



Two Points Taylor's Type Representations for Analytic Complex Functions with Integral Remainders

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Abstract

In this paper we establish some two point weighted Taylor's expansions for analytic functions $f : D \subseteq \mathbb{C} \rightarrow \mathbb{C}$ defined on a convex domain D . Some error bounds for these expansions are also provided. Examples for the complex logarithm and the complex exponential are also given.

1 Introduction

Let $f : D \subseteq \mathbb{C} \rightarrow \mathbb{C}$ be an analytic function on the convex domain D and $z, v \in D$, then we have the following Taylor expansion with integral remainder

$$f(z) = \sum_{k=0}^n \frac{1}{k!} f^{(k)}(v) (z-v)^k + \frac{1}{n!} (z-v)^{n+1} \int_0^1 f^{(n+1)}[(1-s)v + sz] (1-s)^n ds \quad (1)$$

for $n \geq 0$, see for instance [13].

In this paper we establish some two point weighted Taylor expansions for analytic functions $f : D \subseteq \mathbb{C} \rightarrow \mathbb{C}$ defined on a convex domain D . Some error bounds for these expansions are also provided. Examples for the complex logarithm and the complex exponential are given as well.

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Consider the function $f(z) = \text{Log}(z)$ where $\text{Log}(z) = \ln|z| + i \text{Arg}(z)$ and $\text{Arg}(z)$ is such that $-\pi < \text{Arg}(z) \leq \pi$. Log is called the "principal branch" of the complex logarithmic function. The function f is analytic on all of $\mathbb{C}_\ell := \mathbb{C} \setminus \{x + iy : x \leq 0, y = 0\}$ and

$$f^{(k)}(z) = \frac{(-1)^{k-1} (k-1)!}{z^k}, \quad k \geq 1, \quad z \in \mathbb{C}_\ell.$$

Using the representation (1) we then have

$$\begin{aligned} \text{Log}(z) = \text{Log}(v) + \sum_{k=1}^n \frac{(-1)^{k-1}}{k} \left(\frac{z-v}{v} \right)^k \\ + (-1)^n (z-v)^{n+1} \int_0^1 \frac{(1-s)^n ds}{[(1-s)v + sz]^{n+1}} \end{aligned} \quad (2)$$

for all $z, v \in \mathbb{C}_\ell$ with $(1-s)v + sz \in \mathbb{C}_\ell$ for $s \in [0, 1]$.

Consider the complex exponential function $f(z) = \exp(z)$, then by (1) we get

$$\begin{aligned} \exp(z) = \sum_{k=0}^n \frac{1}{k!} (z-v)^k \exp(v) \\ + \frac{1}{n!} (z-v)^{n+1} \int_0^1 (1-s)^n \exp[(1-s)v + sz] ds \end{aligned} \quad (3)$$

for all $z, v \in \mathbb{C}$.

For various inequalities related to Taylor expansions for real functions see [1]-[12].

2 Two Points Taylor Expansions

We have the following two points Taylor expansion with integral remainder:

Theorem 1. *Let $f : D \subseteq \mathbb{C} \rightarrow \mathbb{C}$ be an analytic function on the convex domain D and $z, v, w \in D$, then for all $\lambda \in \mathbb{C}$ we have*

$$\begin{aligned} f(z) = (1-\lambda)f(v) + \lambda f(w) \\ + \sum_{k=1}^n \frac{1}{k!} \left[(1-\lambda) f^{(k)}(v) (z-v)^k + (-1)^k \lambda f^{(k)}(w) (w-z)^k \right] \\ + S_{n,\lambda}(z, v, w), \end{aligned} \quad (4)$$

where the remainder $S_{n,\lambda}(z, v, w)$ is given by

$$\begin{aligned} S_{n,\lambda}(z, v, w) & \\ & := \frac{1}{n!} \left[(1-\lambda)(z-v)^{n+1} \int_0^1 f^{(n+1)}[(1-s)v+sz](1-s)^n ds \right. \\ & \quad \left. + (-1)^{n+1} \lambda (w-z)^{n+1} \int_0^1 f^{(n+1)}[(1-s)z+sw] s^n ds \right]. \end{aligned} \quad (5)$$

Proof. If we replace in (1) v by w , then we get

$$\begin{aligned} f(z) &= \sum_{k=0}^n \frac{1}{k!} f^{(k)}(w)(z-w)^k \\ & \quad + \frac{1}{n!} (z-w)^{n+1} \int_0^1 f^{(n+1)}[(1-s)w+sz](1-s)^n ds \end{aligned} \quad (6)$$

$$\begin{aligned} &= \sum_{k=0}^n \frac{(-1)^k}{k!} f^{(k)}(w)(w-z)^k \\ & \quad + \frac{(-1)^{n+1}}{n!} (w-z)^{n+1} \int_0^1 f^{(n+1)}[(1-s)w+sz](1-s)^n ds \\ & \quad = \sum_{k=0}^n \frac{(-1)^k}{k!} f^{(k)}(w)(w-z)^k \\ & \quad + \frac{(-1)^{n+1}}{n!} (w-z)^{n+1} \int_0^1 f^{(n+1)}[(1-s)z+sw] s^n ds. \end{aligned}$$

Assume that $\lambda \neq 1, 0$. If we multiply (1) by $1-\lambda$ and (6) by λ we get the desired representation (4) with the remainder $S_{n,\lambda}(z, v, w)$ given by (5).

If either $\lambda = 1$ or $\lambda = 0$, then the theorem also holds by the use of Taylor usual expansion. \square

Remark 1. We observe that for $n = 0$ the representation from Theorem 1 becomes

$$f(z) = (1-\lambda)f(v) + \lambda f(w) + S_\lambda(z, v, w), \quad (7)$$

where the remainder $S_\lambda(z, v, w)$ is given by

$$\begin{aligned} S_\lambda(z, v, w) &:= (1-\lambda)(z-v) \int_0^1 f'((1-s)v+sz) ds \\ & \quad - \lambda(w-z) \int_0^1 f'((1-s)z+sw) ds. \end{aligned} \quad (8)$$

Remark 2. If we take in (6) $z = \frac{v+w}{2}$, with $v, w \in D$, then we have for any $\lambda \in \mathbb{C}$ that

$$\begin{aligned} f\left(\frac{v+w}{2}\right) &= (1-\lambda)f(v) + \lambda f(w) \\ &+ \sum_{k=1}^n \frac{1}{2^k k!} \left[(1-\lambda)f^{(k)}(v) + (-1)^k \lambda f^{(k)}(w) \right] (w-v)^k \\ &+ \tilde{S}_{n,\lambda}(v,w), \end{aligned} \quad (9)$$

where the remainder $\tilde{S}_{n,\lambda}(v,w)$ is given by

$$\begin{aligned} \tilde{S}_{n,\lambda}(v,w) & \\ &:= \frac{1}{2^{n+1}n!} (w-v)^{n+1} \left[(1-\lambda) \int_0^1 f^{(n+1)}\left((1-s)v + s\frac{v+w}{2}\right) (1-s)^n ds \right. \\ &\left. + (-1)^{n+1} \lambda \int_0^1 f^{(n+1)}\left((1-s)\frac{v+w}{2} + sw\right) s^n ds \right]. \end{aligned} \quad (10)$$

In particular, for $\lambda = \frac{1}{2}$ in (9) we have

$$\begin{aligned} f\left(\frac{v+w}{2}\right) &= \frac{f(v) + f(w)}{2} \\ &+ \sum_{k=1}^n \frac{1}{2^{k+1}k!} \left[f^{(k)}(v) + (-1)^k f^{(k)}(w) \right] (w-v)^k \\ &+ \tilde{S}_n(v,w), \end{aligned} \quad (11)$$

where the remainder $\tilde{S}_n(v,w)$ is given by

$$\begin{aligned} \tilde{S}_n(v,w) & \\ &:= \frac{1}{2^{n+2}n!} (w-v)^{n+1} \left[\int_0^1 f^{(n+1)}\left((1-s)v + s\frac{v+w}{2}\right) (1-s)^n ds \right. \\ &\left. + (-1)^{n+1} \int_0^1 f^{(n+1)}\left((1-s)\frac{v+w}{2} + sw\right) s^n ds \right]. \end{aligned} \quad (12)$$

Now, by the change of variable in (12) we also get the following representation for the remainder $\tilde{S}_n(v,w)$ as a single integral

$$\begin{aligned} \tilde{S}_n(v,w) &:= \frac{1}{2^{n+2}n!} (w-v)^{n+1} \\ &\times \int_0^1 \left[f^{(n+1)}\left(sv + (1-s)\frac{v+w}{2}\right) + (-1)^{n+1} f^{(n+1)}\left((1-s)\frac{v+w}{2} + sw\right) \right] s^n ds, \end{aligned} \quad (13)$$

for $n \geq 0$.

Corollary 2. *With the assumptions in Theorem 1 we have for each distinct $z, v, w \in D$ with $w \neq v$*

$$\begin{aligned} f(z) &= \frac{1}{w-v} [(w-z)f(v) + (z-v)f(w)] + \frac{(w-z)(z-v)}{w-v} \\ &\quad \times \sum_{k=1}^n \frac{1}{k!} \left\{ (z-v)^{k-1} f^{(k)}(v) + (-1)^k (w-z)^{k-1} f^{(k)}(w) \right\} \\ &\quad + L_n(z, v, w), \end{aligned} \quad (14)$$

where

$$\begin{aligned} L_n(z, v, w) &:= \frac{(w-z)(z-v)}{n!(w-v)} \left[(z-v)^n \int_0^1 f^{(n+1)}((1-s)v + sz) (1-s)^n ds \right. \\ &\quad \left. + (-1)^{n+1} (w-z)^n \int_0^1 f^{(n+1)}((1-s)z + sw) s^n ds \right] \end{aligned}$$

and

$$\begin{aligned} f(z) &= \frac{1}{w-v} [(z-v)f(v) + (w-z)f(w)] \\ &\quad + \frac{1}{w-v} \sum_{k=1}^n \frac{1}{k!} \left\{ (z-v)^{k+1} f^{(k)}(v) + (-1)^k (w-z)^{k+1} f^{(k)}(w) \right\} \\ &\quad + P_n(z, v, w), \end{aligned} \quad (15)$$

where

$$\begin{aligned} P_n(z, v, w) &:= \frac{1}{n!(w-v)} \left[(z-v)^{n+2} \int_0^1 f^{(n+1)}((1-s)v + sz) (1-s)^n ds \right. \\ &\quad \left. + (-1)^{n+1} (w-z)^{n+2} \int_0^1 f^{(n+1)}((1-s)z + sw) s^n ds \right], \end{aligned}$$

respectively.

The proof is obvious, by choosing $\lambda = (z-v)/(w-v)$ and $\lambda = (w-z)/(w-v)$, respectively, in Theorem 1. The details are omitted.

Corollary 3. *With the assumption in Theorem 1 we have for each $\lambda \in [0, 1]$ and any distinct $v, w \in D$ that*

$$\begin{aligned} f((1-\lambda)v + \lambda w) &= (1-\lambda)f(v) + \lambda f(w) + \lambda(1-\lambda) \\ &\quad \times \sum_{k=1}^n \frac{1}{k!} \left[\lambda^{k-1} f^{(k)}(v) + (-1)^k (1-\lambda)^{k-1} f^{(k)}(w) \right] (w-v)^k + S_{n,\lambda}(v, w), \end{aligned} \quad (16)$$

where the remainder $S_{n,\lambda}(v, w)$ is given by

$$\begin{aligned} S_{n,\lambda}(v, w) &:= \frac{1}{n!} (1-\lambda) \lambda (w-v)^{n+1} \left[\lambda^n \int_0^1 f^{(n+1)}((1-s\lambda)v + s\lambda w) (1-s)^n ds \right. \\ &\quad \left. + (-1)^{n+1} (1-\lambda)^n \int_0^1 f^{(n+1)}((1-s-\lambda+s\lambda)v + (\lambda+s-s\lambda)w) s^n ds \right]. \end{aligned} \quad (17)$$

We also have

$$\begin{aligned} f((1-\lambda)w + \lambda v) &= (1-\lambda)f(v) + \lambda f(w) \\ &+ \sum_{k=1}^n \frac{1}{k!} \left[(1-\lambda)^{k+1} f^{(k)}(v) + (-1)^k \lambda^{k+1} f^{(k)}(w) \right] (w-v)^k + P_{n,\lambda}(v, w), \end{aligned} \quad (18)$$

where the remainder $P_{n,\lambda}(v, w)$ is given by

$$\begin{aligned} P_{n,\lambda}(v, w) &:= \frac{1}{n!} (w-v)^{n+1} \left[(1-\lambda)^{n+2} \int_0^1 f^{(n+1)}((1-s+\lambda s)v + (1-\lambda)sw) (1-s)^n ds \right. \\ &\quad \left. + (-1)^{n+1} \lambda^{n+2} \int_0^1 f^{(n+1)}((1-s)\lambda v + (1-\lambda+\lambda s)w) s^n ds \right]. \end{aligned} \quad (19)$$

The case $n = 0$ produces the following simple identities for each distinct $z, v, w \in D$ and $\lambda \in \mathbb{C}$

$$f(z) = (1-\lambda)f(v) + \lambda f(w) + S_\lambda(z, v, w), \quad (20)$$

where the remainder $S_\lambda(z, v, w)$ is given by (8).

We then have for each distinct $z, v, w \in D$

$$f(z) = \frac{1}{w-v} [(w-z)f(v) + (z-v)f(w)] + L(z, v, w), \quad (21)$$

where

$$\begin{aligned} L(z, v, w) &:= \frac{(w-z)(z-v)}{w-v} \left[\int_0^1 f'((1-s)v + sz) ds - \int_0^1 f'((1-s)z + sw) ds \right] \end{aligned} \quad (22)$$

and

$$f(z) = \frac{1}{w-v} [(z-v)f(v) + (w-z)f(w)] + P(z, v, w), \quad (23)$$

where

$$P(z, v, w) := \frac{1}{w-v} \left[(z-v)^2 \int_0^1 f'((1-s)v + sz) ds - (w-z)^2 \int_0^1 f'((1-s)z + sw) ds \right]. \quad (24)$$

We also have for $\lambda \in [0, 1]$

$$f((1-\lambda)v + \lambda w) = (1-\lambda)f(v) + \lambda f(w) + S_\lambda(v, w), \quad (25)$$

where the remainder $S_\lambda(v, w)$ is given by

$$S_\lambda(v, w) := (1-\lambda)\lambda(w-v) \left[\int_0^1 f'((1-s\lambda)v + s\lambda w) ds - \int_0^1 f'((1-s-\lambda+s\lambda)v + (\lambda+s-s\lambda)w) ds \right] \quad (26)$$

and

$$f((1-\lambda)w + \lambda v) = (1-\lambda)f(v) + \lambda f(w) + P_\lambda(v, w), \quad (27)$$

where the remainder $P_\lambda(v, w)$ is given by

$$P_\lambda(v, w) := (w-v) \left[(1-\lambda)^2 \int_0^1 f'((1-s+\lambda s)v + (1-\lambda)sw) ds - \lambda^2 \int_0^1 f'((1-s)\lambda v + (1-\lambda+\lambda s)w) ds \right]. \quad (28)$$

Moreover, if we take in (20) $z = \frac{v+w}{2}$ for each distinct $v, w \in D$ and $\lambda \in \mathbb{C}$, then we have

$$f\left(\frac{v+w}{2}\right) = (1-\lambda)f(v) + \lambda f(w) + S_\lambda(v, w), \quad (29)$$

where the remainder $S_\lambda(v, w)$ is given by

$$S_\lambda(v, w) := \frac{1}{2}(w-v) \times \left[(1-\lambda) \int_0^1 f'((1-s)v + s\frac{v+w}{2}) ds - \lambda \int_0^1 f'((1-s)\frac{v+w}{2} + sw) ds \right]. \quad (30)$$

In particular, for $\lambda = \frac{1}{2}$ we have

$$f\left(\frac{v+w}{2}\right) = \frac{f(v) + f(w)}{2} + S(v, w), \quad (31)$$

where

$$\begin{aligned} S(v, w) &:= \frac{1}{4}(w-v) \\ &\times \left[\int_0^1 f' \left((1-s)v + s\frac{v+w}{2} \right) ds - \int_0^1 f' \left((1-s)\frac{v+w}{2} + sw \right) ds \right]. \end{aligned} \quad (32)$$

Now, assume that $z, v, w \in D \subset \mathbb{C}_\ell$, with D a convex set, then for all $\lambda \in \mathbb{C}$ we have by Theorem 1 for the function $f(z) = \text{Log}(z)$ that

$$\begin{aligned} \text{Log}(z) &= (1-\lambda)\text{Log}(v) + \lambda\text{Log}(w) \\ &+ \sum_{k=1}^n \frac{1}{k} \left[(1-\lambda)(-1)^{k-1} \frac{(z-v)^k}{v^k} - \lambda \frac{(w-z)^k}{w^k} \right] \\ &+ S_{n,\lambda}(z, v, w), \end{aligned} \quad (33)$$

where the remainder $\Lambda_{n,\lambda}(z, v, w)$ is given by

$$\begin{aligned} \Lambda_{n,\lambda}(z, v, w) &:= (1-\lambda)(z-v)^{n+1}(-1)^n \int_0^1 \frac{(1-s)^n}{((1-s)v + sz)^{n+1}} ds \\ &- \lambda(w-z)^{n+1} \int_0^1 \frac{s^n}{((1-s)z + sw)^{n+1}} ds \end{aligned} \quad (34)$$

for $n \geq 0$.

Consider the function $f: \mathbb{C} \rightarrow \mathbb{C}$, $f(z) = \exp z$, then for $z, v, w \in \mathbb{C}$ we have by Theorem 1 that

$$\begin{aligned} \exp z &= (1-\lambda)\exp v + \lambda\exp w \\ &+ \sum_{k=1}^n \frac{1}{k!} \left[(1-\lambda)(z-v)^k \exp v + (-1)^k \lambda(w-z)^k \exp w \right] \\ &+ \Theta_{n,\lambda}(z, v, w), \end{aligned} \quad (35)$$

where the remainder $\Theta_{n,\lambda}(z, v, w)$ is given by

$$\begin{aligned} \Theta_{n,\lambda}(z, v, w) &:= \frac{1}{n!} \left[(1-\lambda)(z-v)^{n+1} \int_0^1 \exp((1-s)v + sz)(1-s)^n ds \right. \\ &\left. + (-1)^{n+1} \lambda(w-z)^{n+1} \int_0^1 \exp((1-s)z + sw)s^n ds \right]. \end{aligned} \quad (36)$$

for $n \geq 0$ and for all $\lambda \in \mathbb{C}$.

3 Some Inequalities

We can state now some results concerning error bounds in approximating an analytic function by two points Taylor expansion:

Theorem 4. *Let $f : D \subseteq \mathbb{C} \rightarrow \mathbb{C}$ be an analytic function on the convex domain D and $z, v, w \in D$, then for all $\lambda \in \mathbb{C}$ we have*

$$\begin{aligned} f(z) &= (1 - \lambda) f(v) + \lambda f(w) \\ &+ \sum_{k=1}^n \frac{1}{k!} \left[(1 - \lambda) f^{(k)}(v) (z - v)^k + (-1)^k \lambda f^{(k)}(w) (w - z)^k \right] \\ &+ S_{n,\lambda}(z, v, w), \end{aligned} \quad (37)$$

and the remainder $S_{n,\lambda}(z, v, w)$ satisfies the inequalities

$$\begin{aligned} &|S_{n,\lambda}(z, v, w)| \\ &\leq \frac{1}{n!} |1 - \lambda| |z - v|^{n+1} \int_0^1 |f^{(n+1)}((1-s)v + sz)| (1-s)^n ds \\ &\quad + \frac{1}{n!} |\lambda| |w - z|^{n+1} \int_0^1 |f^{(n+1)}((1-s)z + sw)| s^n ds \\ &\leq \frac{1}{n!} |1 - \lambda| |z - v|^{n+1} \begin{cases} \frac{1}{n+1} \sup_{s \in [0,1]} |f^{(n+1)}((1-s)v + sz)| \\ \frac{1}{(qn+1)^{1/q}} \left(\int_0^1 |f^{(n+1)}((1-s)v + sz)|^p ds \right)^{1/p} \\ \text{where } p, q > 1 \text{ with } \frac{1}{p} + \frac{1}{q} = 1 \\ \int_0^1 |f^{(n+1)}((1-s)v + sz)| ds \end{cases} \\ &+ \frac{1}{n!} |\lambda| |w - z|^{n+1} \begin{cases} \frac{1}{n+1} \sup_{s \in [0,1]} |f^{(n+1)}((1-s)z + sw)| \\ \frac{1}{(qn+1)^{1/q}} \left(\int_0^1 |f^{(n+1)}((1-s)z + sw)|^p ds \right)^{1/p} \\ \text{where } p, q > 1 \text{ with } \frac{1}{p} + \frac{1}{q} = 1 \\ \int_0^1 |f^{(n+1)}((1-s)z + sw)| ds \end{cases} \quad (38) \end{aligned}$$

for $n \geq 0$.

Proof. Taking the modulus in the representation (5), we get

$$\begin{aligned}
& |S_{n,\lambda}(z, v, w)| \tag{39} \\
& \leq \frac{1}{n!} \left[\left| (1-\lambda)(z-v)^{n+1} \int_0^1 f^{(n+1)}((1-s)v+sz)(1-s)^n ds \right| \right. \\
& \quad \left. + \left| \lambda(w-z)^{n+1} \int_0^1 f^{(n+1)}((1-s)z+sw)s^n ds \right| \right] \\
& \leq |1-\lambda| |z-v|^{n+1} \int_0^1 |f^{(n+1)}((1-s)v+sz)|(1-s)^n ds \\
& \quad + |\lambda| |w-z|^{n+1} \int_0^1 |f^{(n+1)}((1-s)z+sw)|s^n ds.
\end{aligned}$$

By Hölder's integral inequality we have

$$\begin{aligned}
& \int_0^1 |f^{(n+1)}((1-s)v+sz)|(1-s)^n ds \\
& \leq \begin{cases} \sup_{s \in [0,1]} |f^{(n+1)}((1-s)v+sz)| \int_0^1 (1-s)^n ds \\ \left(\int_0^1 |f^{(n+1)}((1-s)v+sz)|^p ds \right)^{1/p} \left(\int_0^1 (1-s)^{qn} ds \right)^{1/q} \\ \text{where } p, q > 1 \text{ with } \frac{1}{p} + \frac{1}{q} = 1 \end{cases} \\
& \leq \begin{cases} \sup_{s \in [0,1]} \{(1-s)^n\} \int_0^1 |f^{(n+1)}((1-s)v+sz)| ds \\ \frac{1}{n+1} \sup_{s \in [0,1]} |f^{(n+1)}((1-s)v+sz)| \\ \frac{1}{(qn+1)^{1/q}} \left(\int_0^1 |f^{(n+1)}((1-s)v+sz)|^p ds \right)^{1/p} \\ \text{where } p, q > 1 \text{ with } \frac{1}{p} + \frac{1}{q} = 1 \end{cases} \\
& \leq \int_0^1 |f^{(n+1)}((1-s)v+sz)| ds
\end{aligned}$$

and

$$\begin{aligned}
& \int_0^1 |f^{(n+1)}((1-s)z + sw)| s^n ds \\
& \leq \begin{cases} \sup_{s \in [0,1]} |f^{(n+1)}((1-s)z + sw)| \int_0^1 s^n ds \\ \left(\int_0^1 |f^{(n+1)}((1-s)z + sw)|^p ds \right)^{1/p} \left(\int_0^1 s^{qn} ds \right)^{1/q} \\ \text{where } p, q > 1 \text{ with } \frac{1}{p} + \frac{1}{q} = 1 \end{cases} \\
& \leq \begin{cases} \sup_{s \in [0,1]} \{s^n\} \int_0^1 |f^{(n+1)}((1-s)z + sw)| ds \\ \frac{1}{n+1} \sup_{s \in [0,1]} |f^{(n+1)}((1-s)z + sw)| \\ \frac{1}{(qn+1)^{1/q}} \left(\int_0^1 |f^{(n+1)}((1-s)z + sw)|^p ds \right)^{1/p} \\ \text{where } p, q > 1 \text{ with } \frac{1}{p} + \frac{1}{q} = 1 \end{cases} \\
& = \begin{cases} \int_0^1 |f^{(n+1)}((1-s)z + sw)| ds, \end{cases}
\end{aligned}$$

which proves the second inequality in (38). \square

Corollary 5. *With the assumptions of Theorem 4 and if*

$$\|f^{(n+1)}\|_{D,\infty} := \sup_{y \in D} |f^{(n+1)}(y)| < \infty,$$

then we have the simple bound

$$|S_{n,\lambda}(z, v, w)| \leq \frac{1}{(n+1)!} \|f^{(n+1)}\|_{D,\infty} \left(|1-\lambda| |z-v|^{n+1} + |\lambda| |w-z|^{n+1} \right) \quad (40)$$

for $n \geq 0$.

Remark 3. *If we take $z = \frac{v+w}{2}$, with $v, w \in D$, then we have for any $\lambda \in \mathbb{C}$ that*

$$\begin{aligned}
f\left(\frac{v+w}{2}\right) &= (1-\lambda)f(v) + \lambda f(w) \quad (41) \\
&+ \sum_{k=1}^n \frac{1}{2^k k!} \left[(1-\lambda)f^{(k)}(v) + (-1)^k \lambda f^{(k)}(w) \right] (w-v)^k \\
&+ \tilde{S}_{n,\lambda}(v, w),
\end{aligned}$$

and if $\|f^{(n+1)}\|_{D,\infty} := \sup_{y \in D} |f^{(n+1)}(y)| < \infty$, then by (40) we get

$$|\tilde{S}_{n,\lambda}(v, w)| \leq \frac{1}{2^{n+1}(n+1)!} \|f^{(n+1)}\|_{D,\infty} (|1-\lambda| + |\lambda|) |w-v|^{n+1}$$

or any $\lambda \in \mathbb{C}$.

In particular, if $\lambda = \frac{1}{2}$, then we have

$$\begin{aligned} f\left(\frac{v+w}{2}\right) &= \frac{f(v) + f(w)}{2} \\ &+ \sum_{k=1}^n \frac{1}{2^{k+1}k!} \left[f^{(k)}(v) + (-1)^k f^{(k)}(w) \right] (w-v)^k \\ &+ \tilde{S}_n(v, w), \end{aligned} \quad (42)$$

and the remainder $\tilde{S}_n(v, w)$ satisfies the bound

$$|\tilde{S}_n(v, w)| \leq \frac{1}{2^{n+1}(n+1)!} \|f^{(n+1)}\|_{D,\infty} |w-v|^{n+1}$$

for $n \geq 0$.

Remark 4. The case $n = 0$ provides some simple inequalities as follows

$$\begin{aligned} &|f(z) - (1-\lambda)f(v) - \lambda f(w)| \\ &\leq |1-\lambda| |z-v| \int_0^1 |f'((1-s)v + sz)| ds + |\lambda| |w-z| \int_0^1 |f'((1-s)z + sw)| ds \\ &\leq |1-\lambda| |z-v| \begin{cases} \sup_{s \in [0,1]} |f'((1-s)v + sz)| \\ \left(\int_0^1 |f'((1-s)v + sz)|^p ds \right)^{1/p} \\ \text{where } p > 1 \\ \int_0^1 |f'((1-s)v + sz)| ds \end{cases} \\ &\quad + |\lambda| |w-z| \begin{cases} \sup_{s \in [0,1]} |f'((1-s)z + sw)| \\ \left(\int_0^1 |f'((1-s)z + sw)|^p ds \right)^{1/p} \\ \text{where } p > 1 \\ \int_0^1 |f'((1-s)z + sw)| ds \end{cases} \end{aligned} \quad (43)$$

where $f : D \subseteq \mathbb{C} \rightarrow \mathbb{C}$ is an analytic function on the convex domain D , $z, v, w \in D$ and $\lambda \in \mathbb{C}$.

If

$$\|f'\|_{D,\infty} := \sup_{y \in D} |f'(y)| < \infty,$$

then we have the simple bound

$$|f(z) - (1 - \lambda)f(v) - \lambda f(w)| \leq \|f'\|_{D,\infty} (|1 - \lambda||z - v| + |\lambda||w - z|) \quad (44)$$

for $z, v, w \in D$ and $\lambda \in \mathbb{C}$.

If we take $z = \frac{v+w}{2}$, with $v, w \in D$, then we have for any $\lambda \in \mathbb{C}$ that

$$\left| f\left(\frac{v+w}{2}\right) - (1 - \lambda)f(v) - \lambda f(w) \right| \leq \frac{1}{2} \|f'\|_{D,\infty} (|1 - \lambda| + |\lambda|) |w - v|, \quad (45)$$

which for $\lambda = \frac{1}{2}$ gives the simple inequality

$$\left| f\left(\frac{v+w}{2}\right) - \frac{f(v) + f(w)}{2} \right| \leq \frac{1}{2} \|f'\|_{D,\infty} |w - v|. \quad (46)$$

If n is even, namely $n = 2m, m \geq 0$, then by (11) we have the representation

$$\begin{aligned} f\left(\frac{v+w}{2}\right) &= \frac{f(v) + f(w)}{2} \\ &+ \sum_{k=1}^{2m} \frac{1}{2^{k+1}k!} \left[f^{(k)}(v) + (-1)^k f^{(k)}(w) \right] (w - v)^k \\ &+ \tilde{S}_{2m}(v, w), \end{aligned} \quad (47)$$

where the remainder $\tilde{S}_{2m}(v, w)$ is given by (13) as

$$\begin{aligned} \tilde{S}_{2m}(v, w) &:= \frac{1}{2^{2m+2} (2m)!} (w - v)^{2m+1} \\ &\times \int_0^1 \left[f^{(2m+1)}\left(sv + (1-s)\frac{v+w}{2}\right) - f^{(2m+1)}\left((1-s)\frac{v+w}{2} + sw\right) \right] s^{2m} ds, \end{aligned} \quad (48)$$

We also have the following result:

Theorem 6. Let $f : D \subseteq \mathbb{C} \rightarrow \mathbb{C}$ be an analytic function on the convex domain D and $v, w \in D$. Let $m \geq 0$ and assume that

$$\left| f^{(2m+1)}(z) - f^{(2m+1)}(y) \right| \leq L_{2m+1} |z - y| \text{ for all } z, y \in D \quad (49)$$

for some $L_{2m+1} > 0$, namely that $f^{(2m+1)}$ is Lipschitzian on D . Then we have the representation (47) and the remainder $\tilde{S}_{2m}(v, w)$ satisfies the bound

$$\left| \tilde{S}_{2m}(v, w) \right| \leq \frac{1}{2^{2m+2} (2m+2) (2m)!} |w-v|^{2m+2} L_{2m+1} \quad (50)$$

Proof. By taking the modulus in (48), we have

$$\begin{aligned} \left| \tilde{S}_{2m}(v, w) \right| &\leq \frac{1}{2^{2m+2} (2m)!} |w-v|^{2m+1} \\ &\times \int_0^1 \left| f^{(2m+1)} \left(sv + (1-s) \frac{v+w}{2} \right) - f^{(2m+1)} \left((1-s) \frac{v+w}{2} + sw \right) \right| s^{2m} ds \\ &\leq \frac{1}{2^{2m+2} (2m)!} |w-v|^{2m+1} \\ &\times L_{2m+1} \int_0^1 \left| sv + (1-s) \frac{v+w}{2} - (1-s) \frac{v+w}{2} - sw \right| s^{2m} ds \\ &= \frac{1}{2^{2m+2} (2m)!} |w-v|^{2m+2} L_{2m+1} \int_0^1 s^{2m+1} ds \\ &= \frac{1}{2^{2m+2} (2m+2) (2m)!} |w-v|^{2m+2} L_{2m+1}, \end{aligned}$$

which proves the desired result (50). \square

Corollary 7. Let $f : D \subseteq \mathbb{C} \rightarrow \mathbb{C}$ be an analytic function on the convex domain D and $v, w \in D$. Assume that

$$|f'(z) - f'(y)| \leq L |z - y| \text{ for all } z, y \in D \quad (51)$$

for some $L > 0$. Then we have the inequality

$$\left| f \left(\frac{v+w}{2} \right) - \frac{f(v) + f(w)}{2} \right| \leq \frac{1}{8} L |w-v|^2. \quad (52)$$

4 Inequalities for Convex Derivatives in Absolute Value

We have:

Theorem 8. Let $f : D \subseteq \mathbb{C} \rightarrow \mathbb{C}$ be an analytic function on the convex domain D and such that for a given $n \geq 0$, $|f^{(n+1)}|$ is convex on D . If $z, v,$

$w \in D$, then for all $\lambda \in \mathbb{C}$ we have

$$\begin{aligned} f(z) &= (1 - \lambda) f(v) + \lambda f(w) \\ &+ \sum_{k=1}^n \frac{1}{k!} \left[(1 - \lambda) f^{(k)}(v) (z - v)^k + (-1)^k \lambda f^{(k)}(w) (w - z)^k \right] \\ &+ S_{n,\lambda}(z, v, w), \end{aligned} \quad (53)$$

and the remainder $S_{n,\lambda}(z, v, w)$ satisfies the inequality

$$\begin{aligned} |S_{n,\lambda}(z, v, w)| &\leq \frac{1}{n!(n+2)} \left[|1 - \lambda| |z - v|^{n+1} \left| f^{(n+1)}(v) \right| \right. \\ &+ \frac{1}{(n+1)} \left[|1 - \lambda| |z - v|^{n+1} + |\lambda| |w - z|^{n+1} \right] \left| f^{(n+1)}(z) \right| \\ &\left. + |\lambda| |w - z|^{n+1} \left| f^{(n+1)}(w) \right| \right]. \end{aligned} \quad (54)$$

Proof. Using the representation (5), we get

$$\begin{aligned} &|S_{n,\lambda}(z, v, w)| \quad (55) \\ &\leq \frac{1}{n!} \left[|1 - \lambda| |z - v|^{n+1} \left| \int_0^1 f^{(n+1)}((1-s)v + sz) (1-s)^n ds \right| \right. \\ &+ \left. |\lambda| |w - z|^{n+1} \left| \int_0^1 f^{(n+1)}((1-s)z + sw) s^n ds \right| \right] \\ &\leq \frac{1}{n!} \left[|1 - \lambda| |z - v|^{n+1} \int_0^1 \left| f^{(n+1)}((1-s)v + sz) \right| (1-s)^n ds \right. \\ &+ \left. |\lambda| |w - z|^{n+1} \int_0^1 \left| f^{(n+1)}((1-s)z + sw) \right| s^n ds \right] \\ &=: A_n(\lambda, w) \end{aligned}$$

By the convexity of $|f^{(n+1)}|$ we have

$$\begin{aligned}
& \int_0^1 \left| f^{(n+1)}((1-s)v + sz) \right| (1-s)^n ds \\
& \leq \int_0^1 \left[(1-s) \left| f^{(n+1)}(v) \right| + s \left| f^{(n+1)}(z) \right| \right] (1-s)^n ds \\
& = \left| f^{(n+1)}(v) \right| \int_0^1 (1-s)^{n+1} ds + \left| f^{(n+1)}(z) \right| \int_0^1 s(1-s)^n ds \\
& = \frac{1}{n+2} \left| f^{(n+1)}(v) \right| + \left| f^{(n+1)}(z) \right| \int_0^1 (1-s)s^n ds \\
& = \frac{1}{n+2} \left| f^{(n+1)}(v) \right| + \left| f^{(n+1)}(z) \right| \left(\frac{1}{n+1} - \frac{1}{n+2} \right) \\
& = \frac{1}{n+2} \left| f^{(n+1)}(v) \right| + \frac{1}{(n+1)(n+2)} \left| f^{(n+1)}(z) \right|
\end{aligned}$$

and

$$\begin{aligned}
& \int_0^1 \left| f^{(n+1)}((1-s)z + sw) \right| s^n ds \\
& \leq \int_0^1 \left[(1-s) \left| f^{(n+1)}(z) \right| + s \left| f^{(n+1)}(w) \right| \right] s^n ds \\
& = \left| f^{(n+1)}(z) \right| \int_0^1 (1-s)s^n ds + \left| f^{(n+1)}(w) \right| \int_0^1 s^{n+1} ds \\
& = \frac{1}{(n+1)(n+2)} \left| f^{(n+1)}(z) \right| + \frac{1}{n+2} \left| f^{(n+1)}(w) \right|.
\end{aligned}$$

Therefore,

$$\begin{aligned}
& A_n(\lambda, w) \\
& \leq \frac{1}{n!} \left[|1-\lambda| |z-v|^{n+1} \left[\frac{1}{n+2} \left| f^{(n+1)}(v) \right| + \frac{1}{(n+1)(n+2)} \left| f^{(n+1)}(z) \right| \right] \right. \\
& \quad \left. + |\lambda| |w-z|^{n+1} \left[\frac{1}{(n+1)(n+2)} \left| f^{(n+1)}(z) \right| + \frac{1}{n+2} \left| f^{(n+1)}(w) \right| \right] \right]
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{n!(n+2)} \left[|1 - \lambda| |z - v|^{n+1} \left[\left| f^{(n+1)}(v) \right| + \frac{1}{(n+1)} \left| f^{(n+1)}(z) \right| \right] \right. \\
&+ \left. |\lambda| |w - z|^{n+1} \left[\frac{1}{(n+1)} \left| f^{(n+1)}(z) \right| + \left| f^{(n+1)}(w) \right| \right] \right] \\
&= \frac{1}{n!(n+2)} \left[|1 - \lambda| |z - v|^{n+1} \left| f^{(n+1)}(v) \right| \right. \\
&+ \frac{1}{(n+1)} \left[|1 - \lambda| |z - v|^{n+1} + |\lambda| |w - z|^{n+1} \right] \left| f^{(n+1)}(z) \right| \\
&+ \left. |\lambda| |w - z|^{n+1} \left| f^{(n+1)}(w) \right| \right],
\end{aligned}$$

which together with (55) produce the desired result (55). \square

Remark 5. Assume that for a given $n \geq 0$, $|f^{(n+1)}|$ is convex on D . If we take in (53) $z = \frac{v+w}{2}$, with $v, w \in D$, then we have for any $\lambda \in \mathbb{C}$ that

$$\begin{aligned}
f\left(\frac{v+w}{2}\right) &= (1-\lambda)f(v) + \lambda f(w) \\
&+ \sum_{k=1}^n \frac{1}{2^k k!} \left[(1-\lambda)f^{(k)}(v) + (-1)^k \lambda f^{(k)}(w) \right] (w-v)^k \\
&+ \tilde{S}_{n,\lambda}(v,w),
\end{aligned} \tag{56}$$

where the remainder $\tilde{S}_{n,\lambda}(v,w)$ satisfies the bound

$$\begin{aligned}
\left| \tilde{S}_{n,\lambda}(v,w) \right| &\leq \frac{1}{2^{n+1} n! (n+2)} |w-v|^{n+1} \left[|1-\lambda| \left| f^{(n+1)}(v) \right| \right. \\
&+ \left. \frac{1}{(n+1)} \left[|1-\lambda| + |\lambda| \right] \left| f^{(n+1)}\left(\frac{v+w}{2}\right) \right| + |\lambda| \left| f^{(n+1)}(w) \right| \right].
\end{aligned} \tag{57}$$

In particular, for $\lambda = \frac{1}{2}$ in (57) we have

$$\begin{aligned}
f\left(\frac{v+w}{2}\right) &= \frac{f(v) + f(w)}{2} \\
&+ \sum_{k=1}^n \frac{1}{2^{k+1} k!} \left[f^{(k)}(v) + (-1)^k f^{(k)}(w) \right] (w-v)^k \\
&+ \tilde{S}_n(v,w),
\end{aligned} \tag{58}$$

where the remainder $\tilde{S}_n(v, w)$ satisfies the bound

$$\begin{aligned} |\tilde{S}_n(v, w)| &\leq \frac{1}{2^{n+1}n!(n+2)} |w-v|^{n+1} \left[\frac{1}{2} |f^{(n+1)}(v)| \right. \\ &\quad \left. + \frac{1}{(n+1)} \left| f^{(n+1)}\left(\frac{v+w}{2}\right) \right| + \frac{1}{2} |f^{(n+1)}(w)| \right]. \end{aligned} \quad (59)$$

Corollary 9. *With the assumption in Theorem 8 we have for each $\lambda \in [0, 1]$ and any distinct $v, w \in D$ that*

$$\begin{aligned} f((1-\lambda)v + \lambda w) &= (1-\lambda)f(v) + \lambda f(w) + \lambda(1-\lambda) \\ &\times \sum_{k=1}^n \frac{1}{k!} \left[\lambda^{k-1} f^{(k)}(v) + (-1)^k (1-\lambda)^{k-1} f^{(k)}(w) \right] (w-v)^k + S_{n,\lambda}(v, w), \end{aligned} \quad (60)$$

where the remainder $S_{n,\lambda}(v, w)$ satisfies the bound

$$\begin{aligned} |S_{n,\lambda}(v, w)| &\leq \frac{1}{n!(n+2)} (1-\lambda)\lambda |w-v|^{n+1} \left[\lambda^n |f^{(n+1)}(v)| \right. \\ &\quad \left. + \frac{1}{(n+1)} \left[\lambda^n + (1-\lambda)^n \right] |f^{(n+1)}((1-\lambda)v + \lambda w)| + (1-\lambda)^n |f^{(n+1)}(w)| \right]. \end{aligned} \quad (61)$$

We also have

$$\begin{aligned} f((1-\lambda)w + \lambda v) &= (1-\lambda)f(v) + \lambda f(w) \\ &+ \sum_{k=1}^n \frac{1}{k!} \left[(1-\lambda)^{k+1} f^{(k)}(v) + (-1)^k \lambda^{k+1} f^{(k)}(w) \right] (w-v)^k + P_{n,\lambda}(v, w), \end{aligned} \quad (62)$$

where the remainder $P_{n,\lambda}(v, w)$ satisfies the bound

$$\begin{aligned} |P_{n,\lambda}(v, w)| &\leq \frac{1}{n!(n+2)} |w-v|^{n+1} \left[(1-\lambda)^{n+2} |f^{(n+1)}(v)| \right. \\ &\quad \left. + \frac{1}{(n+1)} \left[(1-\lambda)^{n+2} + \lambda^{n+2} \right] |f^{(n+1)}((1-\lambda)w + \lambda v)| + \lambda^{n+2} |f^{(n+1)}(w)| \right]. \end{aligned} \quad (63)$$

For $n = 0$, namely if $|f'|$ is convex on D , then by (54) we get

$$\begin{aligned} |f(z) - (1-\lambda)f(v) - \lambda f(w)| &\leq \frac{1}{2} [|1-\lambda| |z-v| |f'(v)| \\ &\quad + [|1-\lambda| |z-v| + |\lambda| |w-z|] |f'(z)| + |\lambda| |w-z| |f'(w)|], \end{aligned} \quad (64)$$

for $z, v, w \in D$ and for all $\lambda \in \mathbb{C}$.

From (57) we get

$$\begin{aligned} \left| f\left(\frac{v+w}{2}\right) - (1-\lambda)f(v) - \lambda f(w) \right| &\leq \frac{1}{4} |w-v| [|1-\lambda| |f'(v)| \\ &+ [|1-\lambda| + |\lambda|] \left| f'\left(\frac{v+w}{2}\right) \right| + |\lambda| |f'(w)|] \end{aligned} \quad (65)$$

for $v, w \in D$ and for all $\lambda \in \mathbb{C}$.

In particular, for $\lambda = \frac{1}{2}$ we get

$$\begin{aligned} \left| f\left(\frac{v+w}{2}\right) - \frac{f(v)+f(w)}{2} \right| \\ \leq \frac{1}{8} |w-v| \left[|f'(v)| + 2 \left| f'\left(\frac{v+w}{2}\right) \right| + |f'(w)| \right] \end{aligned} \quad (66)$$

for $v, w \in D$.

5 Examples for Logarithm and Exponential

Consider the function $f(z) = \text{Log}(z)$ where $\text{Log}(z) = \ln|z| + i \text{Arg}(z)$ and $\text{Arg}(z)$ is such that $-\pi < \text{Arg}(z) \leq \pi$. Log is called the "principal branch" of the complex logarithmic function. The function f is analytic on all of $\mathbb{C}_\ell := \mathbb{C} \setminus \{x+iy : x \leq 0, y = 0\}$ and

$$f^{(k)}(z) = \frac{(-1)^{k-1} (k-1)!}{z^k}, \quad k \geq 1, \quad z \in \mathbb{C}_\ell.$$

Let D be a convex domain in \mathbb{C}_ℓ and assume that $d_D := \inf_{z \in D} |z|$ is a positive and finite number. If $z, v, w \in D \subset \mathbb{C}_\ell$, then by the representation (17) and the inequality (40)

$$\begin{aligned} &|\text{Log}(z) - (1-\lambda)\text{Log}(v) - \lambda\text{Log}(w)| \\ &= \left| \sum_{k=1}^n \frac{1}{k} \left[(1-\lambda)(-1)^{k-1} \frac{(z-v)^k}{v^k} - \lambda \frac{(w-z)^k}{w^k} \right] \right| \\ &\leq \frac{1}{(n+1)d_D^{n+1}} \left(|1-\lambda| |z-v|^{n+1} + |\lambda| |w-z|^{n+1} \right) \end{aligned} \quad (67)$$

for $n \geq 1$ and for $n = 0$ we have

$$|\text{Log}(z) - (1-\lambda)\text{Log}(v) - \lambda\text{Log}(w)| \leq \frac{1}{d_D} (|1-\lambda| |z-v| + |\lambda| |w-z|), \quad (68)$$

for all $\lambda \in \mathbb{C}$.

If $\lambda = \tau \in [0, 1]$, then by (67) we get

$$\begin{aligned} & |\text{Log}(z) - (1 - \tau)\text{Log}(v) - \tau\text{Log}(w) \\ & \quad - \sum_{k=1}^n \frac{1}{k} \left[(1 - \tau)(-1)^{k-1} \frac{(z-v)^k}{v^k} - \tau \frac{(w-z)^k}{w^k} \right]| \\ & \leq \frac{1}{(n+1)d_D^{n+1}} \left((1 - \tau)|z-v|^{n+1} + \tau|w-z|^{n+1} \right) \\ & \leq \frac{1}{(n+1)d_D^{n+1}} \max \left\{ |z-v|^{n+1}, |w-z|^{n+1} \right\} \end{aligned} \quad (69)$$

and for $\tau = \frac{1}{2}$ we get

$$\begin{aligned} & \left| \text{Log}(z) - \frac{\text{Log}(v) + \text{Log}(w)}{2} \right. \\ & \quad \left. - \frac{1}{2} \sum_{k=1}^n \frac{1}{k} \left[(-1)^{k-1} \frac{(z-v)^k}{v^k} - \frac{(w-z)^k}{w^k} \right] \right| \\ & \leq \frac{1}{2(n+1)d_D^{n+1}} \left(|z-v|^{n+1} + |w-z|^{n+1} \right) \end{aligned} \quad (70)$$

for $z, v, w \in D \subset \mathbb{C}_\ell$.

Moreover, if we take $z = \frac{v+w}{2}$ in (70), then we get

$$\begin{aligned} & \left| \text{Log}\left(\frac{v+w}{2}\right) - \frac{\text{Log}(v) + \text{Log}(w)}{2} \right. \\ & \quad \left. - \frac{1}{2} \sum_{k=1}^n \frac{1}{2^k k} \left[\frac{(-1)^{k-1}}{v^k} - \frac{1}{w^k} \right] (w-v)^k \right| \\ & \leq \frac{1}{2^{n+1}(n+1)d_D^{n+1}} |w-v|^{n+1} \end{aligned} \quad (71)$$

for $v, w \in D \subset \mathbb{C}_\ell$.

The case $n = 0$ gives that

$$\left| \text{Log}\left(\frac{v+w}{2}\right) - \frac{\text{Log}(v) + \text{Log}(w)}{2} \right| \leq \frac{1}{2d_D} |w-v| \quad (72)$$

for $v, w \in D \subset \mathbb{C}_\ell$.

For $f(z) = \text{Log}(z)$, $z \in D \subset \mathbb{C}_\ell$, we have

$$|f'(z) - f'(w)| = \left| \frac{1}{z} - \frac{1}{w} \right| = \frac{|w-v|}{|z||w|} \leq \frac{1}{d_D^2} |w-v|$$

showing that f' is Lipschitzian on D with the constant $L = \frac{1}{d_D^2}$.

By the inequality (52) we then get

$$\left| \operatorname{Log} \left(\frac{v+w}{2} \right) - \frac{\operatorname{Log}(v) + \operatorname{Log}(w)}{2} \right| \leq \frac{1}{8d_D^2} |w-v|^2, \quad (73)$$

for $v, w \in D \subset \mathbb{C}_\ell$.

Now consider the exponential function $f(z) = \exp z$. Then

$$|\exp z| = \exp(\operatorname{Re} z)$$

and

$$|\exp((1-t)z + tw)| \leq (1-t)|\exp z| + t|\exp w|$$

for any $z, w \in \mathbb{C}$ and $t \in [0, 1]$, showing that f is convex in absolute value.

Now let D be a convex domain in \mathbb{C} and assume that $E_D := \sup_{z \in D} [\exp(\operatorname{Re} z)] < \infty$. If we use the representation (35) and the inequality (40), we have

$$\begin{aligned} & |\exp z - (1-\lambda)\exp v - \lambda\exp w \\ & - \sum_{k=1}^n \frac{1}{k!} \left[(1-\lambda)(z-v)^k \exp v + (-1)^k \lambda (w-z)^k \exp w \right] \\ & \leq \frac{1}{(n+1)!} E_D \left(|1-\lambda| |z-v|^{n+1} + |\lambda| |w-z|^{n+1} \right) \end{aligned} \quad (74)$$

for all $z, v, w \in D \subset \mathbb{C}$ and $n \geq 1$.

For $n = 0$ we have the simpler inequality

$$|\exp z - (1-\lambda)\exp v - \lambda\exp w| \leq E_D (|1-\lambda| |z-v| + |\lambda| |w-z|) \quad (75)$$

for all $z, v, w \in D \subset \mathbb{C}$.

If $\lambda = \tau \in [0, 1]$, then by (74) we get

$$\begin{aligned} & |\exp z - (1-\tau)\exp v - \tau\exp w \\ & - \sum_{k=1}^n \frac{1}{k!} \left[(1-\tau)(z-v)^k \exp v + (-1)^k \tau (w-z)^k \exp w \right] \\ & \leq \frac{1}{(n+1)!} E_D \left((1-\tau) |z-v|^{n+1} + \tau |w-z|^{n+1} \right) \\ & \leq \frac{1}{(n+1)!} E_D \max \left\{ |z-v|^{n+1}, |w-z|^{n+1} \right\} \end{aligned} \quad (76)$$

for all $z, v, w \in D \subset \mathbb{C}$ and for $\tau = \frac{1}{2}$ we get

$$\left| \exp z - \frac{\exp v + \exp w}{2} - \frac{1}{2} \sum_{k=1}^n \frac{1}{k!} \left[(z-v)^k \exp v + (-1)^k (w-z)^k \exp w \right] \right| \leq \frac{1}{2(n+1)!} E_D \left(|z-v|^{n+1} + |w-z|^{n+1} \right) \quad (77)$$

for all $z, v, w \in D \subset \mathbb{C}$.

Moreover, if we take $z = \frac{v+w}{2}$ in (77), then we get

$$\left| \exp \left(\frac{v+w}{2} \right) - \frac{\exp v + \exp w}{2} - \frac{1}{2} \sum_{k=1}^n \frac{1}{2^k k!} \left[\exp v + (-1)^k \exp w \right] (w-v)^k \right| \leq \frac{1}{2^{n+1} (n+1)!} E_D |w-v|^{n+1} \quad (78)$$

for all $v, w \in D \subset \mathbb{C}$.

The case $n = 0$ gives that

$$\left| \exp \left(\frac{v+w}{2} \right) - \frac{\exp v + \exp w}{2} \right| \leq \frac{1}{2} E_D |w-v| \quad (79)$$

for all $v, w \in D \subset \mathbb{C}$.

The function $f(z) = \exp z$ is Lipschitzian on D with the constant $L = E_D$, then by (52) we get

$$\left| \exp \left(\frac{v+w}{2} \right) - \frac{\exp v + \exp w}{2} \right| \leq \frac{1}{8} E_D |w-v|^2. \quad (80)$$

for all $v, w \in D \subset \mathbb{C}$.

By the convexity in modulus of the complex function and by (66) we also have

$$\left| \exp \left(\frac{v+w}{2} \right) - \frac{\exp v + \exp w}{2} \right| \leq \frac{1}{8} |w-v| \left[\exp(\operatorname{Re} v) + 2 \exp \left(\operatorname{Re} \left(\frac{v+w}{2} \right) \right) + \exp(\operatorname{Re} w) \right] \quad (81)$$

for all $v, w \in \mathbb{C}$.

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