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Bounds for the minimum distance function

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Abstract

Let I be a homogeneous ideal in a polynomial ring S. In this paper, we extend the study of the asymptotic behavior of the minimum distance function δ_I of I and give bounds for its stabilization point, r_I, when I is an F-pure or a square-free monomial ideal. These bounds are related with the dimension and the Castelnuovo–Mumford regularity of I.

Keywords: Minimum distance, Castelnuovo–Mumford regularity, monomial ideal.

1 Introduction

In this manuscript we study the minimum distance function δ_I of a homogeneous ideal I contained in a polynomial ring $S = \mathbb{K}[x_1, \ldots, x_n]$ over a field $\mathbb{K}$. This minimum distance function for ideals was introduced by the second-named and third-named authors together with Martínez-Bernal [MBPV17] to obtain an algebraic formulation of the minimum distance of projective Reed–Muller-type codes over finite fields.

If I is an unmixed radical graded ideal and its associate primes are generated by linear forms, then δ_I is non-increasing [MBPV17]. In our first result, we extend this property to any radical ideal.

Theorem A (Theorem 3.4)**.** Suppose that $I \subseteq S$ is a radical ideal. Then, $\delta_I(d)$ is a non-increasing function.

Key Words: Minimum distance, Castelnuovo–Mumford regularity, monomial ideal.
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The previous result allow us to define the regularity index of δ_I, r_I, as the value where δ_I stabilizes. If $\dim(S/I) = 1$, previous work shows that $r_I \leq \operatorname{reg}(S/I)$ [GSRTR02, RMSV11], where $\operatorname{reg}(S/I)$ is the Castelnuovo–Mumford regularity of S/I. This motivated the authors to conjecture that this relation holds in greater generality.

Conjecture B ([NnBPV18])**.** Let $I \subseteq S$ be a radical homogeneous ideal whose associated primes are generated by linear forms. Then, $r_I \leq \operatorname{reg}(S/I)$.

This conjecture was previously showed for edge ideals associated to Cohen–Macaulay bipartite graphs [NnBPV18] and when $\dim(S/I) = 1$ [GSRTR02, RMSV11]. However, the conjecture does not hold in general. Jaramillo and the third-named author provided an example of a monomial edge ideal I such that $r_I > \operatorname{reg}(S/I)$ [JV21]. In this work, we find bounds for r_I for square-free monomial ideals.

Theorem C (Theorem 4.5 & 5.7)**.** Let $I \subseteq S$ be a square-free monomial ideal. Then, $r_I \leq \dim(S/I)$. Moreover, if I is shellable or Gorenstein, then $r_I \leq \operatorname{reg}(S/I)$.

We also prove a similar result for ideals such that S/I is a F-pure ring. These class of rings play an important role in the study of singularities in prime characteristic [HR76].

Theorem D (Theorem 5.5 & 5.6)**.** Suppose that $\mathbb{K}$ is a field of prime characteristic. Let $I \subseteq S$ be an ideal such that S/I if F-pure. Then, $r_I \leq \dim(S/I)$. Moreover, if I is Gorenstein, then $r_I \leq \operatorname{reg}(S/I)$.

2 Preliminaries

In this section we recall some well known notion and results that are needed throughout this manuscript.

Let $S = \mathbb{K}[x_1, \dots, x_n] = \bigoplus_{t=0}^{\infty} S_t$ be a polynomial ring over a field $\mathbb{K}$ with the standard grading and let $I \neq (0)$ be a homogeneous ideal of S. Let d denote the Krull dimension of $R = S/I$.

The *Hilbert function* of S/I, denoted H_I, is given by

$$H_I(t) = \dim_{\mathbb{K}}(R_{\leq t}) = \dim_{\mathbb{K}}(S_{\leq t}/I_{\leq t}) = \dim_{\mathbb{K}}(S_{\leq t}) - \dim_{\mathbb{K}}(I_{\leq t}),$$

where $I_{\leq t} = I \cap S_{\leq t}$. By a classical theorem of Hilbert there is a unique polynomial $h_I(t) \in \mathbb{Q}[t]$ of degree d such that $H_I(t) = h_I(t)$ for $t \gg 0$.

The *Hilbert-Samuel multiplicity* or *degree* of R, denoted $\mathrm{e}(R)$, is the positive integer defined by $\mathrm{e}(R) = d! \lim_{t\to\infty} H_I(t)/t^d$.

Given an integer $t \geq 1$, let $\mathcal{F}_t$ be the set of all zero-divisors of S/I not in I of degree $t \geq 1$. That is

$$\mathcal{F}_t = \{\, f \in S_t \,|\, f \notin I,\ (I\colon f) \neq I\}.$$

We note that $(I\colon f) \neq I$ is equivalent to $f \in \mathfrak{p}$ for some $\mathfrak{p} \in \mathrm{Ass}_S(S/I)$, $Ass_S(S/I)$ is the set of associated primes of S/I.

Definition 2.1. The *minimum distance function* of I is the function $\delta_I\colon \mathbb{N}_+ \to \mathbb{Z}$ given by

$$\delta_I(t) = \begin{cases} \mathrm{e}(S/I) - \max\{\mathrm{e}(S/(I,f))|\, f \in \mathcal{F}_t\} & \text{if } \mathcal{F}_t \neq \emptyset, \\ \mathrm{e}(S/I) & \text{if } \mathcal{F}_t = \emptyset. \end{cases}$$

Definition 2.2. Let $I \subseteq S$ be a graded ideal and let $\mathbb{F}_\star$ be the minimal graded free resolution of S/I as an S-module:

$$\mathbb{F}_\star:\quad 0 \to \bigoplus_j S(-j)^{\beta_{gj}} \to \cdots \to \bigoplus_j S(-j)^{\beta_{1j}} \to S \to S/I \to 0.$$

The *Castelnuovo–Mumford regularity* of S/I, *regularity* of S/I for short, is defined as

$$\mathrm{reg}(S/I) = \max\{j - i|\, \beta_{ij} \neq 0\}.$$

The following result shows the asymptotic behavior of δ_I for a particular case of graded ideals.

An ideal $I \subseteq S$ is called *unmixed* if all its associated primes have the same height, in other case I is *mixed*.

Theorem 2.3 ([MBPV17, Theorem 3.8])**.** *Let $I \subseteq S$ be an unmixed radical homogeneous ideal. If all the associated primes of I are generated by linear forms, then there is an integer $r_0 \geq 1$ such that*

$$\delta_I(1) > \delta_I(2) > \cdots > \delta_I(r_0) = \delta_I(d) = 1 \quad \textit{for} \quad d \geq r_0.$$

The integer r_0 where the stabilization occurs is called the *regularity index* of δ_I and is denoted by r_I. In Section 3, we show that one can define this index for any radical ideal.

Local cohomology Let R be a commutative Noetherian ring with identity and let I be a homogeneous ideal generated by the forms $f_1, \ldots, f_\ell \in R$. Consider the Čech complex, $\check{\mathrm{C}}^\star(\bar{f}; R)$:

$$0 \to R \to \bigoplus_i R_{f_i} \to \bigoplus_{i,j} R_{f_i f_j} \to \cdots \to R_{f_1,\ldots,f_\ell} \to 0.$$

where $\check{\mathrm{C}}^i(\bar{f}; R) = \bigoplus_{1 \le j_1 \le \ldots \le j_i \le \ell} R_{f_{j_1},\ldots,f_{j_i}}$ and the homomorphism in every summand is a localization map with appropriate sign.

Definition 2.4. Let M be a graded R-modue. The *i-th local cohomology of M with support in I* is defined as

$$H^i_I(M) = H^i(\check{\mathrm{C}}^\star(\bar{f}; R) \otimes_R M).$$

Remark 2.5. Since M is a graded R-module and I is homogeneous the local cohomology module $H^i_I(M)$ is graded.

Remark 2.6. If $\phi : M \to N$ is a homogeneous R-module homomorphism of degree t, then the induced R-module map $H^i_I(M) \to H^i_I(N)$ is homogeneous of degree t.

Theorem 2.7 (Grothendieck's Vanishing Theorem)**.** *Let M be an R-module of dimension d. Then, $H^i_I(M) = 0$ for all $i > d$.*

Theorem 2.8 (Grothendieck's Non-Vanishing Theorem)**.** *Let M be a finitely generated R-module of dimension d. Then, $H^d_I(M) \neq 0$.*

Definition 2.9. Let M be an R-module with dimension d. The *a_i-invariants*, $a_i(M)$, for $i = 0, \ldots, d$ are defined as follows. If $H^i_{\mathfrak{m}}(M) \neq 0$,

$$a_i(M) = \max\{\alpha \mid H^i_{\mathfrak{m}}(M)_\alpha \neq 0\},$$

for $0 \le i \le d$, where $H^i_{\mathfrak{m}}(M)$ denotes the local cohomology module with support in the maximal ideal $\mathfrak{m}$. If $H^i_{\mathfrak{m}}(M) = 0$, we set $a_i(M) = -\infty$.

If $d = \dim(M)$, then, $a_d(M)$, is often just called the *a-invariant* of M.

The a-invariant, is a classical invariant [GW78], and is closely related to the Castelnuovo-Mumford regularity.

Definition 2.10. Let R be a positively graded ring and let M be a finitely generated R-module. The *Castelnuovo–Mumford regularity of M*, $\mathrm{reg}(M)$, is defined as

$$\mathrm{reg}(M) = \max\{a_i(M) + i \mid 1 \le i \le d\}.$$

Remark 2.11. *If M is a standard graded module of dimension d, then the a-invariant is related to the Castelnuovo–Mumford regularity, via the inequeality $a(M) + d \leq \operatorname{reg}(M)$ which is equality in the case Cohen–Macaulay.*

Definition 2.12. Suppose that R has prime characteristic p. The Frobenius map $F : R \to R$ is defined by $r \mapsto r^p$.

Remark 2.13. *If R is reduced, R^{1/p^e} the ring of the p^e-th roots of R is well defined, and $R \subseteq R^{1/p^e}$.*

3 Asymptotic behavior of the minimum distance function

In this section we prove that the minimum distance function δ_I is non-increasing. Then, the notion of regularity index of δ_I is well defined. We also find what is the stable value of the minimum distance function. We start this section establishing notation.

Notation 3.1. Given an ideal $I \subseteq S$, we set

$$\begin{aligned}\mathcal{A}(I) &= \{\mathfrak{p} \in \operatorname{Ass}_S(S/I) \mid \dim(S/I) = \dim(S/\mathfrak{p})\};\\ \mathcal{V}(I) &= \{\mathfrak{p} \in \operatorname{Spec}(S) \mid I \subseteq \mathfrak{p}\};\\ \mathcal{D}(I) &= \operatorname{Spec}(S) \setminus \mathcal{V}(I).\end{aligned}$$

Remark 3.2. *For an ideal $I \subseteq S$, we have*

$$\mathrm{e}(S/I) = \sum_{\mathfrak{p} \in \mathcal{A}(I)} \lambda_{S_\mathfrak{p}}(S_\mathfrak{p}/IS_\mathfrak{p})\, \mathrm{e}(S/\mathfrak{p}),$$

where $\lambda_{S_\mathfrak{p}}(S_\mathfrak{p}/IS_\mathfrak{p})$ denotes the length of $S_\mathfrak{p}/IS_\mathfrak{p}$ as $S_\mathfrak{p}$-module, by the additivity formula (see for instance [HS06, Theorem 11.2.4]). In particular, if I is radical, then

$$\mathrm{e}(S/I) = \sum_{\mathfrak{p} \in \mathcal{A}(I)} \mathrm{e}(S/\mathfrak{p}).$$

Lemma 3.3. *Suppose that I is a radical ideal. Let $f \in \mathcal{F}_t$ such that* $\dim(S/(I,f)) = \dim(S/I)$*. Then, $\mathcal{A}((I,f)) = \mathcal{A}(I) \cap \mathcal{V}(f)$. Furthermore,*

$$\mathrm{e}(S/(I,f)) = \sum_{\mathfrak{p} \in \mathcal{A}(I) \cap \mathcal{V}(f)} \mathrm{e}(S/\mathfrak{p}).$$

Proof. Let $J = (I,f)$. Let Q be an associated prime of J. Since $I \subseteq J$, there exists an associated prime $\mathfrak{p}$ of I such that $\mathfrak{p} \subseteq Q$. If $\dim(S/Q) = \dim(S/J) = \dim(S/I)$, then $\mathfrak{p} = Q$. Thus, $\mathcal{A}(J) \subseteq \mathcal{A}(I) \cap \mathcal{V}(f)$.

Let $Q \in \mathcal{A}(I) \cap \mathcal{V}(f)$. Then, $J \subseteq Q$ and $\dim(S/Q) = \dim(S/J)$. Then, Q is a minimal prime of S/J. Thus, $Q \in \operatorname{Ass}_S(S/J)$, and so, $Q \in \mathcal{A}(J)$.

We note that J is not necessarily radical. However, $JS_{\mathfrak{p}} = IS_{\mathfrak{p}}$ for every $\mathfrak{p} \in \mathcal{A}(I) \cap \mathcal{V}(f)$. Thus, $\lambda_{S_{\mathfrak{p}}}(S_{\mathfrak{p}}/IS_{\mathfrak{p}}) = 1$ for every $\mathfrak{p} \in \mathcal{A}(I) \cap \mathcal{V}(f)$. Then,

$$\mathrm{e}(S/(I,f)) = \sum_{\mathfrak{p} \in \mathcal{A}(I) \cap \mathcal{V}(f)} \mathrm{e}(S/\mathfrak{p})$$

by the additivity formula. □

We now show that the minimum distance function is non-increasing.

Theorem 3.4. *Suppose that I is a radical ideal. Then, $\delta_I(d)$ is a non-increasing function.*

Proof. If $\mathcal{F}_t = \emptyset$ for every $t \geq 1$, then $\delta_I(t) = \mathrm{e}(S/I)$, which is the maximum value. We note that this case is equivalent to I being a prime ideal.

We now assume that $\mathcal{F}_t \neq \emptyset$ for some $t \in \mathbb{N}$. We note that in this case $\dim(S/I) \neq 0$, otherwise, $I = \mathfrak{m}$ and so $\mathcal{F}_t = \emptyset$. Let $f \in \mathcal{F}_t$ such that $\delta_I(t) = \mathrm{e}(S/I) - \mathrm{e}(S/(I,f))$. Then,

$$\begin{aligned}
\delta_I(t) &= \mathrm{e}(S/I) - \mathrm{e}(S/(I,f)) \\
&= \sum_{\mathfrak{p} \in \mathcal{A}(I)} \mathrm{e}(S/\mathfrak{p}) - \sum_{\mathfrak{p} \in \mathcal{A}((I,f))} \mathrm{e}(S/\mathfrak{p}) \\
&= \sum_{\mathfrak{p} \in \mathcal{A}(I)} \mathrm{e}(S/\mathfrak{p}) - \sum_{\mathfrak{p} \in \mathcal{A}(I) \cap \mathcal{V}(f)} \mathrm{e}(S/\mathfrak{p}) \\
&= \sum_{\mathfrak{p} \in \mathcal{A}(I) \cap \mathcal{D}(f)} \mathrm{e}(S/\mathfrak{p}).
\end{aligned}$$

Since I is radical and $\dim(S/I) > 0$, we have that $\mathfrak{m}$ is not an associated prime. Then, $\mathfrak{m}f \not\subseteq I$ because $f \notin I$. We conclude that there exists $i = 1, \ldots, n$ such that $x_i f \notin I$. In particular, $x_i f \in \mathcal{F}_{t+1}$ and $\mathcal{F}_{t+1} \neq \emptyset$. Then,

$$\begin{aligned}
\delta_I(t+1) &\leq \sum_{\mathfrak{p} \in \mathcal{A}(I) \cap \mathcal{D}(x_i f)} \mathrm{e}(S/\mathfrak{p}) \\
&= \sum_{\mathfrak{p} \in \mathcal{A}(I) \cap \mathcal{D}(x_i) \cap \mathcal{D}(f)} \mathrm{e}(S/\mathfrak{p}) \\
&\leq \sum_{\mathfrak{p} \in \mathcal{A}(I) \cap \mathcal{D}(f)} \mathrm{e}(S/\mathfrak{p}) = \delta_I(t).
\end{aligned}$$

□

Thanks to the previous theorem we have that the minimum distance function eventually stabilizes, and it has a regularity index.

Definition 3.5. Suppose that I is a radical ideal. The *regularity index of δ_I*, denoted by r_I, is defined by

$$r_I = \min\{s \in \mathbb{N} \mid \ \delta_I(s) = \lim_{t\to\infty} \delta_I(t)\}.$$

Proposition 3.6. *Suppose that I is a radical ideal. Then,*

$$\delta_I(t) = \min\{\ \mathrm{e}(S/\mathfrak{p}) \mid \mathfrak{p} \in \mathrm{Ass}_S(S/I)\ \}$$

for $t \gg 0$ if I is unmixed and $\delta_I(t) = 0$ for $t \gg 0$ otherwise.

Proof. We first assume that I is mixed. Let J_1 be the intersection of the minimal primes of I of dimension $\dim(S/I)$ and let J_2 be the intersection of the minimal primes of I of dimension smaller than $\dim(S/I)$. Let $f \in J_1 \setminus I$. Let $\alpha = \deg(f)$. In particular, $f \in \mathcal{F}_\alpha$. We note that $\dim(S/I) = \dim(S/(I,f))$ and $\mathcal{A}(I) = \mathcal{A}(I,f)$, and so, $\mathrm{e}(S/(I,f)) = \mathrm{e}(S/I)$. We conclude that $\delta_I(t) = 0$. Since δ_I is nondecresing by Theorem 3.4, we obtain that $\delta_I(t) = 0$ for $t \geq \alpha$.

We now assume that I is unmixed. Then, $\mathcal{A}(I) = \mathrm{Ass}_S(S/I)$. If I is a prime ideal, then $\delta_I(t) = \mathrm{e}(S/I)$ for every $t \in \mathbb{N}$, and our claim follows. We assume that $\dim(S/I) > 0$ and that I is not a prime ideal. For every $f \in \mathcal{F}_t$, there exists a prime ideal $\mathfrak{p}$ such that $f \notin \mathfrak{p}$. Then,

$$\mathrm{e}(S/I) - \mathrm{e}(S/(I,f)) \geq \mathrm{e}(S/\mathfrak{p}) \geq \min\{\ \mathrm{e}(S/\mathfrak{p}) \mid \mathfrak{p} \in \mathrm{Ass}_S(S/I)\ \}.$$

We conclude that $\delta_I(t) \geq \min\{\ \mathrm{e}(S/\mathfrak{p}) \mid \mathfrak{p} \in \mathrm{Ass}_S(S/I)\ \}$. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_\ell$ denote the associated primes of I in an order such that $e(S/\mathfrak{p}_i) \leq \mathrm{e}(S/\mathfrak{p}_j)$ for $i \leq j$. Let $f \in \mathfrak{p}_2 \cap \ldots \cap \mathfrak{p}_\ell \setminus I$. Let $\alpha = \deg(f)$. We have that $f \in \mathcal{F}_\alpha$. Then, $\delta_I(\alpha) \leq \mathrm{e}(S/I) - \mathrm{e}(S/(I,f)) = \mathrm{e}(S/\mathfrak{p}_1)$. Since δ is nondecresing by Theorem 3.4, we obtain that $\delta_I(t) \leq \mathrm{e}(S/\mathfrak{p}_1)$ for $t \geq \alpha$. We conclude that $\delta_I(t) = \mathrm{e}(S/\mathfrak{p}_1)$ for $t \geq \alpha$. □

Proposition 3.7. *Let I be a mixed radical ideal. Let J_1 be the intersection of the minimal primes of I of dimension* $\dim(S/I)$ *and let J_2 be the intersection of the minimal primes of I of dimension smaller than* $\dim(S/I)$. *Then, $r_I =$* $\min\{t \mid [J_1/I]_t \neq 0\}$.

Proof. As in the proof of Proposition 3.6, we have that $\delta_I(t) = 0$ for $t \geq \min\{t \mid [J_1/I]_t \neq 0\}$. We conclude that $r_I \leq \min\{t \mid [J_1/I]_t \neq 0\}$.

Let $f \in \left(\bigcup_{\mathfrak{p}\in \mathrm{Ass}_S(S/I)} \mathfrak{p}\right) \setminus I$ of degree strictly less than $\min\{t \mid [J_1/I]_t \neq 0\}$. Then, $f \notin J_1$, and so, there exists a prime ideal $\mathfrak{p}$ such that $f \notin \mathfrak{p}$ and

$\dim(S/\mathfrak{p}) = \dim(S/I)$. Then, $e(S/I) - \mathrm{e}(S/(I,f)) \geq \mathrm{e}(S/\mathfrak{p})$. We conclude that $\delta_I(t) > 0$. Then, $r_I \geq \min\{t \mid [J_1/I]_t \neq 0\}$. □

Proposition 3.8. *Suppose that $I \subset S$ is an unmixed radical ideal with associated primes $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ and $\mathfrak{q}_1, \dots, \mathfrak{q}_s$ such that $\mathrm{e}(S/\mathfrak{p}_i) = \min\{\mathrm{e}(S/Q) \mid Q \in \mathrm{Ass}_S(S/I)\}$. And let $J_i = \left(\bigcap_{j\neq i} \mathfrak{p}_j\right) \bigcap \left(\bigcap_{j=1}^{s} \mathfrak{q}_j\right)$. Then, $r_I = \min\{t \mid \exists i$ such that $[J_i/I]_t \neq 0\}$.*

Proof. Set $\mathrm{e} = \mathrm{e}(S/\mathfrak{p}_i)$. We have that $\delta_I(t) = e$, for $t \geq \min\{t \mid \exists i$ such that $[J_i/I]_t \neq 0\}$, as in the proof of Proposition 3.6. We conclude that $r_I \leq \min\{t \mid \exists i$ such that $[J_i/I]_t \neq 0\}$.

Let $f \in \left(\bigcup_{\mathfrak{p}\in \mathrm{Ass}_S(S/I)} \mathfrak{p}\right) \setminus I$ of degree strictly less than $\min\{t \mid \exists i$ such that $[J_i/I]_t \neq 0\}$. Then, $f \notin J_i$ for every i, and so, either $f \notin \mathfrak{q}_j$ or f does not belong to two different primes $\mathfrak{p}_i$ and $\mathfrak{p}_j$. In both cases, $\dim(S/I) = \dim(S/(I,f))$. In the first case, $\mathrm{e}(S/I) - \mathrm{e}(S/(I,f)) \geq \mathrm{e}(S/\mathfrak{q}_j) > e$. In the second case, $\mathrm{e}(S/I) - \mathrm{e}(S/(I,f)) \geq 2\,\mathrm{e} > \mathrm{e}$. We conclude that $\delta_I(t) > \mathrm{e}$. Then, $r_I \geq \min\{t \mid \exists i$ such that $[J_i/I]_t \neq 0\}$. □

4 Stanley–Reisner ideals associated to a shellable simplicial complex

In this section we use the shellability condition to relate the regularity index of a Stanley–Reisner ideal of a shellable simplicial complex, I_Δ, with the Castelnuovo–Mumford regularity.

Definition 4.1. A *simplicial complex* on a vertex set $X = \{x_1, x_2, \dots, x_n\}$ is a collection of subsets of X, called *faces*, satisfying that $\{x_i\} \in \Delta$ for every $i \in [n]$ and, if $\sigma \in \Delta$ and $\theta \subseteq \sigma$ then $\theta \in \Delta$. A face of Δ not properly contained in another face of Δ is called a *facet*.

A face $\sigma \in \Delta$ of cardinality $\mid \sigma \mid = i + 1$ has *dimension* i and is called an *i-face* of Δ. The *dimension* of Δ is $\dim \Delta = \max\{\dim \sigma \mid \sigma \in \Delta\}$, or if $\Delta = \{\}$ is the void complex, which has no faces. We say that Δ is *pure* if all its facets have the same dimension.

Let Δ be a simplicial complex of dimension d with the vertex set $[n] = \{1, 2, \dots, n\}$, and let $\mathbb{K}$ be a field. The square-free monomial ideal I_Δ in the polynomial ring $S = \mathbb{K}[x_1, \dots, x_n]$ is generated by the monomials $x^\sigma = \prod_{i\in\sigma} x_i$ which σ is a non-face in Δ.

The simplicial complex Δ is said Cohen-Macaulay when the quotient ring $\mathbb{K}[\Delta] = S/I_\Delta$, called Stanley–Reisner ring of Δ, is Cohen-Macaulay.

Definition 4.2. A pure simplicial complex Δ of dimension d is *shellable* if the facets of Δ can be order $\sigma_1, \ldots, \sigma_s$ such that

$$\bar{\sigma_i} \bigcap \left(\bigcup_{j=1}^{i-1} \bar{\sigma_j} \right)$$

is pure of dimension $d-1$ for all $i \geq 2$. Here $\bar{\sigma_i} = \{\sigma \in \Delta \mid \sigma \subseteq \sigma_i\}$. If Δ is pure shellable, $\sigma_1, \ldots, \sigma_s$ is called a shelling.

Theorem 4.3 ([Vil15, Theorem 6.3.23]). *Let Δ be a simplicial complex. If Δ is pure shellable, then Δ is Cohen–Macaulay over any field $\mathbb{K}$.*

Remark 4.4. *Let Δ be a simplicial complex. Suppose that Δ is pure shellable with ordered facets $\sigma_1, \ldots, \sigma_s$. We have that $\sigma_1, \ldots, \sigma_i$ is a shelling for the simplicial complex associated to $\sigma_1 \cup \ldots \cup \sigma_i$. Let $P_{\sigma_j} = (x_t \mid t \notin \sigma_j)$. Then,*

$$S/P_{\sigma_1} \cap \ldots \cap P_{\sigma_i},$$

is Cohen.Macaulay for every $i = 1, \ldots s$ by Theorem 4.3.

We are now able to show one of our main results.

Theorem 4.5. *Let $I = I_\Delta$ be the Stanley–Reisner ideal of a shellable simplicial complex, with $\dim(S/I_\Delta) = d$. Then $r_I \leq \operatorname{reg}(S/I)$.*

Proof. Let $d = \dim(S/I_\Delta)$. Since Δ is shellable, S/I_Δ is a Cohen-Macaulay ring by Theorem 4.3. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_\ell$ denote the associate primes of I. For $1 \leq i \leq \ell$, we set $R_i = S/\mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \cdots \cap \mathfrak{p}_i$ and $J_i = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_i$. We have that R_i is Cohen-Macaulay of dimension d, because J_i is a shelling of I_Δ.

We have the following short exact sequence;

$$0 \longrightarrow J_{i-1}/J_i \longrightarrow R_i \longrightarrow R_{i-1} \longrightarrow 0$$

for $2 \leq i \leq \ell$.

We note that $\dim(R_{i-1}) = d$, and so, $H^j_{\mathfrak{m}}(R_{i-1}) = 0$ for all $j > d$. Then, the short exact sequence induces a long exact sequence as follows:

$$\begin{aligned} 0 \to H^0_{\mathfrak{m}}(J_{i-1}/J_i) \to H^0_{\mathfrak{m}}(R_i) \to H^0_{\mathfrak{m}}(R_{i-1}) \to \cdots \\ \to H^d_{\mathfrak{m}}(J_{i-1}/J_i) \to H^d_{\mathfrak{m}}(R_i) \to H^d_{\mathfrak{m}}(R_{i-1}) \to 0. \end{aligned}$$

Since R_{i-1} and R_i are Cohen-Macaulay rings, we have that a short exact sequence reduces to

$$0 \to H^d_{\mathfrak{m}}(J_{i-1}/J_i) \to H^d_{\mathfrak{m}}(R_i) \to H^d_{\mathfrak{m}}(R_{i-1}) \to 0,$$

because all the other local cohomology modules vanish. Then, J_{i-1}/J_i is also Cohen-Macaulay of dimension d. We have that $a_d(J_{i-1}/J_i) \leq a_d(R_i)$, and

$$\operatorname{reg}(J_{i-1}/J_i) = a_d(J_{i-1}/J_i) + d \leq a_d(R_i) + d = \operatorname{reg}(R_i).$$

for $2 \leq i \leq \ell$. By Proposition 3.8,

$$r_I \leq \min\{t \mid [J_{\ell-1}/J_\ell]_t \neq 0\} \leq \operatorname{reg}(J_{\ell-1}/J_\ell).$$

Then, $r_I \leq \operatorname{reg}(R_\ell) = \operatorname{reg}(S/I_\Delta)$. □

5 Results related to F-purity

Definition 5.1. Let R be a Noetherian ring of prime characteristic p, and $F : R \to R$ be the Frobenius map. We say that R is F-pure if for every R-module, M, we have that

$$M \otimes_R R \xrightarrow{1_M \otimes_R F} M \otimes_R R$$

is injective. We say that R is F-finite if R is finitely generated as R^p-module.

Definition 5.2. Suppose that $\mathbb{K}$ has prime characteristic, $\mathbb{K}$ is F-finite, and that I is a radical ideal. Then, we set

- $F^e_* S/I := \{F^e_* f \mid f \in S/I\} \cong S/I$ as a Abelian groups, but the action of S/I on $F^e_* S/I$: is given by $rF^e_* f = F^e_* f^{p^e} f$.
- $\mathfrak{m}_e = \{f \in S/I \mid \phi(F^e_* f) \in \mathfrak{m}\ \forall \phi : F^e_* S/I \to S/I\}$ [AE05].
- $b_e = \max\{t \mid \mathfrak{m}^t \not\subseteq \mathfrak{m}_e\}$.
- $\operatorname{fpt}(S/I) = \lim\limits_{e\to\infty} \frac{b_e}{p^e}$ [TW04].

Theorem 5.3 ([DSNnB18, Theorem B])**.** *Suppose that $\mathbb{K}$ has prime characteristic. If S/I is a F-pure ring, then $a_i(S/I) \leq -\operatorname{fpt}(S/I)$. Furthermore, if S/I is a Gorenstein ring, then* $\operatorname{reg}(S/I) = \dim(S/I) - \operatorname{fpt}(S/I)$.

Remark 5.4. *Suppose that $\mathbb{K}$ has prime characteristic, $\mathbb{K}$ is F-finite, and that S/I is a F-pure ring. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_\ell$ be the minimal primes of I, and $J_i = \bigcap_{i\neq j} \mathfrak{p}_j$. Then, S/J_i is F-pure [Sch10, Corollary 4.8]. Furthermore,* $\operatorname{fpt}(S/I) \leq \operatorname{fpt}(S/J_i)$ *[DSNnB18, Theorem 4.7], because $J_i \cdot S/I$ is a compatible ideal for S/I.*

Theorem 5.5. *Suppose that $\mathbb{K}$ has prime characteristic. If S/I is a F-pure ring, then $r_I \leq \dim(S/I)$.*

Proof. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_\ell$ be the minimal primes of I. For $i = 1, \dots, \ell$, we set $J_i = \bigcap_{i \neq j} \mathfrak{p}_j$. We have a short exact sequence

$$0 \to J_i/I \to S/I \to S/J_i \to 0.$$

This induces a long exact sequence

$$0 \to H^0_{\mathfrak{m}}(J_i/I) \to H^0_{\mathfrak{m}}(S/I) \to H^0_{\mathfrak{m}}(S/J_i) \to H^1_{\mathfrak{m}}(J_i/I) \to \dots.$$

Since both S/J_i and S/I are F-pure, we have that $a_j(S/J_i) \leq 0$ and $a_j(S/I) \leq 0$ for every j. Then, $a_j(J_i/I) \leq 0$ for every j [HR76, Proposition 2.4]. Then,

$$\begin{aligned}
\min\{t \mid [J_i/I]_t \neq 0\} &\leq \max\{\ell \mid \beta_{0,\ell}(J_i/I) \neq 0\} \\
&\leq \operatorname{reg}(J_i/I) \\
&= \max\{a_j(J_i/I) + j\} \\
&\leq \dim(S/I).
\end{aligned}$$

□

Theorem 5.6. *Suppose that $\mathbb{K}$ has prime characteristic. If S/I is a F-pure ring and Gorenstein, then $r_I \leq \operatorname{reg}(S/I)$.*

Proof. We first assume that $\mathbb{K}$ is F-finite. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_\ell$ be the minimal primes of I. For $i = 1, \dots, \ell$, we set $J_i = \bigcap_{i \neq j} \mathfrak{p}_j$. We have a short exact sequence

$$0 \to J_i/I \to S/I \to S/J_i \to 0.$$

This induces a long exact sequence

$$0 \to H^0_{\mathfrak{m}}(J_i/I) \to H^0_{\mathfrak{m}}(S/I) \to H^0_{\mathfrak{m}}(S/J_i) \to H^1_{\mathfrak{m}}(J_i/I) \to \dots.$$

Since both S/J_i and S/I are F-pure, we have that $a_j(S/J_i) \leq -\operatorname{fpt}(S/J_i)$ and $a_j(S/I) \leq -\operatorname{fpt}(S/I)$ for every j. Then,

$$a_j(J_i/I) \leq \max\{-\operatorname{fpt}(S/J_i), -\operatorname{fpt}(S/I)\} \leq -\operatorname{fpt}(S/I)$$

for every j by Theorem 5.3. Then,

$$\begin{aligned}
\min\{t \mid [J_i/I]_t \neq 0\} &\leq \max\{\ell \mid \beta_{0,\ell}(J_i/I) \neq 0\} \\
&\leq \operatorname{reg}(J_i/I) \\
&= \max\{a_j(J_i/I) + j\} \\
&= \max\{-\operatorname{fpt}(S/I) + j\} \\
&\leq \dim(S/I) - \operatorname{fpt}(S/I) \\
&= \operatorname{reg}(S/I) \text{ by Theorem 5.3.}
\end{aligned}$$

The result for non F-finite fields follows from taking the product $\otimes_{\mathbb{K}}\overline{\mathbb{K}}$, because the numerical invariants do not change after field extensions. In addition, F-purity is stable for field extensions. $\square$

Theorem 5.7. *Let $\mathbb{K}$ be any field and I is a square-free monomial ideal. Then, $r_I \leq \dim(S/I)$. If S/I is a Gorenstein ring, then $r_I \leq \operatorname{reg}(S/I)$.*

Proof. If $\mathbb{K}$ has prime characteristic, the result follows from Theorems 5.5 and 5.6.

We now assume that $\mathbb{K}$ has characteristic zero. Since field extensions do not affect whether a ring is Gorenstein and their dimension, without loss of generality we can assume that $\mathbb{K} = \mathbb{Q}$. Let $A = \mathbb{Z}[x_1, \dots, x_n]$ and I_A the monomial ideal generated by the monomials in I. We have that $r_I = r_{I_A \otimes_{\mathbb{Z}} \mathbb{F}_p}$ by Propositions 3.7 and 3.8, since $\dim(S/I) = \dim(A \otimes_{\mathbb{Z}} \mathbb{Q}/I_A \otimes_{\mathbb{Z}} \mathbb{Q}) = \dim(\mathbb{F}_p[x_1, \dots, x_n]/I_A \otimes_{\mathbb{Z}} \mathbb{F}_p)$. Then,

$$r_I = r_{I_A \otimes_{\mathbb{Z}} \mathbb{F}_p} \leq \dim(\mathbb{F}_p[x_1, \dots, x_n]/I_A \otimes_{\mathbb{Z}} \mathbb{F}_p) = \operatorname{reg}(S/I)$$

by Theorem 5.6, because Stanley–Reisner rings in prime characteristic are F-pure.

We have that

$$\operatorname{reg}_S(S/I) = \operatorname{reg}_{A \otimes_{\mathbb{Z}} \mathbb{Q}}(A \otimes_{\mathbb{Z}} \mathbb{Q}/I_A \otimes_{\mathbb{Z}} \mathbb{Q}) = \operatorname{reg}_{A \otimes_{\mathbb{Z}} \mathbb{F}_p}(A/I_A \otimes_{\mathbb{Z}} \mathbb{F}_p)$$

and $A/J \otimes_{\mathbb{Z}} \mathbb{F}_p$ is Gorenstein for $p \gg 0$ [HH, Theorem 2.3.5]. Then, the result follows from Theorem 5.6, because Stanley–Reisner rings in prime characteristic are F-pure. $\square$

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