer solution.*

Proof. Let $y > 1$. Since $1 + m + m^2 \equiv -m \pmod{(m+1)^2}$, by taking (2) modulo $(m+1)^2$ we write

$$m^x \equiv (-m)^z \pmod{(m+1)^2}$$

$$m^{|x-z|} \equiv (-1)^z \pmod{(m+1)^2}$$

$$(-1)^{|x-z|} + (-1)^{|x-z-1|}(m+1)|x-z| \equiv (-1)^z \pmod{(m+1)^2}.$$

Taking into account x is even, by 3.1, we get that

$$|x-z| \equiv 0 \pmod{(m+1)}.$$

In particular $x \equiv z \pmod 3 \Rightarrow x \equiv 0 \pmod 3$. Thus by Lemma 3.1, all x, y and z are divisible by 3. Hence equation (2) is of the form $A^3 + B^3 = C^3$, which is famous Fermat's Last Theorem and it is proved that it has no positive integer solutions [35]. □

Proof of Theorem 1.1. From Lemma 3.1 we know that x is even and y is odd. Let $x = 2X$. So we rewrite (2) as

$$(m^X)^2 + (m+1)((m+1)^{\frac{y-1}{2}})^2 = (1+m+m^2)^z. \tag{5}$$

Thus $(U, V, Z) = (m^X, (m+1)^{\frac{y-1}{2}}, z)$ is a positive integer solution of the equation

$$U^2 + (m+1)V^2 = (1+m+m^2)^Z. \tag{6}$$

Thus from Theorem 3, there exist a positive integer solution (u_1, v_1, z_1) of (6) such that

$$z = z_1 t, \quad t \in \mathbb{N}, \tag{7}$$

$$m^X + (m+1)^{\frac{y-1}{2}}\sqrt{-(m+1)} = \lambda_1(u_1 + \lambda_2 v_1\sqrt{-(m+1)})^t, \tag{8}$$

where $\gcd(u_1, v_1) = 1$, $\lambda_1, \lambda_2 \in \{\pm 1\}$ and $z_1 \mid h(-4D)$. By taking the complex conjugate of (8) and subtracting it from (8) we get that

$$(m+1)^{(\frac{y-1}{2})} = v_1 \left|\frac{\alpha^t - \beta^t}{\alpha - \beta}\right| = v_1|L_t| \tag{9}$$

where

$$\alpha = u_1 + v_1\sqrt{-(m+1)}, \qquad \beta = u_1 - v_1\sqrt{-(m+1)}.$$

Note that $\gcd(\alpha+\beta, \alpha\beta) = \gcd\left(2u_1, (1+m+m^2)^{z_1}\right) = 1$ since $1 + m + m^2$ is odd and $\gcd(u_1, v_1) = 1$. Let $r := \dfrac{\alpha}{\beta}$. Then r satisfies the equation

$(1+m+m^2)^{z_1}r^2 - 2(u_1^2 - (m+1)v_1^2)r + (1+m+m^2)^{z_1} = 0$ and neither $2(u_1^2 - (m+1)v_1^2) = 0$ nor $\frac{2(u_1^2-(m+1)v_1^2)}{(1+m+m^2)^{z_1}} = \pm 1$. So $r := \frac{\alpha}{\beta}$ is not a root of unity. Thus L_t is a Lucas sequence. From (9) we see that L_t has no primitive divisors. For $t \geq 5$ and $t \neq 6$, one can easily check that

$$(\alpha, \beta) = \left(\frac{2u_1 + \sqrt{-4v_1^2(m+1)}}{2}, \frac{2u_1 - \sqrt{-4v_1^2(m+1)}}{2}\right)$$

does not match with any pair given in Theorem 2.3 by taking into account u_1 and v_1 are relatively prime and $u_1^2 + (m+1)v_1^2 = (1+m+m^2)^{z_1}$. Thus it remains to check the equation (8) for the cases $t = 2, 3, 4, 6$ and $t = 1$. If $2 \mid t$, then from (8) we write that

$$m^X + (m+1)^{\frac{y-1}{2}}\sqrt{-(m+1)} = \lambda_1(U_2 + V_2\sqrt{-(m+1)})^2 \tag{10}$$

where

$$(u_1 + \lambda_2 v_1\sqrt{-(m+1)})^{\frac{t}{2}} = (U_2 + V_2\sqrt{-(m+1)}). \tag{11}$$

From (10) we have that

$$m^X = \lambda_1(U_2^2 - (m+1)V_2^2) \qquad \text{and} \qquad (m+1)^{\frac{y-1}{2}} = 2\lambda_1 U_2 V_2 \tag{12}$$

Since $\gcd(m^X, (m+1)^{\frac{y-1}{2}}) = 1$, from (12) we see that $|U_2| = 1$, and hence $|V_2| = \frac{1}{2}(m+1)^{\frac{y-1}{2}}$. So it follows that

$$m^X = \frac{1}{4}(m+1)^y - 1, \tag{13}$$

that is $4m^X + 4 = (1+m)^y$. Taking this equation modulo m we get that $4 \equiv 1 \pmod{m}$, that is $m = 3$. For $m = 3$, the equation (2) turns into the equation $3^x + 4^y = 13^z$ which is proved that it has only the positive integer solution $(x, y, z) = (2, 1, 1)$ [9], but it leads a contradiction with choose of t as even and $z = z_1 t$. So we have $2 \nmid t$. Thus either $t = 3$ or $t = 1$. If $t = 3$ then from (8) we get that

$$m^X = \lambda_1 u_1(u_1^2 - 3v_1^2(m+1)) \qquad \text{and} \tag{14}$$

$$(m+1)^{\frac{y-1}{2}} = \lambda_1\lambda_2 v_1(3u_1^2 - v_1^2(m+1)). \tag{15}$$

Assume that $y > 1$. By Lemma 3.3 we need to check only the case $3 \nmid m+1$. So from (14) we see that $\gcd(3u_1, m+1) = 1$. Thus from (15) we have that

$$3u_1^2 - v_1^2(m+1) = \pm 1 \tag{16}$$

and hence $v_1 = (m+1)^{\frac{y-1}{2}}$. Then, since $u_1^2 \equiv 1 \pmod{m+1}$ by (6), by taking (16) modulo $m+1$ we find that $3 \equiv \pm 1 \pmod{m+1}$ which implies $m = 3$. For $m = 3$, we have already mentioned that this equation $3^x + 4^y = 13^z$ has only positive integer solution $(x, y, z) = (2, 1, 1)$ which also gives a contradiction since $3 \mid z$. So $y = 1$. But $y = 1$ also implies that $3u_1^2 - v_1^2(m+1) = \pm 1$ which leads to same contradiction as before. Thus $3 \nmid t$ and so we arrive at $t = 1$. Hence from (7) and Lemma 2.1 we find that

$$z < \frac{4}{\pi}\sqrt{(m+1)}\log\left(2e\sqrt{(m+1)}\right).$$

If $z < m+1$ then the result follows from Lemma 3.2. So we take $m+1 \leq z$ and therefore from the inequality

$$m+1 \leq z < \frac{4}{\pi}\sqrt{(m+1)}\log\left(2e\sqrt{(m+1)}\right)$$

we find that $m < 17$. Thus all variables are bounded as $x, y < z$. So with a short computer program in Maple we check all variables in the range $2 \leq m \leq 17$ and we find that (2) has no solutions other than $(x, y, z) = (2, 1, 1)$. This completes the proof. $\square$

Discussion: After seeing x is even from Lemma 3.1 the equation (2) can be viewed as

$$a^X + b^y = (a+b)^z.$$

It seems that, despite its benign appearance, it is not proved yet that this equation has only the positive integer solution $(X, y, z) = (1, 1, 1)$ with some exceptional case. In fact this is the context of Conjecture 1.3 in [34].

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Murat ALAN,
Yildiz Technical University
Department of Mathematics,
Davutpasa Campus, 34210, Esenler, Istanbul, Turkey.
e-mail: alan@yildiz.edu.tr