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Generalized Helicoidal Surfaces in Euclidean 5-space

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Abstract

In this paper, we study generalized helicoidal surfaces in Euclidean 5-space. We obtain the necessary and sufficient conditions for generalized helicoidal surfaces in Euclidean 5-space to be minimal, flat or of zero normal curvature tensor, which are ordinary differential equations. We solve those equations and discuss the completeness of the surfaces.

1 Introduction

In differential geometry, one of the well-known surfaces is helicoidal surfaces. Helicoidal surfaces are a generalization of rotational surfaces. These surfaces are invariant by a subgroup of the group of isometries of the ambient space, called helicoidal group whose elements can be seen as a composition of a translation with a rotation for a given axis. In [4], the authors studied the space of all helicoidal surfaces in Euclidean 3-space which have constant mean curvatures or constant Gaussian curvatures. This space behaves as a circular cylinder, where a given generator corresponds to the rotational surfaces and each parallel corresponds to a periodic family of helicoidal surfaces. In [2], the cases with prescribed mean curvature or Gauss curvature have been studied.

Helicoidal surfaces were studied by many researchers in different spaces. In [6], authors constructed linear Weingarten helicoidal surfaces in Minkowski 3-space under the cubic screw motion. In [5], the authors constructed a helicoidal

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surface with a light-like axis with prescribed mean curvature or Gauss curvature given by smooth function in Minkowski 3-space and solved an open problem left in [3]. Also, in [7], the authors classify all helicoidal non-degenerate surfaces in Minkowski 3-space with constant mean curvature whose generating curve is the graph of a polynomial or a Lorentzian circle.

Besides, in [1], the authors studied rotational surfaces in higher dimensional Euclidean spaces. They obtained some results related with the curvature properties of these surfaces. Also they give examples of rotational surfaces in Euclidean 5-space.

In this paper, we study generalized helicoidal surfaces in Euclidean 5-space. We obtain the necessary and sufficient conditions for generalized helicoidal surfaces in Euclidean 5-space to be minimal, flat or of zero normal curvature tensor, which are ordinary differential equations. We solve those equations and discuss the completeness of the surfaces.

2 Preliminaries

Let $\mathbb{E}^5$ be the 5–dimensional Euclidean space with standard coordinate system $\{x_1, x_2, x_3, x_4, x_5\}$ and the metric tensor g has the form

$$g = \sum_{i=1}^{5} (dx_i)^2 = (dx_1)^2 + (dx_2)^2 + (dx_3)^2 + (dx_4)^2 + (dx_5)^2 .$$

Let M be a surface immersed in the 5-dimensional Euclidean space $\mathbb{E}^5$. We denote the Levi-Civita connections of $\mathbb{E}^5$ and M by $\widetilde{\nabla}$ and ∇, respectively. Let e_1, e_2, e_3, e_4, e_5 be an adapted local orthonormal frame in $\mathbb{E}^5$ such that e_1, e_2 are tangent to M and e_3, e_4, e_5 are normal to M. We know that

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X,Y)$$

and

$$\widetilde{\nabla}_X \xi = -A_\xi X + {}^\perp\nabla_X\, \xi$$

where $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma\left(TM^\perp\right)$. Then h is the second fundamental form, A_ξ is the shape operator, and ${}^\perp\nabla$ is the normal connection. We note that

$$\langle h(X,Y), \xi \rangle = \langle A_\xi X, Y \rangle .$$

The normal curvature tensor ${}^\perp R$ is defined by

$${}^\perp R(X,Y)\,\xi = {}^\perp\nabla_X\, {}^\perp\nabla_Y\, \xi - {}^\perp\nabla_Y\, {}^\perp\nabla_X\, \xi - {}^\perp\nabla_{[X,Y]}\xi ,$$

where $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(TM^{\perp})$. Taking the normal part of the following equation

$$\widetilde{\nabla}_X \widetilde{\nabla}_Y \xi - \widetilde{\nabla}_Y \widetilde{\nabla}_X \xi - \widetilde{\nabla}_{[X,Y]} \xi = 0$$

where $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(TM^{\perp})$, we get the Ricci equation

$$\left\langle {}^{\perp}R(X, Y)\xi, \eta \right\rangle = \langle A_\eta X, A_\xi Y \rangle - \langle A_\xi X, A_\eta Y \rangle$$

where $\eta \in \Gamma(TM^{\perp})$.

We may use the ranges of indices:

$$1 \leq i, j, k, ... \leq 2, \quad 3 \leq r, s, t, ... \leq 5, \quad 1 \leq A, B, C, ... \leq 5.$$

Set

$$h_{ij}^s = \langle h(e_i, e_j), e_s \rangle$$

and

$$R_{rij}^s = \left\langle {}^{\perp}R(e_i, e_j) e_r, e_s \right\rangle,$$

which are the components of the second fundamental form h and the normal curvature tensor ${}^{\perp}R$, respectively.

By the Ricci equation, the normal curvature tensor satisfies

$$R_{rij}^s = \left(\langle A_{e_s} e_i, A_{e_r} e_j \rangle - \langle A_{e_r} e_i, A_{e_s} e_j \rangle \right).$$

Noting that

$$A_{e_s} e_i = \sum_k h_{ik}^s e_k,$$

we obtain

$$R_{rij}^s = \sum_k \left(h_{ik}^s h_{jk}^r - h_{jk}^s h_{ik}^r \right).$$

Also the mean curvature vector H of M in $\mathbb{E}^5$ is defined by

$$H = \frac{1}{2} \sum_s \left(h_{11}^s + h_{22}^s \right) e_s.$$

A surface M is called minimal if $H = 0$ identically.

The Gauss curvature K of M in $\mathbb{E}^5$ is given by

$$K = \sum_s \left(h_{11}^s h_{22}^s - \left(h_{12}^s \right)^2 \right).$$

A surface M is called flat if $K = 0$ identically.

3 Generalized helicoidal surfaces in $\mathbb{E}^5$

In this section, we discuss the geometric properties of a generalized helicoidal surface M in $\mathbb{E}^5$ with the parametrization

$$M: \quad F(t,u) = (\alpha(t)\cos u, \alpha(t)\sin u, \beta(t)\cos u, \beta(t)\sin u, u) \tag{3.1}$$

where $(\alpha(t))^2 + (\beta(t))^2 > 0$ and $(\alpha'(t))^2 + (\beta'(t))^2 > 0$. Then we have

$$\begin{aligned} F_t &= (\alpha'(t)\cos u, \alpha'(t)\sin u, \beta'(t)\cos u, \beta'(t)\sin u, 0), \\ F_u &= (-\alpha(t)\sin u, \alpha(t)\cos u, -\beta(t)\sin u, \beta(t)\cos u, 1) \end{aligned}$$

and

$$\langle F_t, F_t\rangle = (\alpha'(t))^2 + (\beta'(t))^2, \quad \langle F_t, F_u\rangle = 0, \quad \langle F_u, F_u\rangle = 1 + \alpha^2(t) + \beta^2(t).$$

Then we can choose the followings:

$$\begin{aligned} e_1 &= \frac{1}{\sqrt{(\alpha')^2 + (\beta')^2}} F_t = \frac{1}{\sqrt{(\alpha')^2 + (\beta')^2}} (\alpha'\cos u, \alpha'\sin u, \beta'\cos u, \beta'\sin u, 0), \\ e_2 &= \frac{1}{\sqrt{1+\alpha^2+\beta^2}} F_u = \frac{1}{\sqrt{1+\alpha^2+\beta^2}} (-\alpha\sin u, \alpha\cos u, -\beta\sin u, \beta\cos u, 1), \\ e_3 &= \frac{1}{\sqrt{\alpha^2+\beta^2}} (\beta\sin u, -\beta\cos u, -\alpha\sin u, \alpha\cos u, 0), \\ e_4 &= \frac{1}{\sqrt{(\alpha')^2 + (\beta')^2}} (-\beta'\cos u, -\beta'\sin u, \alpha'\cos u, \alpha'\sin u, 0), \\ e_5 &= \frac{1}{\sqrt{\alpha^2+\beta^2}\sqrt{1+\alpha^2+\beta^2}} \left(-\alpha\sin u, \alpha\cos u, -\beta\sin u, \beta\cos u, -\alpha^2-\beta^2\right). \end{aligned}$$

Here $\{e_1, e_2\}$ is an orthonormal frame field on M and $\{e_3, e_4, e_5\}$ is a normal orthonormal frame field to M.

Also we can easily obtain that

$$\begin{aligned} \widetilde{\nabla}_{e_1} e_1 &= \frac{(\beta'\alpha'' - \alpha'\beta'')}{\left((\alpha')^2 + (\beta')^2\right)^2} (\beta'\cos u, \beta'\sin u, -\alpha'\cos u, -\alpha'\sin u, 0), \\ \widetilde{\nabla}_{e_2} e_1 &= \frac{1}{\sqrt{(\alpha')^2 + (\beta')^2}\sqrt{1+\alpha^2+\beta^2}} (-\alpha'\sin u, \alpha'\cos u, -\beta'\sin u, \beta'\cos u, 0), \\ \widetilde{\nabla}_{e_2} e_2 &= \frac{1}{1+\alpha^2+\beta^2} (-\alpha\cos u, -\alpha\sin u, -\beta\cos u, -\beta\sin u, 0). \end{aligned}$$

The components of the second fundamental form h are given as follows

$$h_{11}^4 = \frac{-\beta'\alpha'' + \alpha'\beta''}{\left((\alpha')^2 + (\beta')^2\right)^{3/2}}, \quad h_{12}^3 = \frac{-\beta\alpha' + \alpha\beta'}{\sqrt{(\alpha')^2 + (\beta')^2}\sqrt{1+\alpha^2+\beta^2}\sqrt{\alpha^2+\beta^2}},$$

$$h_{12}^5 = \frac{\alpha\alpha' + \beta\beta'}{(1+\alpha^2+\beta^2)\sqrt{(\alpha')^2 + (\beta')^2}\sqrt{\alpha^2+\beta^2}}, \quad h_{22}^4 = \frac{-\beta\alpha' + \alpha\beta'}{(1+\alpha^2+\beta^2)\sqrt{(\alpha')^2 + (\beta')^2}},$$

$$h_{11}^3 = h_{11}^5 = h_{12}^4 = h_{22}^3 = h_{22}^5 = 0.$$

Then we get the following theorem and corollary.

Theorem 1. *Let M be generalized helicoidal surface parametrized by* (3.1)*. Then the mean curvature vector H of M is given by*

$$H = \frac{1}{2}\left(\frac{-\beta'\alpha'' + \alpha'\beta''}{\left((\alpha')^2 + (\beta')^2\right)^{3/2}} + \frac{-\beta\alpha' + \alpha\beta'}{(1+\alpha^2+\beta^2)\sqrt{(\alpha')^2 + (\beta')^2}}\right)e_4.$$

Corollary 1. *Let M be generalized helicoidal surface parametrized by* (3.1)*. Then M is minimal if and only if*

$$\frac{\alpha'\beta'' - \beta'\alpha''}{(\alpha')^2 + (\beta')^2} = \frac{\beta\alpha' - \alpha\beta'}{(1+\alpha^2+\beta^2)}. \tag{3.2}$$

Let $\beta(t) = t$ in the equation (3.2). Then the minimal surface equation is

$$\left(\alpha^2 + t^2 + 1\right)\alpha'' + (t\alpha' - \alpha)\left((\alpha')^2 + 1\right) = 0. \tag{3.3}$$

If $\alpha(t)$ is a linear function, that is, $\alpha(t) = pt+q$, then from the above equation, we have $q = 0$ and $\alpha(t) = pt$. Then the surface M is a helicoid in a 3-dimensional subspace of $\mathbb{E}^5$. So, in the following, we will consider the case where $\alpha(t)$ is a nonlinear function.

Multiplying (3.3) by $2\alpha'/\left((\alpha')^2 + 1\right)^2$, we can get

$$\left(t^2\frac{(\alpha')^2}{(\alpha')^2+1}\right)' - \left(\frac{\alpha^2+1}{(\alpha')^2+1}\right)' = 0.$$

Thus we have

$$t^2\frac{(\alpha')^2}{(\alpha')^2+1} - \frac{\alpha^2+1}{(\alpha')^2+1} = c_1$$

for a constant c_1. Then

$$\alpha' = \pm\sqrt{\frac{\alpha^2+1+c_1}{t^2-c_1}}$$

and

$$\frac{\alpha'}{\sqrt{\alpha^2+1+c_1}} = \pm\frac{1}{\sqrt{t^2-c_1}}. \tag{3.4}$$

Changing t to $-t$ if necessary, we may only consider the $(+)$ case.

(i) When $c_1 = 0$, we have

$$\frac{\alpha'}{\sqrt{\alpha^2+1}} = \frac{1}{t}.$$

Integrating it we have

$$\log\left|\sqrt{\alpha^2+1}+\alpha\right| = \log|t| + c_2$$

for a constant c_2 and

$$\sqrt{\alpha^2+1}+\alpha = c_3 t$$

where $c_3 \neq 0$ is constant. Thus we get

$$\alpha = \frac{1}{2}\left(c_3 t - \frac{1}{c_3 t}\right),$$

and its graph is a hyperbola. The corresponding surface M is a complete minimal surface.

(ii) When $c_1 \neq 0$, integrating the equation (3.4), we have

$$\log\left|\sqrt{\alpha^2+1+c_1}+\alpha\right| = \log\left|\sqrt{t^2-c_1}+t\right| + c_2$$

for a constant c_2, and

$$\sqrt{\alpha^2+1+c_1}+\alpha = c_3\left(\sqrt{t^2-c_1}+t\right)$$

where $c_3 \neq 0$ is constant. Thus we get

$$\alpha = \frac{1}{2c_1c_3}\left[\left(c_1c_3^2-1-c_1\right)t + \left(c_1c_3^2+1+c_1\right)\sqrt{t^2-c_1}\right].$$

Since $\alpha(t)$ is not a linear function, we have $c_1c_3^2+1+c_1 \neq 0$. When $c_1 < 0$, the function $\alpha(t)$ is defined for any $t \in \mathbb{R}$ and its graph is a hyperbola. So we have a complete minimal surface.

When $c_1 > 0$, we return to the equation (3.4) and rewrite it as

$$\frac{t'}{\sqrt{t^2 - c_1}} = \frac{1}{\sqrt{\alpha^2 + 1 + c_1}}, \quad t' = \frac{dt}{d\alpha}.$$

Integrating it with respect to α, we have

$$\log|\sqrt{t^2 - c_1} + t| = \log\left|\sqrt{\alpha^2 + 1 + c_1} + \alpha\right| + c_2$$

for constant c_2, and

$$\sqrt{t^2 - c_1} + t = c_3\left(\sqrt{\alpha^2 + 1 + c_1} + \alpha\right)$$

where $c_3 \neq 0$ is constant. Hence

$$t = \frac{1}{2c_3(1 + c_1)}\left[\left(c_3^2 + c_1c_3^2 - c_1\right)\alpha + \left(c_3^2 + c_1c_3^2 + c_1\right)\sqrt{\alpha^2 + 1 + c_1}\right].$$

Since $t(\alpha)$ is not a linear function, we have $c_3^2 + c_1c_3^2 + c_1 \neq 0$. The function $t(\alpha)$ is defined for any $\alpha \in \mathbb{R}$ and its graph is hyperbola. So we get a complete minimal surface also in this case.

Theorem 2. *The nonlinear solution of the minimal surface equation* (3.3) *is given by*

$$\alpha = \frac{1}{2}\left(c_3t - \frac{1}{c_3t}\right),$$

$$\alpha = \frac{1}{2c_1c_3}\left[\left(c_1c_3^2 - 1 - c_1\right)t + \left(c_1c_3^2 + 1 + c_1\right)\sqrt{t^2 - c_1}\right], \quad (c_1 < 0)$$

or

$$t = \frac{1}{2c_3(1 + c_1)}\left[\left(c_3^2 + c_1c_3^2 - c_1\right)\alpha + \left(c_3^2 + c_1c_3^2 + c_1\right)\sqrt{\alpha^2 + 1 + c_1}\right], \quad (c_1 > 0),$$

where c_1 and $c_3 \neq 0$ are constants. The corresponding surface M is a complete minimal surface in any case.

In the following theorem, we give the Gauss curvature of the surface (3.1).

Theorem 3. *Let M be generalized helicoidal surface parametrized by* (3.1). *Then the Gauss curvature K of M is given by*

$$K = \frac{(\alpha'\beta'' - \beta'\alpha'')(\alpha\beta' - \beta\alpha')}{\left((\alpha')^2 + (\beta')^2\right)^2(1 + \alpha^2 + \beta^2)} - \frac{(\alpha\beta' - \beta\alpha')^2\left(1 + \alpha^2 + \beta^2\right) + (\alpha\alpha' + \beta\beta')^2}{\left((\alpha')^2 + (\beta')^2\right)(1 + \alpha^2 + \beta^2)^2(\alpha^2 + \beta^2)}. \tag{3.5}$$

Corollary 2. *Let M be generalized helicoidal surface parametrized by* (3.1). *Then M is flat if and only if*

$$\frac{\left(\alpha'\beta''-\beta'\alpha''\right)\left(\alpha\beta'-\beta\alpha'\right)}{\left(\left(\alpha'\right)^2+\left(\beta'\right)^2\right)}=\frac{\left(\alpha\beta'-\beta\alpha'\right)^2\left(1+\alpha^2+\beta^2\right)+\left(\alpha\alpha'+\beta\beta'\right)^2}{\left(1+\alpha^2+\beta^2\right)\left(\alpha^2+\beta^2\right)}.$$

To study flat surfaces, it is convenient to let

$$\alpha\left(t\right)=P\left(t\right)\cos\left(Q\left(t\right)\right)\quad\text{and}\quad\beta\left(t\right)=P\left(t\right)\sin\left(Q\left(t\right)\right)\tag{3.6}$$

where $P\left(t\right)>0$ and $Q\left(t\right)$ are nonconstant smooth functions. From the equation (3.5), we have

$$K=\frac{-\left(P'\right)^4+P^4\left(P'\right)^2\left(Q'\right)^2-P^3\left(1+P^2\right)\left(Q'\right)^2P''+P^3\left(1+P^2\right)P'Q'Q''}{\left(1+P^2\right)^2\left(\left(P'\right)^2+P^2\left(Q'\right)^2\right)^2}.$$

Then the surface M is flat if and only if

$$-\left(P'\right)^4+P^4\left(P'\right)^2\left(Q'\right)^2-P^3\left(1+P^2\right)\left(Q'\right)^2P''+P^3\left(1+P^2\right)P'Q'Q''=0.\tag{3.7}$$

Here for $Q\left(t\right)=t$, the equation (3.7) is rewritten as

$$-\left(P'\right)^4+P^4\left(P'\right)^2-P^3\left(1+P^2\right)P''=0.\tag{3.8}$$

Multiplying the above equation by $2PP'=\left(P^2\right)'$, we have

$$-P^4\left(1+P^2\right)\left(\left(P'\right)^2\right)'+P^4\left(P^2\right)'\left(P'\right)^2-\left(P^2\right)'\left(P'\right)^4=0.$$

Dividing by $P^4\left(P'\right)^4$, we get

$$\left(\frac{1+P^2}{\left(P'\right)^2}\right)'+\left(\frac{1}{P^2}\right)'=0.$$

So

$$\frac{1+P^2}{\left(P'\right)^2}+\frac{1}{P^2}=\frac{1}{c_1^2}$$

for a positive constant c_1. Then

$$P'=\pm c_1P\sqrt{\frac{1+P^2}{P^2-c_1^2}}.$$

Thus we find

$$\frac{dt}{dP}=\pm\frac{1}{c_1P}\sqrt{\frac{P^2-c_1^2}{1+P^2}}. \tag{3.9}$$

Then

$$t\left(P\right)=\pm\int\frac{1}{c_1P}\sqrt{\frac{P^2-c_1^2}{1+P^2}}dP.$$

Let

$$I:=\int\frac{1}{c_1P}\sqrt{\frac{P^2-c_1^2}{1+P^2}}dP.$$

Set

$$\sqrt{\frac{P^2-c_1^2}{1+P^2}}=:s.$$

Then

$$P^2=\frac{s^2+c_1^2}{1-s^2}$$

and

$$PdP=\frac{1+c_1^2}{\left(1-s^2\right)^2}sds.$$

So we have

$$\begin{aligned}
I &= \frac{1+c_1^2}{c_1}\int\frac{s^2}{\left(1-s^2\right)\left(s^2+c_1^2\right)}ds=\frac{1}{2c_1}\int\left(\frac{1}{1+s}+\frac{1}{1-s}-\frac{2c_1^2}{s^2+c_1^2}\right)ds \\
&= \frac{1}{2c_1}\log\left(\frac{1+s}{1-s}\right)-\arctan\left(\frac{s}{c_1}\right)+c_2 \\
&= \frac{1}{2c_1}\log\left(\frac{\sqrt{1+P^2}+\sqrt{P^2-c_1^2}}{\sqrt{1+P^2}-\sqrt{P^2-c_1^2}}\right)-\arctan\left(\frac{1}{c_1}\sqrt{\frac{P^2-c_1^2}{1+P^2}}\right)+c_2 \\
&= \frac{1}{c_1}\log\left(\sqrt{1+P^2}+\sqrt{P^2-c_1^2}\right)-\arctan\left(\frac{1}{c_1}\sqrt{\frac{P^2-c_1^2}{1+P^2}}\right) \\
&\quad -\frac{1}{2c_1}\log\left(1+c_1^2\right)+c_2,
\end{aligned}$$

where c_2 is a constant.

Let

$$\Phi\left(P\right):=\frac{1}{c_1}\log\left(\sqrt{1+P^2}+\sqrt{P^2-c_1^2}\right)-\arctan\left(\frac{1}{c_1}\sqrt{\frac{P^2-c_1^2}{1+P^2}}\right).$$

We denote by $t_+(P)$ and $t_-(P)$ the solutions of (3.9) in the $(+)$ and $(-)$ cases, respectively. Then

$$t_+(P) = \Phi(P) + c_3 \quad \text{and} \quad t_-(P) = -\Phi(P) + c_4$$

for some constants c_3 and c_4. The function $t_+(P)$ is an increasing function on (c_1, ∞) and

$$\lim_{P\to\infty} t_+(P) = \infty, \quad \lim_{P\to c_1^+} t_+(P) = \frac{1}{2c_1}\log\left(1 + c_1^2\right) + c_3, \quad \lim_{P\to c_1^+} t'_+(P) = 0.$$

Similarly, $t_-(P)$ is a decreasing function on (c_1, ∞) and

$$\lim_{P\to\infty} t_-(P) = -\infty, \quad \lim_{P\to c_1^+} t_-(P) = -\frac{1}{2c_1}\log\left(1 + c_1^2\right) + c_4, \quad \lim_{P\to c_1^+} t'_-(P) = 0.$$

We choose c_4 such that

$$\frac{1}{2c_1}\log\left(1 + c_1^2\right) + c_3 = -\frac{1}{2c_1}\log\left(1 + c_1^2\right) + c_4.$$

The curves $(P\cos(t_+(P)), P\sin(t_+(P)))$ and $(P\cos(t_-(P)), P\sin(t_-(P)))$ can be connected continuously, but it is not a regular curve. So the corresponding surface M cannot be extended as a complete flat surface.

Theorem 4. *The solution of the flat surface equation* (3.8) *is given by*

$$t_+(P) = \Phi(P) + c_3 \quad \textit{or} \quad t_-(P) = -\Phi(P) + c_4,$$

where

$$\Phi(P) = \frac{1}{c_1}\log\left(\sqrt{1 + P^2} + \sqrt{P^2 - c_1^2}\right) - \arctan\left(\frac{1}{c_1}\sqrt{\frac{P^2 - c_1^2}{1 + P^2}}\right)$$

and $c_1 > 0$, c_3, c_4 *are constants. The corresponding surface M cannot be extended as a complete flat surface.*

In the following theorem, we consider the case where the normal curvature tensor of M is identically zero.

Theorem 5. *Let M be generalized helicoidal surface parametrized by* (3.1). *Then the normal curvature tensor of M is identically zero if and only if*

$$\left((\alpha')^2 + (\beta')^2\right)(\beta\alpha' - \alpha\beta') + (\alpha'\beta'' - \beta'\alpha'')\left(1 + \alpha^2 + \beta^2\right) = 0. \tag{3.10}$$

Proof. We have

$$\begin{aligned}
R^3_{412} &= h^3_{11}h^4_{21} - h^3_{21}h^4_{11} + h^3_{12}h^4_{22} - h^3_{22}h^4_{12} \\
&= \frac{(\beta\alpha' - \alpha\beta')\left[\left((\alpha')^2 + (\beta')^2\right)(\beta\alpha' - \alpha\beta') + (\alpha'\beta'' - \beta'\alpha'')\left(1 + \alpha^2 + \beta^2\right)\right]}{\sqrt{\alpha^2 + \beta^2}\,(1 + \alpha^2 + \beta^2)^{3/2}\left((\alpha')^2 + (\beta')^2\right)^2}, \\
R^3_{512} &= h^3_{11}h^5_{21} - h^3_{21}h^5_{11} + h^3_{12}h^5_{22} - h^3_{22}h^5_{12} = 0, \\
R^4_{512} &= h^4_{11}h^5_{21} - h^4_{21}h^5_{11} + h^4_{12}h^5_{22} - h^4_{22}h^5_{12} \\
&= \frac{(\alpha\alpha' + \beta\beta')\left[\left((\alpha')^2 + (\beta')^2\right)(\beta\alpha' - \alpha\beta') + (\alpha'\beta'' - \beta'\alpha'')\left(1 + \alpha^2 + \beta^2\right)\right]}{\sqrt{\alpha^2 + \beta^2}\,(1 + \alpha^2 + \beta^2)^2\left((\alpha')^2 + (\beta')^2\right)^2}.
\end{aligned}$$

Thus ${}^{\perp}R = 0$ if and only if

$$\left((\alpha')^2 + (\beta')^2\right)(\beta\alpha' - \alpha\beta') + (\alpha'\beta'' - \beta'\alpha'')\left(1 + \alpha^2 + \beta^2\right) = 0.$$

□

To study surfaces with zero normal curvature tensor, it is convenient to let

$$\alpha(t) = P(t)\cos(Q(t)) \quad \text{and} \quad \beta(t) = P(t)\sin(Q(t))$$

where $P(t) > 0$ and $Q(t)$ are nonconstant smooth functions. From the equation (3.10), we have

$$PQ'\left(P''\left(1 + P^2\right) - P\left(Q'\right)^2\right) - \left(2 + P^2\right)\left(P'\right)^2 Q' - \left(1 + P^2\right)PP'Q'' = 0. \tag{3.11}$$

Here for $Q(t) = t$, the equation (3.11) is rewritten as

$$PP''\left(1 + P^2\right) - \left(2 + P^2\right)\left(P'\right)^2 - P^2 = 0. \tag{3.12}$$

Multiplying by $2PP' = \left(P^2\right)'$, we have

$$P^2\left(1 + P^2\right)\left(\left(P'\right)^2\right)' - \left(2 + P^2\right)\left(P^2\right)'\left(P'\right)^2 - P^2\left(P^2\right)' = 0.$$

Dividing by P^6, we get

$$\left(\frac{1 + P^2}{P^4}\left(P'\right)^2\right)' + \left(\frac{1}{P^2}\right)' = 0.$$

So
$$\frac{1+P^2}{P^4}\left(P'\right)^2+\frac{1}{P^2}=\frac{1}{c_1^2}$$
for a positive constant c_1. Then
$$P'=\pm\frac{P}{c_1}\sqrt{\frac{P^2-c_1^2}{1+P^2}}.$$
Thus we find
$$\frac{dt}{dP}=\pm\frac{c_1}{P}\sqrt{\frac{1+P^2}{P^2-c_1^2}}. \tag{3.13}$$
Then
$$t\left(P\right)=\pm\int\frac{c_1}{P}\sqrt{\frac{1+P^2}{P^2-c_1^2}}dP.$$
Let
$$I:=\int\frac{c_1}{P}\sqrt{\frac{1+P^2}{P^2-c_1^2}}dP.$$
Set
$$\sqrt{\frac{1+P^2}{P^2-c_1^2}}=:s.$$
Then
$$P^2=\frac{1+c_1^2s^2}{s^2-1}$$
and
$$PdP=-\frac{1+c_1^2}{\left(s^2-1\right)^2}sds.$$
So we have
$$\begin{aligned}
I &= -c_1\left(1+c_1^2\right)\int\frac{s^2}{\left(s^2-1\right)\left(1+c_1^2s^2\right)}ds\\
&= \frac{c_1}{2}\int\left(\frac{1}{s+1}-\frac{1}{s-1}-\frac{2}{1+c_1^2s^2}\right)ds\\
&= \frac{c_1}{2}\log\left(\frac{s+1}{s-1}\right)-\arctan\left(c_1s\right)+c_2\\
&= \frac{c_1}{2}\log\left(\frac{\sqrt{1+P^2}+\sqrt{P^2-c_1^2}}{\sqrt{1+P^2}-\sqrt{P^2-c_1^2}}\right)-\arctan\left(c_1\sqrt{\frac{1+P^2}{P^2-c_1^2}}\right)+c_2
\end{aligned}$$
$$=c_1\log\left(\sqrt{1+P^2}+\sqrt{P^2-c_1^2}\right)-\arctan\left(c_1\sqrt{\frac{1+P^2}{P^2-c_1^2}}\right)-\frac{c_1}{2}\log\left(1+c_1^2\right)+c_2,$$

where c_2 is a constant.

Let

$$\Psi(P) := c_1 \log\left(\sqrt{1+P^2} + \sqrt{P^2 - c_1^2}\right) - \arctan\left(c_1 \sqrt{\frac{1+P^2}{P^2 - c_1^2}}\right).$$

We denote by $t_+(P)$ and $t_-(P)$ the solutions of (3.13) in the $(+)$ and $(-)$ cases, respectively. Then

$$t_+(P) = \Psi(P) + c_3 \quad \text{and} \quad t_-(P) = -\Psi(P) + c_4$$

for some constants c_3 and c_4.

The function $t_+(P)$ is an increasing function on (c_1, ∞) and

$$\lim_{P\to\infty} t_+(P) = \infty, \quad \lim_{P\to c_1^+} t_+(P) = \frac{c_1}{2}\log\left(1+c_1^2\right) - \frac{\pi}{2} + c_3, \quad \lim_{P\to c_1^+} t'_+(P) = \infty.$$

Similarly, $t_-(P)$ is a decreasing function on (c_1, ∞) and

$$\lim_{P\to\infty} t_-(P) = -\infty, \ \lim_{P\to c_1^+} t_-(P) = -\frac{c_1}{2}\log\left(1+c_1^2\right) + \frac{\pi}{2} + c_4, \ \lim_{P\to c_1^+} t'_-(P) = -\infty.$$

We choose c_4 such that

$$\frac{c_1}{2}\log\left(1+c_1^2\right) - \frac{\pi}{2} + c_3 = -\frac{c_1}{2}\log\left(1+c_1^2\right) + \frac{\pi}{2} + c_4 =: t_0.$$

Let $P_+(t)$ denote the inverse function of $t_+(P)$. It is an increasing function on (t_0, ∞) and

$$\lim_{t\to\infty} P_+(t) = \infty, \quad \lim_{t\to t_0^+} P_+(t) = c_1 \quad \lim_{t\to t_0^+} P'_+(t) = 0.$$

Let $P_-(t)$ denote the inverse function of $t_-(P)$. It is a decreasing function on $(-\infty, t_0)$ and

$$\lim_{t\to-\infty} P_-(t) = \infty, \quad \lim_{t\to t_0^-} P_-(t) = c_1 \quad \lim_{t\to t_0^-} P'_-(t) = 0.$$

Now we define a function $P(t)$ on $\mathbb{R}$ such that $P(t) = P_+(t)$ for $t > t_0$, $P(t_0) = c_1$ and $P(t) = P_-(t)$ for $t < t_0$. Then $P(t)$ is a C^1 function on $\mathbb{R}$ such that $P'(t) = P'_+(t)$ for $t > t_0$, $P'(t_0) = 0$ and $P'(t) = P'_-(t)$ for $t < t_0$. For $t \neq t_0$, it satisfies

$$P''(t) = \frac{\left(2 + P(t)^2\right)\left(P'(t)\right)^2 + P(t)^2}{P(t)\left(1 + P(t)^2\right)}.$$

Then we have
$$\lim_{t\to t_0} P''(t) = \frac{c_1}{1+c_1^2}.$$
Thus we find that $P(t)$ is a C^2 function on $\mathbb{R}$, which satisfies the zero normal curvature tensor equation. So the corresponding surface M is a complete surface with zero normal curvature tensor.

Theorem 6. *The solution of the zero normal curvature tensor equation* (3.12) *is given by*
$$t_+(P) = \Psi(P) + c_3 \quad or \quad t_-(P) = -\Psi(P) + c_4$$
where
$$\Psi(P) = c_1 \log\left(\sqrt{1+P^2} + \sqrt{P^2 - c_1^2}\right) - \arctan\left(c_1\sqrt{\frac{1+P^2}{P^2-c_1^2}}\right)$$
and $c_1 > 0$, c_3, c_4 *are constants. The corresponding surface* M *can be extended as a complete surface with zero normal curvature tensor.*

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