



Notes on the zero-divisor graph and annihilating-ideal graph of a reduced ring

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Abstract

We translate some graph properties of $\mathbb{A}\mathbb{G}(R)$ and $\Gamma(R)$ to some topological properties of Zariski topology. We prove that the facts “(1) The zero ideal of R is an anti fixed-place ideal. (2) $\text{Min}(R)$ does not have any isolated point. (3) $\text{Rad}(\mathbb{A}\mathbb{G}(R)) = 3$. (4) $\text{Rad}(\Gamma(R)) = 3$. (5) $\Gamma(R)$ is triangulated (6) $\mathbb{A}\mathbb{G}(R)$ is triangulated.” are equivalent. Also, we show that if the zero ideal of a ring R is a fixed-place ideal, then $\text{dt}_t(\mathbb{A}\mathbb{G}(R)) = |\mathcal{B}(R)|$ and also if in addition $|\text{Min}(R)| > 2$, then $\text{dt}(\mathbb{A}\mathbb{G}(R)) = |\mathcal{B}(R)|$. Finally, it is shown that $\text{dt}(\mathbb{A}\mathbb{G}(R))$ is finite if and only if $\text{dt}_t(\mathbb{A}\mathbb{G}(R))$ is finite if and only if $\text{Min}(R)$ is finite.

1 Preliminary

Researchers try to define a graph illustration for some kind of mathematical aspects. For example see [3] in the lattice literature, [13] in the measure literature and [16] in the real-valued continuous functions literature. The study of translating graph properties to algebraic properties is an interesting subject for mathematicians. The concept of zero-divisor graph of a commutative was initiated in [17]. In this article the authors let all elements of the commutative ring be vertices of the graph. The authors in [12] studied the zero-divisor graph whose vertices are the non-zero zero-divisor elements. Studying of this graph has been continued in several articles; see [24, 11, 4, 5, 28, 29]. Also, first the

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annihilating-ideal graph is introduced and studied in [18] and then it has been studied in several articles; see [19, 10, 2, 1, 26, 21, 27]. In those researches some graph properties of the graphs of $\mathbb{A}G(R)$ and $\Gamma(R)$ is characterized by some ring properties of R . In this article, we study more graph properties, for example the radius, triangulation and domination and total domination number of these graphs by some ring properties and topological properties of Zariski topology. We find the impossible assumption “ $\Gamma(R)$ is a triangulated graph” in [29, Corollary 3.3] and some mistakes in the proof of [27, Theorem 2.2]. In [27, Theorem 2.2], it is shown that the total domination of $\mathbb{A}G(R)$ is equal to the cardinal of the family all maximal elements of $\mathbb{A}(R)^*$. By giving a counterexample, we show that [27, Theorem 2.2] is not true. Finally, by giving some facts, we correct these mistakes.

In the next section, first we recall some pertinent definitions and then give a retract of the annihilating ideal graph. Sec. 3 is devoted to the translating graph properties of these graphs to Zariski topology. Also, we note an impossible assumption in [29, Corollary 3.3]. In Sec. 4, we characterize the radius of $\Gamma(X)$ and $\mathbb{A}G(X)$. In the last section, the domination number of the annihilating-ideal graph is studied and we note some mistakes in [27, Theorem 2.2].

2 Introduction

Throughout this article, R is a *commutative unitary reduced ring*. By $\text{Spec}(R)$, we mean the set of all prime ideals of R . A *semi-prime ideal* means an ideal which is the intersection of a family of prime ideals. R is called a *reduced ring* if the zero ideal of R is semi-prime. For each ideal I of R and each subset S of R , we denote the ideal $\{x \in R : Sx \subseteq I\}$ by $(I : S)$. When $I = \{0\}$, we write $\text{Ann}(S)$ instead of $(\{0\} : S)$ and call it the *annihilator* of S . Also, we write $\text{Ann}(a)$ instead of $\text{Ann}(\{a\})$. A prime ideal P is said to be a *minimal prime ideal* over an ideal I if there are not any prime ideal strictly contained in P that contains I . By $\text{Min}(I)$, we mean the set of all minimal prime ideals over I . We use $\text{Min}(R)$ instead of $\text{Min}(\{0\})$. A prime ideal P is called a *Bourbaki associated prime divisor* of an ideal I if $(I : x) = P$, for some $x \in R$. We denote the set of all Bourbaki associated prime divisors of an ideal I by $\mathcal{B}(I)$. It is easy to see that $\mathcal{B}(I) \subseteq \text{Min}(I)$, for any ideal I of a ring R . We use $\mathcal{B}(R)$ instead of $\mathcal{B}(\{0\})$. Let I be a semi-prime ideal, then $P_{\circ} \in \text{Min}(I)$ is called *irredundant* with respect to I if $I \neq \bigcap_{P_{\circ} \neq P \in \text{Min}(I)} P$. If I is equal to the intersection of all irredundant ideals with respect to I , then we call it a *fixed-place ideal*. In this case, exactly by [6, Theorem 2.1], we have $I = \bigcap \mathcal{B}(I)$. If $\mathcal{B}(I) = \emptyset$, then I is called an *anti-fixed place ideal*. We use $\mathcal{B}(R)$ instead of $\mathcal{B}(\{0\})$. For more information about the fixed-place ideals and anti fixed-place

ideals, see [6, 7].

Let $G = \langle V(G), E(G) \rangle$ be an undirected graph. A vertex is called a *pendant vertex* if it is adjacent to just one vertex. For each pair of vertices u and v in $V(G)$, the length of the shortest path between u and v , denoted by $d(u, v)$, is called the *distance* between u and v . The *eccentricity* of a vertex u of G is denoted by $\text{ecc}(u)$ and is defined to be maximum of $\{d(u, v) : v \in G\}$. The minimum of $\{\text{ecc}(u) : u \in G\}$, denoted by $\text{Rad}(G)$, is called the *radius* of G . We say G is *triangulated* if each vertex of G is a vertex of some triangle. Two vertices u and v are called *orthogonal* if u and v are adjacent and there are not any vertex which is adjacent to the both vertices u and v . A *graph homomorphism* φ from a graph $G = \langle V(G), E(G) \rangle$ to a graph $H = \langle V(H), E(H) \rangle$ is a map from $V(G)$ to $V(H)$ that $\{u, v\} \in E(G)$ implies $\{f(u), f(v)\} \in E(H)$, for all pairs of vertices $u, v \in V(G)$. A *retraction* is a homomorphism φ from a graph G to a subgraph H of G such that $\varphi(v) = v$, for each vertex $v \in V(H)$. In this case, the subgraph H is called a *retract* of G . A subset D of vertices of a graph is called a *dominating set* if every vertex of the graph is either in D or adjacent to some vertex of D . Also, a *total dominating set* of a graph is a family S of vertices of the graph such that every vertex is adjacent to some vertex of S . The *domination number* and *total domination number* of a graph is the minimum cardinality of dominating and total dominating sets of the graph, respectively. We denote the domination and total domination numbers of a graph G by $\text{dt}(G)$ and $\text{dt}_t(G)$, respectively. For every $u, v \in V(G)$, we denote the length of the shortest cycle containing u and v by $gi(u, v)$.

Suppose that I and a are an ideal and element of R , respectively. If $\text{Ann}(I) \neq \{0\}$, then I is called *annihilating-ideal* and if $\text{Ann}(a) \neq \{0\}$, then a is called a *zero-divisor element*. Let $\mathbb{A}(R)^*$ be the family of all non-zero annihilating-ideals and $Z(R)^*$ be the family of all non-zero zero-divisor elements of R . $\mathbb{A}\mathbb{G}(R)$ is a graph with the vertices $\mathbb{A}(R)^*$ and two distinct vertices I and J are adjacent if $IJ = \{0\}$. Also, $\Gamma(R)$ is a graph with vertices $Z(R)^*$ and two distinct vertices a and b are adjacent if $ab = 0$. $\mathbb{A}\mathbb{G}(R)$ and $\Gamma(R)$ are called the *annihilating-ideal graph* and the *zero-divisor graph* of R , respectively.

Throughout this article, all $Y \subseteq \text{Spec}(R)$ is considered by Zariski topology; i.e., by assuming as a base for the closed sets of Y , the sets $h_Y(a) = \{P \in Y : a \in P\}$. Hence, the closed sets of Y are of the form $h_Y(I) = \bigcap_{a \in I} h_Y(a) = \{P \in Y : I \subseteq P\}$, for some ideal I in R . Also, we set $h_Y^c(I) = Y \setminus h_Y(I)$. When $Y = \text{Min}(R)$, we write h_m instead of h_Y . A point $P \in \text{Spec}(R)$ is called a *quasi-isolated* point, if P is an isolated point of $\text{Min}(R)$. By [23, Theorem 2.3 and Corollary 2.4], the space $\text{Min}(R)$ is a Hausdorff space in which $\{h_m^c(a) : a \in R\}$ is a base of clopen sets.

In this research, $C(X)$ denotes the ring of all real-valued continuous functions on a Tychonoff space X and we abbreviate $\mathbb{A}(C(X))^*$ and $\mathbb{A}\mathbb{G}(C(X))$ by $\mathbb{A}(X)^*$ and $\mathbb{A}\mathbb{G}(X)$, respectively. We denote the set of all isolated points of X , by $I(X)$. A space X is called almost discrete, if $\overline{I(X)} = X$.

The reader is referred to [14, 30, 31, 22, 20] for undefined terms and notations.

For each subset S of R , let P_S be the intersection of all minimal prime ideals containing S . An ideal I in R is said to be *strongly z° -ideal* (or briefly *sz° -ideal*) if $P_F \subseteq I$, for every finite subset F of I . Since the intersection of a family of strong z° -ideals is a strong z° -ideal, the smallest strong z° -ideal containing an ideal I exists, and we denote it by I_{sz° . For more details about the strong z° -ideals, see [25, 9, 8].

Lemma 2.1. *Let I and J be two vertices of $\mathbb{A}\mathbb{G}(R)$. I is adjacent to J if and only if I_{sz° is adjacent to J_{sz° .*

Proof. \Rightarrow). Suppose that $a \in I_{sz^\circ}$ and $b \in J_{sz^\circ}$, then by [8, Proposition 7.5], finite subsets F of I and G of J exist such that $h_m(G) \subseteq h_m(a)$ and $h_m(H) \subseteq h_m(b)$. Since I is adjacent to J , it follows that $IJ = \{0\}$, so $GH = \{0\}$, this implies that $\text{Min}(R) = h_m(GH) = h_m(G) \cup h_m(H) \subseteq h_m(a) \cup h_m(b) = h_m(ab)$, thus $h_m(ab) = \text{Min}(R)$, hence $ab \in kh_m(ab) = \{0\}$, and, therefore $ab = 0$. This shows that $I_{sz^\circ}J_{sz^\circ} = \{0\}$ and thus I_{sz° is adjacent to J_{sz° .

\Leftarrow). It is clear. □

Proposition 2.2. *The family of all sz° -ideals of $\mathbb{A}(R)^*$ is a retract of $\mathbb{A}\mathbb{G}(R)$.*

Proof. Suppose that $I \in \mathbb{A}(R)^*$, so $J \in \mathbb{A}(R)^*$ exists such that $IJ = \{0\}$. By Lemma 2.1, we have I_{sz° is adjacent to J_{sz° . Since $0 \neq I \subseteq I_{sz^\circ} \subseteq \text{Ann}(J_{sz^\circ}) \subseteq \text{Ann}(J) \neq R$, it follows that $I_{sz^\circ} \in \mathbb{A}(R)^*$. This shows that φ from $\mathbb{A}(R)^*$ to the family of all sz° -ideals of $\mathbb{A}(R)^*$, defined by $\varphi(I) = I_{sz^\circ}$, is a retraction, and, therefore the family of all sz° -ideals of $\mathbb{A}(R)^*$ is a retract of $\mathbb{A}\mathbb{G}(R)$. □

3 Zariski Topology

In this section, we give Zariski topological characterization of elements of $\Gamma(R)$ and $\mathbb{A}\mathbb{G}(R)$, then we characterize the adjacency, distance, orthogonality, eccentricity and triangulation. Also, it is shown that the radius of these graphs are strictly greater than 1.

Lemma 3.1. *Let $Y \subseteq \text{Spec}(R)$ and $\bigcap Y = \{0\}$. Then*

(a) *$a = 0$ if and only if $h_Y(a) = Y$, for each $a \in R$.*

- (b) $\text{Ann}(a) \neq 0$ if and only if $\overline{h_Y^c(a)} \neq Y$, for each $a \in R$.
- (c) $I = \{0\}$ if and only if $h_Y(I) = Y$, for each ideal I of R .
- (d) I is an annihilating-ideal if and only if $\overline{h_Y^c(I)} \neq Y$.
- (e) For each $a, b \in Z(R)^*$, a is adjacent to b if and only if $h_Y^c(a) \cap h_Y^c(b) = \emptyset$.
- (f) For each $I, J \in \mathbb{A}(R)^*$, I is adjacent to J if and only if $h_Y^c(I) \cap h_Y^c(J) = \emptyset$.

Proof. (a) and (c). Since $\bigcap Y = \{0\}$, they are clear.

(b). Since $\text{Ann}(a) = kh_Y^c(a)$, it follows that $\text{Ann}(a) \neq \{0\}$ if and only if $kh_Y^c(a) \neq \{0\}$, and it is equivalent to say that $kh_Y^c(a) \neq Y$, because $\bigcap Y = \{0\}$, and, therefore they are equivalent to $\overline{h_Y^c(I)} \neq Y$.

(d). The proof is analogously similar to the proof of part (b).

(e) and (f). They are evident. \square

In [29, Proposition 2.2], the concept of distance in $\Gamma(R)$ is characterized by Zariski topology on $\text{Spec}(R)$. In the following proposition, we generalize this characterization by every reduced family of prime ideals and also we characterize the concept of distance in $\mathbb{A}\mathbb{G}$.

Proposition 3.2. *Let $I, J \in \mathbb{A}(R)^*$, $a, b \in Z(R)^*$, $Y \subseteq \text{Spec}(R)$ and $\bigcap Y = \{0\}$. Then*

- (a) $d(a, b) = 1$ if and only if $h_Y^c(a) \cap h_Y^c(b) = \emptyset$.
- (b) $d(a, b) = 2$ if and only if $h_Y^c(a) \cap h_Y^c(b) \neq \emptyset$ and $h_Y^c(a) \cup h_Y^c(b)$ is not dense in Y .
- (c) $d(a, b) = 3$ if and only if $h_Y^c(a) \cap h_Y^c(b) \neq \emptyset$ and $h_Y^c(a) \cup h_Y^c(b)$ is dense in Y .
- (d) $d(I, J) = 1$ if and only if $h_Y^c(I) \cap h_Y^c(J) = \emptyset$.
- (e) $d(I, J) = 2$ if and only if $h_Y^c(I) \cap h_Y^c(J) \neq \emptyset$ and $h_Y^c(I) \cup h_Y^c(J)$ is not dense in Y .
- (f) $d(I, J) = 3$ if and only if $h_Y^c(I) \cap h_Y^c(J) \neq \emptyset$ and $h_Y^c(I) \cup h_Y^c(J)$ is dense in Y .

Proof. (a) and (d). They are clear, by Lemma 3.1.

(b \Rightarrow). By Lemma 3.1, we have $h_Y^c(a) \cap h_Y^c(b) \neq \emptyset$. By the assumption, there is an element $c \in Z(R)^*$ such that c is adjacent to the both vertices a and b . Now Lemma 3.1 implies that

$$h_Y^c(a) \cap h_Y^c(c) = h_Y^c(a) \cap h_Y^c(c) = \emptyset \quad \Rightarrow \quad h_Y^c(a) \cup h_Y^c(b) \subseteq h_Y(c) \quad (*)$$

Since $c \neq 0$, by Lemma 3.1, it follows that $h_Y(c) \neq Y$. Since $h_Y(c)$ is closed, by (*), it follows that $h_Y^c(a) \cup h_Y^c(b)$ is not dense in Y .

(b \Leftarrow). Part (a) concludes that $d(a, b) > 1$. Since $\{h_Y^c(c) : c \in R\}$ is a base for Zariski topology, by the assumption, it follows that there is some $c \in R$ such that $h_Y^c(a) \cup h_Y^c(b) \subseteq h_Y(c) \subset Y$, so $h_Y^c(a) \cap h_Y^c(c) = h_Y^c(a) \cap h_Y^c(c) = \emptyset$, $Y \neq h_Y(a)$ and $\overline{h_Y^c(c)} \neq Y$, hence $c \in Z(R)^*$ and c is adjacent to the both vertices a and b and thus $d(a, b) = 2$.

(c). It deduces from parts (a), (b) and [12, Theorem 2.2].

(e). By the fact that $\{h_Y^c(K) : K \text{ is an ideal of } R\}$ is a base for Zariski topology, it is similar to part (b)

(f). It concludes from parts (d), (e) and [18, Theorem 7.1]. \square

Theorem 3.3. *Let $I, J \in \mathbb{A}(R)^*$, $a, b \in Z(R)^*$, $Y \subseteq \text{Spec}(R)$ and $\bigcap Y = \{0\}$. Then*

- (a) *Two vertices I and J are orthogonal if and only if $h_Y^c(I) \cap h_Y^c(J) = \emptyset$ and $\overline{h_Y^c(I) \cup h_Y^c(J)} = Y$.*
- (b) *Two vertices a and b are orthogonal if and only if $h_Y^c(a) \cap h_Y^c(b) = \emptyset$ and $\overline{h_Y^c(a) \cup h_Y^c(b)} = Y$.*

Proof. (a \Rightarrow). By the assumption and Lemma 3.1, we have I is adjacent to J and thus $h_Y^c(I) \cap h_Y^c(J) = \emptyset$. If $\overline{h_Y^c(I) \cup h_Y^c(J)} \neq Y$, since $\{h_Y^c(K) : K \text{ is an ideal of } R\}$ is a base for Zariski topology, it follows that there is some ideal K of R such that $h_Y^c(K) \cap [\overline{h_Y^c(I) \cup h_Y^c(J)}] = \emptyset$, so $h_Y^c(K) \cap h_Y^c(I) = h_Y^c(K) \cap h_Y^c(J) = \emptyset$, $h_Y^c(K) \neq Y$ and $\overline{h_Y^c(K)} \neq Y$, hence $K \in \mathbb{A}(R)^*$, by Lemma 3.1, and K is adjacent to the both vertices I and J , by Lemma 3.1, which contradicts the assumption and thus $\overline{h_Y^c(I) \cup h_Y^c(J)} = Y$.

(a \Leftarrow). By the assumption and Lemma 3.1, we have $h_Y^c(I) \cap h_Y^c(J) = \emptyset$. On the contrary, suppose that there is an $K \in \mathbb{A}(R)^*$ such that K is adjacent to the both vertices I and J , then $h_Y^c(K) \cap [h_Y^c(I) \cup h_Y^c(J)] = [h_Y^c(K) \cap h_Y^c(I)] \cup [h_Y^c(K) \cap h_Y^c(J)] = \emptyset$, by Lemma 3.1. Since $K \in \mathbb{A}(R)^*$, by Lemma 3.1, it follows that $h_Y^c(K) \neq \emptyset$, and, therefore $\overline{h_Y^c(I) \cup h_Y^c(J)} \neq Y$, which contradicts the assumption.

(b). By the fact that $\{h_Y^c(c) : c \in R\}$ is a base for Zariski topology, the proof is similar to part (a). \square

Suppose that $\bigcap Y = \{0\}$. Since for every $I \in \mathbb{A}(R)^*$, it follows that I and $\text{Ann}(I)$ are orthogonal and thus, we have $\overline{h_Y^c(I) \cap h_Y^c(\text{Ann}(I))} = Y$, by the above theorem. Similarly, for every $a \in Z(R)^*$ and $b \in \text{Ann}(a)$, we have $\overline{h_Y^c(a) \cup h_Y^c(b)} = Y$.

Theorem 3.4. *Suppose that $I \in \mathbb{A}(R)^*$, $a \in Z(R)^*$, $Y \subseteq \text{Min}(R)$ and $\bigcap Y = \{0\}$. Then*

- (a) $\text{ecc}(I) > 1$, for every $I \in \mathbb{A}(R)^*$.
- (b) $\text{ecc}(I) = 2$ if and only if $h_Y^c(I)$ is singleton.
- (c) $\text{ecc}(I) = 3$ if and only if $h_Y^c(I)$ is not singleton.
- (d) $\text{ecc}(a) > 1$, for every $a \in Z(R)^*$.
- (e) $\text{ecc}(a) = 2$ if and only if $h_Y^c(a)$ is singleton.
- (f) $\text{ecc}(a) = 3$ if and only if $h_Y^c(a)$ is not singleton.

Proof. Since R is not an integral domain and $\bigcap Y = \{0\}$, it follows that $|Y| \geq 2$.

(c \Rightarrow). By the assumption there is some $J \in \mathbb{A}(R)^*$ such that $d(I, J) = 3$. Lemma 3.2 implies that $h_Y^c(I) \cap h_Y^c(J) = \emptyset$ and $\overline{h_Y^c(I) \cup h_Y^c(J)} = Y$. On the contrary, suppose that $h_Y^c(I)$ is singleton, then $h_Y^c(I) \subseteq h_Y^c(J)$, and, therefore $\overline{h_Y^c(J)} = \overline{h_Y^c(I) \cup h_Y^c(J)} = Y$. Hence $J \notin \mathbb{A}(R)^*$, by Lemma 3.1, which is a contradiction.

(c \Leftarrow). By the assumption, there are distinct prime ideals P and Q in $h_Y^c(I)$. Since $Y \subseteq \text{Min}(R)$ is Hausdorff and $\{h_Y^c(K) : K \text{ is an ideal of } R\}$ is a base for Y , there are ideals J and K such that $h_Y^c(J), h_Y^c(K) \subseteq h_Y^c(I)$, $P \in h_Y^c(J)$, $Q \in h_Y^c(K)$ and $h_Y^c(J) \cap h_Y^c(K) = \emptyset$. Thus

$$\begin{aligned} h_Y^c(J + \text{Ann}(I)) \cap h_Y^c(K) &= [h_Y^c(J) \cup h_Y^c(\text{Ann}(I))] \cap h_Y^c(K) \\ &\subseteq [h_Y^c(J) \cap h_Y^c(K)] \cup [h_Y^c(\text{Ann}(I)) \cap h_Y^c(K)] = \emptyset. \end{aligned}$$

Hence $h_Y^c(J + \text{Ann}(I)) \neq Y$ and $\overline{h_Y^c(J + \text{Ann}(I))} \neq Y$, and, therefore we have $J + \text{Ann}(I) \in \mathbb{A}(R)^*$. Since

$$h_Y^c(I) \cap h_Y^c(J + \text{Ann}(I)) \supseteq h_Y^c(I) \cap h_Y^c(J) = h_Y^c(I) \neq \emptyset$$

and

$$\overline{h_Y^c(I) \cap h_Y^c(J + \text{Ann}(I))} \supseteq \overline{h_Y^c(I) \cap h_Y^c(\text{Ann}(I))} = Y,$$

by Proposition 3.2, we have $d(I, J + \text{Ann}(I)) = 3$, and, therefore $\text{ecc}(I) = 3$, by [18, Theorem 7.1].

(a). Suppose that there is some $I \in \mathbb{A}(R)^*$ such that $\text{ecc}(I) = 1$. Part (c) concludes that $h_Y^c(I)$ is singleton, so there is some $P \in Y$ such that $h_Y^c(I) = \{P\}$, hence $\text{Ann}(I) = P$ and thus $\{0\} \neq I \subseteq \text{Ann}(P)$. Since $\text{ecc}(I) = 1$, it follows that I is adjacent to $\text{Ann}(P)$, consequently $I\text{Ann}(P) = \{0\}$, this implies that for every $a \in I$, we have $a^2 \in I\text{Ann}(P) = \{0\}$, and, therefore $a^2 = 0$. Since R is reduced, it follows that $a = 0$ and consequently $I = \{0\}$, which is a contradiction.

(b). By parts (a), (c) and [18, Theorem 7.1], it is clear.

The proof of parts (d), (e) and (f) are similar to parts (a), (b) and (c), respectively. \square

The following corollary is an immediate consequence of the above theorem.

Corollary 3.5. $\text{Rad}\Gamma(R) > 1$ and $\text{Rad}\mathbb{A}\mathbb{G}(R) > 1$.

Lemma 3.6. *Let $a \in Z(R)^*$, $I \in \mathbb{A}(R)^*$, $Y \subseteq \text{Min}(R)$ and $\bigcap Y = \{0\}$. Then*

- (a) *a is a vertex of a triangle if and only if $h_Y(a)$ is not singleton.*
- (b) *I is a vertex of a triangle if and only if $h_Y(I)^\circ$ is not singleton.*

Proof. (a \Rightarrow). By the assumption, there are vertices $b, c \in \mathbb{A}(R)^*$ such that a, b and c are pairwise distinct adjacent vertices. Thus $h_Y^c(a)$, $h_Y^c(b)$ and $h_Y^c(c)$ are pairwise disjoint nonempty sets, by Lemma 3.1, hence $h_Y^c(b) \cup h_Y^c(c) \subseteq h_Y(a)$ and $|h_Y^c(b) \cup h_Y^c(c)| \geq 2$. Since $h_Y^c(b) \cup h_Y^c(c)$ is open, it follows that $h_Y(a)$ is not singleton.

(a \Leftarrow). Suppose that P and Q are distinct elements of $h_Y(a)$. Since $Y \subseteq \text{Min}(R)$ is Hausdorff, $h_Y(a)$ is open and $\{h_Y^c(x) : x \in R\}$ is a base for Y , it follows that there are $b, c \in R$ such that $P \in h_Y^c(b) \subseteq h_Y(a)$, $Q \in h_Y^c(c) \subseteq h_Y(a)$ and $h_Y^c(b) \cap h_Y^c(c) = \emptyset$, and, therefore $h_Y^c(a)$, $h_Y^c(b)$ and $h_Y^c(c)$ are pairwise disjoint nonempty sets which are not dense in Y . Now Lemma 3.1 implies that $b, c \in \mathbb{A}(R)^*$ and Theorem 3.1 concludes that a, b and c are pairwise distinct adjacent vertices and thus a is a vertex of a triangle.

(b). It is similar to part (a). \square

Proposition 3.7. *Suppose that $a, b \in Z(R)^*$ are not pendant vertices, $Y \subseteq \text{Min}(R)$ and $\bigcap Y = \{0\}$. Then*

- (a) *$h_Y^c(a) \cap h_Y^c(b) = \emptyset$ and $\overline{h_Y^c(a) \cup h_Y^c(b)} \neq Y$ if and only if $\text{gi}(a, b) = 3$.*
- (b) *If $2 \notin Z(R)$, $h_Y^c(a) \cap h_Y^c(b) = \emptyset$ and $\overline{h_Y^c(a) \cup h_Y^c(b)} = Y$, then $\text{gi}(a, b) = 4$.*
- (c) *Suppose that $h_Y^c(a) \cap h_Y^c(b) \neq \emptyset$. Then $\overline{h_Y^c(a) \cup h_Y^c(b)} \neq Y$ if and only if $\text{gi}(a, b) = 4$.*

- (d) Suppose that $2 \notin Z(R)$ and $h_Y^c(a) \cap h_Y^c(b) \neq \emptyset$. Then $\overline{h_Y^c(a) \cup h_Y^c(b)} = Y$ if and only if $\text{gi}(a, b) = 6$.

Proof. By Lemma 3.1, the proof is similar to [29, Theorem 3.4]. \square

Theorem 3.8. Suppose that $I, J \in \mathbb{A}(R)^*$ and they are not pendant vertices. The following statements hold.

- (a) $h_Y^c(I) \cap h_Y^c(J) = \emptyset$ and $\overline{h_Y^c(I) \cup h_Y^c(J)} \neq Y$ if and only if $\text{gi}(I, J) = 3$.
- (b) If $h_Y^c(I) \cap h_Y^c(J) = \emptyset$ and $\overline{h_Y^c(I) \cup h_Y^c(J)} = Y$, then $\text{gi}(I, J) = 4$.
- (c) If $h_Y^c(I) \cap h_Y^c(J) \neq \emptyset$ and $\overline{h_Y^c(I)} = \overline{h_Y^c(J)}$, then $\text{gi}(I, J) = 4$.
- (d) If $h_Y^c(I) \cap h_Y^c(J) \neq \emptyset$ and $\overline{h_Y^c(I)} \neq \overline{h_Y^c(J)}$ and $Y \setminus \overline{h_Y^c(I) \cup h_Y^c(J)}$ is not singleton, then $\text{gi}(I, J) = 4$.
- (e) If $h_Y^c(I) \cap h_Y^c(J) \neq \emptyset$, $\overline{h_Y^c(I)} \neq \overline{h_Y^c(J)}$ and $Y \setminus \overline{h_Y^c(I) \cup h_Y^c(J)}$ is singleton, then $4 \leq \text{gi}(I, J) \leq 5$.
- (f) If $\text{gi}(I, J) = 5$, then $h_Y^c(I) \cap h_Y^c(J) \neq \emptyset$, $\overline{h_Y^c(I)} \neq \overline{h_Y^c(J)}$ and $Y \setminus \overline{h_Y^c(I) \cup h_Y^c(J)}$ is singleton.

Proof. (a \Rightarrow). By Lemma 3.1, I is adjacent to J and by Theorem 3.3, I and J are not orthogonal. Thus $\text{gi}(I, J) = 3$.

(a \Leftarrow). By the assumption, I is adjacent to J and, I and J are not orthogonal, so by Lemma 3.1, we have $h_Y^c(I) \cap h_Y^c(J) = \emptyset$ and, by Proposition 3.3, we have $\overline{h_Y^c(I) \cup h_Y^c(J)} \neq Y$.

(b). By the assumption, we have $IJ = \{0\}$ and we can see easily that $h_Y(I)^\circ \cap h_Y(J)^\circ = \emptyset$. We know that $h_Y^c(\text{Ann}(I)) \subseteq h_Y(I)^\circ$ and $h_Y^c(\text{Ann}(J)) \subseteq h_Y(J)^\circ$, so $h_Y^c(\text{Ann}(I)) \cap h_Y^c(\text{Ann}(J)) = \emptyset$. Now Lemma 3.1 concludes that $\text{Ann}(I)\text{Ann}(J) = \{0\}$. Since I and J are not pendant vertices, there are $I_1, J_1 \in \mathbb{A}(X)^*$ such that I is adjacent to $I_1 \neq J$ and J is adjacent to $J_1 \neq I$, so $II_1 = JJ_1 = \{0\}$, thus $I_1 \subseteq \text{Ann}(I)$ and $J_1 \subseteq \text{Ann}(J)$, hence $I_1J_1 \subseteq \text{Ann}(I)\text{Ann}(J) = \{0\}$, and, therefore $I_1J_1 = \{0\}$. Consequently, $I - J - J_1 - I_1 - I$ is a cycle. The assumption and part (a) conclude that $\text{gi}(I, J) > 3$ and thus $\text{gi}(I, J) = 4$.

(c). We conclude from the assumption and part (a) that $\text{gi}(I, J) \geq 4$. Clearly $\text{Ann}(I), \text{Ann}(J) \in \mathbb{A}(R)^*$. Since $\overline{h_Y^c(I)} = \overline{h_Y^c(J)}$, it follows that $h_Y^c(I) \cap h_Y^c(\text{Ann}(J)) \subseteq \overline{h_Y^c(I)} \cap h_Y(J)^\circ = \overline{h_Y^c(I)} \cap \overline{h_Y^c(J)}^c = \overline{h_Y^c(I)} \cap \overline{h_Y^c(I)}^c = \emptyset$, so, by Lemma 3.1, we have $I\text{Ann}(J) = \{0\}$. Similarly, we can show that $J\text{Ann}(I) = \{0\}$. If $\text{Ann}(I) \neq \text{Ann}(J)$, then $I - \text{Ann}(I) - J - \text{Ann}(J) - I$ is a cycle, and, therefore $\text{gi}(I, J) = 4$. Now we suppose that $\text{Ann}(I) = \text{Ann}(J)$. Since I is adjacent to $\text{Ann}(I)$ and I is not a pendant vertex, it follows there

is some vertex $I_1 \in \mathbb{A}(X)^*$ distinct from $\text{Ann}(I)$ such that I is adjacent to I_1 , then $I_1 I = \{0\}$, so $I_1 \subseteq \text{Ann}(I) = \text{Ann}(J)$, and, therefore $I_1 J = \{0\}$. Consequently, $I - \text{Ann}(I) = \text{Ann}(J) - J - I_1 - I$ is a cycle and thus $\text{gi}(I, J) = 4$.

(d). We can conclude from the assumption and part (a) that $\text{gi}(I, J) \geq 4$. Since $\{h_Y^c(K) : K \text{ is an ideal of } R\}$ is a base for Y , Y is Hausdorff and $Y \setminus \overline{h_Y^c(I) \cap h_Y^c(J)}$ is not singleton, it follows that there are two distinct ideals K_1 and K_2 such that $h_Y^c(K_1) \cap (h_Y^c(I) \cup h_Y^c(J)) = h_Y^c(K_2) \cap (h_Y^c(I) \cup h_Y^c(J)) = \emptyset$. Hence $h_Y^c(I) \cap h_Y^c(K_1) = h_Y^c(K_1) \cap h_Y^c(J) = h_Y^c(J) \cap h_Y^c(K_2) = h_Y^c(K_2) \cap h_Y^c(I) = \emptyset$. Then, by Lemma 3.1, we have $K_1, K_2 \in \mathbb{A}(R)^*$ and $I - K_1 - J - K_2 - I$ is a cycle. Consequently, $\text{gi}(I, J) = 4$.

(e). By part (a), we have $\text{gi}(I, J) \geq 4$. Since $Y \setminus \overline{h_Y^c(I) \cup h_Y^c(J)} \neq Y$ and $\{h_Y^c(K) : K \text{ is an ideal of } R\}$ is a base for Y , it follows that there is some ideal K_1 of R such that $h_Y^c(K_1) \cap [h_Y^c(I) \cup h_Y^c(J)] = \emptyset$, so $h_Y^c(K_1) \cap h_Y^c(I) = h_Y^c(K_1) \cap h_Y^c(J) = \emptyset$. By Lemma 3.1, we have $K_1 \in \mathbb{A}(R)^*$ and K_1 is adjacent to the both vertices I and J . If there is some $K_2 \in \mathbb{A}(R)^*$ distinct from K_1 such that $h_Y^c(K_1) = h_Y^c(K_2)$, then K_2 also is adjacent to the both vertices I and J . Thus $\text{gi}(I, J) = 4$. Now suppose that $h_Y^c(K) = h_Y^c(K_1)$ implies that $K = K_1$. If $h_Y^c(I) \subseteq \overline{h_Y^c(J)}$, then $\overline{h_Y^c(I)} \subseteq \overline{h_Y^c(J)}$, so $Y \setminus \overline{h_Y^c(I) \cup h_Y^c(J)} = Y \setminus \overline{h_Y^c(J)}$, and, therefore, by the assumption, $Y \setminus \overline{h_Y^c(J)}$ is singleton. Since J is not a pendant vertex, there is some vertex K_2 such that K_2 is adjacent to J , thus, by Lemma 3.1, we have $\overline{h_Y^c(K_2) \cap h_Y^c(J)} = \emptyset$, so $h_Y^c(K_2) \cap \overline{h_Y^c(J)} = \emptyset$ and thus $h_Y^c(K_2) \subseteq Y \setminus \overline{h_Y^c(J)}$. By Lemma 3.1, we have $h_Y^c(K_2) \neq \emptyset$ and therefore $h_Y^c(K_2) = Y \setminus \overline{h_Y^c(J)}$. Similarly, we can show that $h_Y^c(K_1) = Y \setminus \overline{h_Y^c(J)}$, hence $h_Y^c(K_1) = h_Y^c(K_2)$, which is a contradiction. Hence $h_Y^c(I) \not\subseteq \overline{h_Y^c(J)}$. Similarly, one can show that $h_Y^c(J) \not\subseteq \overline{h_Y^c(I)}$ and thus $h_Y^c(I) \setminus \overline{h_Y^c(J)}$ and $h_Y^c(J) \setminus \overline{h_Y^c(I)}$ are disjoint nonempty open sets. Since $\{h_Y^c(K) : K \text{ is an ideal of } R\}$ is a base for Y , there are distinct ideals K_2 and K_3 such that $h_Y^c(K_2) \subseteq h_Y^c(I) \setminus \overline{h_Y^c(J)}$ and $h_Y^c(K_3) \subseteq h_Y^c(J) \setminus \overline{h_Y^c(I)}$. Consequently, $h_Y^c(J) \cap h_Y^c(K_2) = h_Y^c(K_2) \cap h_Y^c(K_3) = h_Y^c(K_3) \cap h_Y^c(I) = \emptyset$. By Lemma 3.1, we have $K_2, K_3 \in \mathbb{A}(R)^*$ and $I - K_1 - J - K_2 - K_3 - I$ is a cycle, and, therefore $\text{gi}(I, J) \leq 5$.

(f). It is clear, by parts (a)-(e). \square

Suppose that $R = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$, $I = \{0\} \times \mathbb{Z} \times \mathbb{Z} \times \{0\}$, $J = \mathbb{Z} \times \{0\} \times \mathbb{Z} \times \{0\}$, $R' = \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$, $I' = \{0\} \times \mathbb{R} \times \mathbb{R} \times \{0\}$ and $J' = \mathbb{R} \times \{0\} \times \mathbb{R} \times \{0\}$. Then the both pair vertices $I, J \in \mathbb{A}(R)^*$ and $I', J' \in \mathbb{A}(R')^*$ satisfy the conditions of part (e) of the above theorem but it is seen readily that $\text{gi}(I, J) = 4$ and $\text{gi}(I', J') = 5$.

Now we can conclude the following corollary from the above theorem and [6, Corollary 4.2].

Corollary 3.9. *If there are $I, J \in \mathbb{A}(R)^*$ such that $\text{gi}(I, J) = 5$, then the*

following equivalent conditions hold

- (a) $\text{Min}(R)$ has an isolated point.
- (b) $\mathcal{B}(R) \neq \emptyset$.

4 Radius and Triangulation

This section is devoted to study of the radius and the triangulation of $\Gamma(R)$ and $\mathbb{A}\mathbb{G}(R)$. We show that the concept of the anti fixed-place ideal plays the main role in this studying.

Theorem 4.1. *The following statements are equivalent.*

- (a) $\text{Rad}\Gamma(R) = 3$.
- (b) $\text{Rad}\mathbb{A}\mathbb{G}(R) = 3$.
- (c) *The zero ideal of R is an anti fixed-place ideal.*
- (d) *The $\text{Min}(R)$ does not have any isolated point.*

Proof. (a) \Rightarrow (b). Suppose that $\text{Rad}\mathbb{A}\mathbb{G}(R) \neq 3$, then, by Corollary 3.5 and [19, Theorem 1.4], there is some $I \in \mathbb{A}(R)^*$ such that $\text{ecc}(I) = 2$, hence, by Theorem 3.4, there is some $P \in \text{Min}(R)$ such that $h_m^c(I) = \{P\}$ and thus $\text{Ann}(I) = P$. Set $0 \neq a \in I$, then $\emptyset \neq h_m^c(a) \subseteq h_m^c(I) = \{P\}$, so $h_m^c(a) = \{P\}$, and, therefore $\text{ecc}(a) = 2$, by Theorem 3.4. Consequently, $\text{Rad}\Gamma(R) \neq 3$.

(b) \Rightarrow (c). Suppose that the zero ideal of R is not an anti fixed-place ideal, then there is an affiliated prime ideal P , hence $a \in Z(R)^*$ exists such that $\text{Ann}(a) = P$, this implies that $\langle a \rangle \in \mathbb{A}(R)^*$ and $h_m^c(\langle a \rangle) = h_m^c(a) = \{P\}$, and, therefore $\text{Rad}\mathbb{A}\mathbb{G}(R) \neq 3$, by Theorem 3.4.

(c) \Rightarrow (a). Suppose that $\text{Rad}\Gamma(R) \neq 3$. Then Corollary 3.5 and [18] imply that some $a \in Z(R)^*$ exists such that $\text{ecc}(a) = 2$, hence, by Theorem 3.5, there is some $P \in \text{Min}(R)$ such that $h_m^c(a) = \{P\}$, thus $\text{Ann}(a) = P$, hence P is an affiliated prime ideal, so $P \in \mathcal{B}(R) \neq \emptyset$, and, therefore the zero ideal of R is not an anti fixed-place ideal.

(c) \Leftrightarrow (d). It implies from [6, Corollary 4.3]. \square

The following corollary is an immediate consequence of [19, Theorem 1.4], the above theorem and Corollary 3.5.

Corollary 4.2. *The following statements are equivalent.*

- (a) $\text{Rad}\Gamma(R) = 2$.

- (b) $\text{Rad}\mathbb{A}\mathbb{G}(R) = 2$.
- (c) *The zero ideal of R is not an anti fixed-place ideal.*
- (d) *The $\text{Min}(R)$ has an isolated point.*

Now we can conclude the following corollary from the above theorem and corollary.

Corollary 4.3. $\text{Rad}\Gamma(R) = \text{Rad}\mathbb{A}\mathbb{G}(R)$.

Corollary 4.4. *Suppose that X is a Tychonoff topological space. Then*

$$\text{Rad}\Gamma(X) = \text{Rad}\mathbb{A}\mathbb{G}(X) = \begin{cases} 2 & \text{If } X \text{ has an isolated point.} \\ 3 & \text{If } X \text{ does not have any isolated point.} \end{cases}$$

Proof. It concludes from [6, Corollary 5.4], Theorem 4.1 and Corollary 4.2. \square

Theorem 4.5. *The following statements are equivalent.*

- (a) *The zero ideal of R is an anti fixed-place ideal.*
- (b) $\Gamma(R)$ *is triangulated.*
- (c) $\text{Min}(R)$ *does not have any isolated point.*

Proof. (a) \Rightarrow (b). Suppose that $\Gamma(R)$ is not triangulated, then $a \in Z(R)^*$ exists such that a is not a vertex of any triangle, so, by Lemma 3.6, we have $h_m(a)$ is singleton, hence there is a $P \in \text{Min}(R)$ such that $h_m(a) = \{P\}$. Since $h_m(a)$ is open and $\{h_m^c(x) : x \in R\}$ is a base for Y , it follows that $b \in R$ exists such that $P \in h_m^c(b) \subseteq h_m(a) = \{P\}$, thus $h_m^c(b) = \{P\}$, and, therefore $\text{Ann}(b) = P$. It shows that P is an affiliated prime ideal, hence $P \in \mathcal{B}(R) \neq \emptyset$ and consequently the zero ideal is not an anti fixed-place ideal.

(b) \Rightarrow (c). By [29, Theorem 3.1], the space $\text{Spec}(R)$ does not have any quasi-isolated point, i.e., $\text{Min}(R)$ does not have any isolated point.

(c) \Rightarrow (a). It concludes from [6, Corollary 4.3]. \square

Theorem 4.6. *The following statements are equivalent.*

- (a) *The zero ideal of R is an anti fixed-place ideal.*
- (b) $\mathbb{A}\mathbb{G}(R)$ *is triangulated.*
- (c) $\text{Min}(R)$ *does not have any isolated point.*

Proof. (a) \Rightarrow (b). It is similar to the proof of part (a) \Rightarrow (b) of the previous theorem.

(b) \Rightarrow (a). Suppose that the zero ideal of R is not an anti fixed-place ideal. Then $P \in \mathcal{B}(R) \neq \emptyset$ exists, hence P is an affiliated prime ideal, so there is some $a \in R$ such that $\text{Ann}(a) = P$ and thus $h_m^c(a) = \{P\}$. This implies that $\{P\}$ is open in $\text{Min}(R)$, therefore $h_m^c(P) = \text{Min}(R) \setminus \{P\}$ is closed and consequently $\overline{h_m^c(P)} = \text{Min}(R) \setminus \{P\}$. Thus $h_m(P)^\circ = \left(\overline{h_m^c(P)}\right)^c = \{P\}$. Now Lemma 3.6 concludes that P is not a vertex of any triangle, and, therefore $\mathbb{A}\mathbb{G}(R)$ is not triangulated.

(a) \Leftrightarrow (c). It is clear, by [6, Corollary 4.3]. \square

In [29, Corollary 3.3], it is asserted that “Let R be a reduced ring and let $\text{Spec}(R)$ be finite. Then $\Gamma(R)$ is a triangulated graph if and only if $\text{Spec}(R)$ has no isolated points.”. If $\text{Spec}(R)$ is finite, then $\text{Min}(R)$ is finite, so the zero ideal of R is fixed-place, and, therefore it is not anti fixed-place. Hence, by the above theorem, $\Gamma(R)$ is not triangulated. Hence the assumption “ $\Gamma(R)$ is a triangulated graph” in this assertion is not possible.

Now we can conclude the following corollary from the above theorems.

Corollary 4.7. $\Gamma(R)$ is triangulated if and only if $\mathbb{A}\mathbb{G}(R)$ is triangulated.

Now we can conclude easily from Theorem 4.5 and [6, Corollary 5.4] that $\Gamma(X)$ is triangulated if and only if X does not have any isolated point. This fact is shown in [15, Proposition 2.1]. Also, we can conclude easily from Theorem 4.6 and [6, Corollary 5.4] that $\mathbb{A}\mathbb{G}(X)$ is triangulated if and only if X does not have any isolated point. This fact also is shown in [16, Theorem 4.5].

If $\text{Min}(R)$ is finite, then the zero ideal of R is fixed-place, hence it is not anti fixed-place, and, therefore $\text{Rad}\Gamma(R) = \text{Rad}\mathbb{A}\mathbb{G}(R) = 2$, by Corollary 4.2.

Suppose that D is an integral domain and R be an arbitrary ring. Then $\{0\} \times R \in \mathcal{B}(D \times R) \neq \emptyset$, so the zero ideal of $D \times R$ is not an anti fixed-place ideal, thus, by Corollary 4.2 and Theorems 4.5 and 4.6, $\text{Rad}\Gamma(R) = \text{Rad}\mathbb{A}\mathbb{G}(R) = 2$ and the graphs $\mathbb{A}\mathbb{G}(R)$ and $\Gamma(R)$ are not triangulated.

5 Domination Number

The main purpose of this section is studying of domination numbers of $\mathbb{A}\mathbb{G}(R)$ and then $\mathbb{A}\mathbb{G}(X)$. In this studying, we employ the Bourbaki associated prime divisor of the zero ideal and the fixed-place ideal notions.

Lemma 5.1. *Let I be an ideal in $\mathbb{A}(R)^*$. The following statements are equivalent.*

- (a) I is prime.
 (b) I is a maximal element of $\mathbb{A}(R)^*$.
 (c) I is a Bourbaki associated prime divisor of the zero ideal of R .

Proof. (a) \Rightarrow (b). Suppose that $I \subseteq J$ and $J \in \mathbb{A}(R)^*$, then $0 \neq a \in \text{Ann}(J)$ exists. Since R is a reduced ring, it follows that $a \notin J$, then $a \notin I$ and $aJ \subseteq I$, hence $J \subseteq I$ and thus $I = J$. Consequently, I is a maximal element of $\mathbb{A}(R)^*$.

(b) \Rightarrow (c). Since $I \in \mathbb{A}(R)^*$, it follows that there is some $0 \neq a \in R$ such that $\text{Ann}(a) = I$. Suppose that $xy \in I$ and $x \notin I$, then $I = \text{Ann}(a) \subseteq \text{Ann}(ax)$, so $y \in \text{Ann}(ax) \subseteq \text{Ann}(a) = I$, by the maximality of I , hence I is prime, and, therefore I is a Bourbaki associated prime divisor of the zero ideal.

(c) \Rightarrow (a). It is clear. \square

Proposition 5.2. *The following statements hold.*

- (a) Suppose that $I \in \mathbb{A}(R)^*$. I is contained in some maximal element of $\mathbb{A}(R)^*$ if and only if $\text{Min}(I) \cap \mathcal{B}(R) \neq \emptyset$.
 (b) Every element of $\mathbb{A}(R)^*$ is contained in some maximal element of $\mathbb{A}(R)^*$ if and only if the zero ideal of R is a fixed-place ideal.
 (c) $\mathbb{A}(R)^*$ does not have any maximal element if and only if the zero ideal of R is an anti fixed-place ideal.

Proof. (a \Rightarrow). By Lemma 5.1, some $P \in \mathcal{B}(R)$ exists such that $I \subseteq P$, since $P \in \text{Min}(R)$, it follows that $P \in \text{Min}(I)$, and, therefore $P \in \mathcal{B}(R) \cap \text{Min}(R) \neq \emptyset$.

(a \Leftarrow). It is clear, by Lemma 5.1.

(b \Rightarrow). On the contrary, suppose that $\bigcap_{P \in \mathcal{B}(R)} P \neq \{0\}$ and thus there is some $0 \neq a \in \bigcap_{P \in \mathcal{B}(R)} P$. Then

$$\text{Ann}(a) = (0 : a) = \left(\bigcap_{P \in \text{Min}(R)} P : a \right) = \bigcap_{a \notin P \in \text{Min}(R)} P$$

By the assumption, there is some $P_o \in \mathcal{B}(R)$ such that $\text{Ann}(a) \subseteq P_o$, then $\bigcap_{a \notin P \in \text{Min}(R)} P \subseteq P_o$, and, therefore

$$\bigcap_{P_o \neq P \in \text{Min}(R)} P \subseteq \bigcap_{a \notin P \in \text{Min}(R)} P \subseteq P_o$$

$$\Rightarrow \{0\} = \left(\bigcap_{P \circ \neq P \in \text{Min}(R)} P \right) \cap P \circ = \bigcap_{P \circ \neq P \in \text{Min}(R)} P$$

which is a contradiction.

(b \Leftarrow). By the assumption, $\bigcap_{P \in \mathcal{B}(R)} P = \{0\}$. So

$$\text{Ann}(I) = (0 : I) = \left(\bigcap_{P \in \mathcal{B}(R)} P : I \right) = \bigcap_{P \in \mathcal{B}(R)} (P : I) = \bigcap_{I \not\subseteq P \in \mathcal{B}(R)} P$$

Hence $P \in \mathcal{B}(R)$ exists such that $I \subseteq P$ and thus, by Lemma 5.1. It completes the proof.

(c). It is evident, by Lemma 5.1. \square

In the proof of [27, Theorem 2.2], it is claimed that ‘‘By Zorn’s Lemma, it is clear that if $\mathbb{A}(R)^* \neq \emptyset$, then $\mathbb{A}(R)^*$ has a maximal element’’ and ‘‘for every $J \in \mathbb{A}(R)^*$, there exists $I \in M$ [the family of all maximal elements of $\mathbb{A}(R)^*$] such that $J \subseteq I$ ’’. But the chain of elements of $\mathbb{A}(R)^*$ need not have an upper bound and thus $\mathbb{A}(R)^*$ does not satisfy the conditions of Zorn’s Lemma. By the above proposition, an ideal of $\mathbb{A}(R)^*$ need not contained in a maximal element of $\mathbb{A}(R)^*$. In the following example, we give a counterexample.

Example 5.3. *Since \mathbb{R} does not have any isolated point, by [6, Corollary 5.4], the zero ideal of $C(\mathbb{R})$ is an anti fixed-place ideal and therefore $\mathcal{B}(C(\mathbb{R})) = \emptyset$. Then Lemma 5.1 implies that the family of all maximal elements of $\mathbb{A}(C(\mathbb{R}))^*$ is empty.*

Also, in [27, Theorem 2.2], it is asserted that ‘‘Let M be the set of all maximal elements of the set $\mathbb{A}(R)^*$. If $|M| > 1$, then $\text{dt}_t \mathbb{A}\mathbb{G}(R) = M$ ’’. In Example 5.8, we show that this assertion is not true.

In the following theorem, we show that the cardinal of $\mathcal{B}(R)$ is a lower bound of $\text{dt}_t(\mathbb{A}\mathbb{G}(R))$. Also, Theorem 5.5 shows that if the zero ideal of R is fixed-place, then $|\mathcal{B}(R)| = \text{dt}_t(\mathbb{A}\mathbb{G}(R))$. Finally, we know that if $\text{Min}(R)$ is a finite set (for example R is a Noetherian ring), then the zero ideal of R is a fixed-place ideal and thus $|\text{Min}(R)| = |\mathcal{B}(R)| = \text{dt}_t(R)$.

Theorem 5.4. *The following hold.*

- (a) $|\mathcal{B}(R)| \leq \text{dt}_t(\mathbb{A}\mathbb{G}(R))$.
- (b) *If $|\text{Min}(R)| > 2$, then $|\mathcal{B}(R)| \leq \text{dt}(\mathbb{A}\mathbb{G}(R))$.*

Proof. (a). Suppose that D is a total dominating set of $\mathbb{A}\mathbb{G}(R)$. For each $P \in \mathcal{B}(R)$, there is some $I_P \in D$, such that I_P is adjacent to P , so $PI_P = \{0\}$, hence $P \subseteq \text{Ann}(I_P)$ and thus $P = \text{Ann}(I_P)$, by Lemma 5.1. Now suppose

that $I_P = I_Q$, for some $P, Q \in \mathcal{B}(R)$, then $P = \text{Ann}(I_P) = \text{Ann}(I_Q) = Q$ and thus the map $P \rightsquigarrow I_P$ is one-to-one. This implies that $|\mathcal{B}(R)| \leq |D|$ and consequently $|\mathcal{B}(R)| \leq \text{dt}_t(\mathbb{A}\mathbb{G}(R))$.

(b). Let D be a dominating set. For each $P \in \mathcal{B}(R)$, if $P \in D$, then we set $K_P = P$ and if $P \notin D$, there is some $K_P \in D$ such that K_P is adjacent to P . Suppose that $K_P = K_Q$, for some $P, Q \in \mathcal{B}(R)$. If $P, Q \in D$, then $P = K_P = K_Q = Q$. If $P, Q \notin D$, then P and Q are adjacent to K_P and K_Q , respectively, so $PK_P = QK_Q = \{0\}$, thus $P \subseteq \text{Ann}(K_P)$ and $Q \subseteq \text{Ann}(K_Q)$, and, therefore $P = \text{Ann}(K_P) = \text{Ann}(K_Q) = Q$, by Lemma 5.1. Finally, without loss of generality, we assume $P \in D$ and $Q \notin D$, then $P = K_P$ and K_Q is adjacent to Q , so P is adjacent to Q and thus $PQ = \{0\}$. Hence for each $P' \in \text{Min}(R)$, we have $PQ = \{0\} \subseteq P'$, and, therefore either $P \subseteq P'$ or $Q \subseteq P'$, and, therefore, by Lemma 5.1, either $P = P'$ or $Q = P'$. This implies that $|\text{Min}(R)| \leq 2$, which contradicts the assumption. Consequently, the map $P \rightsquigarrow K_P$ is one-to-one. This implies that $|\mathcal{B}(R)| \leq |D|$ and thus $|\mathcal{B}(R)| \leq \text{dt}_t(\mathbb{A}\mathbb{G}(R))$. \square

Theorem 5.5. *If the zero ideal of R is a fixed-place ideal, then*

- (a) $\text{dt}_t(\mathbb{A}\mathbb{G}(R)) = |\mathcal{B}(R)|$.
- (b) *If $|\text{Min}(R)| > 2$, then $\text{dt}_t(\mathbb{A}\mathbb{G}(R)) = |\mathcal{B}(R)|$.*

Proof. (a). By the above theorem, it is sufficient to show that $\text{dt}_t(\mathbb{A}\mathbb{G}(R)) \leq |\mathcal{B}(R)|$. For every $P \in \mathcal{B}(R)$, pick $a_P \in R$, such that $\text{Ann}(a_P) = P$. For each $K \in \mathbb{A}(R)^*$, by the assumption and Proposition 5.2, there is some $P \in \mathcal{B}(R)$ such that $K \subseteq P = \text{Ann}(a_P)$, so $Ra_P K = \{0\}$, and, therefore K is adjacent to Ra_P . This implies that $\{Ra_P : P \in \mathcal{B}(R)\}$ is a dominating set and consequently $\text{dt}_t(\mathbb{A}\mathbb{G}(R)) \leq |\mathcal{B}(R)|$.

(b). By the fact that $\text{dt}_t(\mathbb{A}\mathbb{G}(R)) \leq \text{dt}_t(\mathbb{A}\mathbb{G}(R))$, it follows from part (a) and the above theorem. \square

We recall that if $\text{Min}(R)$ is finite, then the zero ideal of R is a fixed-place ideal and $\text{Min}(R) = \mathcal{B}(R)$. Thus [26, Theorem 2.4] and [27, Theorem 2.4] are immediate consequences of the above theorem. Also, we can conclude the following corollary from the above theorem and [6, Theorems 5.2 and 5.5].

Corollary 5.6. *Suppose that X is an almost discrete space. Then*

- (a) $\text{dt}_t(\mathbb{A}\mathbb{G}(X)) = |I(X)|$.
- (b) *If $|X| > 2$, then $\text{dt}_t(\mathbb{A}\mathbb{G}(X)) = |I(X)|$.*

Theorem 5.7. *If the zero ideal of a ring R is not a fixed-place ideal, then $\text{dt}_t(\mathbb{A}\mathbb{G}(R))$ and $\text{dt}_t(\mathbb{A}\mathbb{G}(R))$ are infinite.*

Proof. Suppose that D is a dominating set of $\mathbb{A}\mathbb{G}(R)$. By Proposition 5.2, there is some $J_1 \in \mathbb{A}(R)^*$ which is not contained in a maximal element of $\mathbb{A}(R)^*$. If $J_1 \in D$, then we set $I_1 = K_1 = J_1$. If $J_1 \notin D$, then there is some vertex $I_1 \in D$ which is adjacent to J_1 , then $J_1 I_1 = \{0\}$, so $J_1 \subseteq \text{Ann}(I_1)$. In this case, we set $K_1 = \text{Ann}(I_1)$. Since J_1 is not contained in a maximal element of $\mathbb{A}(R)^*$ and $J_1 \subseteq K_1$, there is some $J_2 \in \mathbb{A}(R)^*$ such that $K_1 \subset J_2$. Similarly, we can find $K_2 \in \mathbb{A}(R)^*$ in which either $I_2 = K_2 \in D$ or $K_2 = \text{Ann}(I_2)$, for some $I_2 \in D$. By induction, we have the following chain

$$J_1 \subseteq K_1 \subset J_2 \subseteq K_2 \subset \dots \subset J_n \subseteq K_n \subset \dots$$

Now suppose that $n \neq m$, then $K_n \neq K_m$. Without loss of generality, we assume $n < m$, hence we have four cases

- case 1: If $I_n = K_n$ and $I_m = K_m$, then it is evident that $I_n \neq I_m$.
- case 2: If $K_n = \text{Ann}(I_n)$ and $K_m = \text{Ann}(I_m)$, so it is clear that $I_n \neq I_m$.
- case 3: If $K_n = I_n$ and $K_m = \text{Ann}(I_m)$, then $I_n \subset \text{Ann}(I_m)$, so we have $I_n I_m = \{0\}$. Hence $I_n \neq I_m$, because otherwise, $I_n^2 = \{0\}$ and therefore $I_n = \{0\}$, which is a contradiction.
- case 4: If $K_n = \text{Ann}(I_n)$ and $K_m = I_m$, then $\text{Ann}(I_n) \subset I_m$, so we have $\text{Ann}(I_m) \subseteq \text{Ann}(\text{Ann}(I_n))$. Hence $I_n \neq I_m$, because otherwise, similar to case 3, $\text{Ann}(I_n) = \{0\}$, which is a contradiction.

Since $\{I_n : n \in \mathbb{N}\} \subseteq D$, it follows that D is infinite and consequently $\text{dt}(\mathbb{A}\mathbb{G}(R))$ is infinite. Hence $\text{dt}_t(\mathbb{A}\mathbb{G}(R))$ is infinite, by the fact that

$$\text{dt}(\mathbb{A}\mathbb{G}(R)) \leq \text{dt}_t(\mathbb{A}\mathbb{G}(R)). \quad \square$$

Now by the above theorem, $\text{dt}_t(\mathbb{A}\mathbb{G}(C(\mathbb{R})))$ and $\text{dt}(\mathbb{A}\mathbb{G}(C(\mathbb{R})))$ are infinite, so the inequality in Theorem 5.4, can be proper.

Example 5.8. Suppose that $R = C(\mathbb{R}) \times \mathbb{R} \times \mathbb{R}$. It is easy to see that if M is the family of all maximal elements of $\mathbb{A}(R)^*$, then $M = \mathcal{B}(C(\mathbb{R}) \times \mathbb{R} \times \mathbb{R}) = \{C(\mathbb{R}) \times \mathbb{R} \times \{0\}, C(\mathbb{R}) \times \{0\} \times \mathbb{R}\}$, by Lemma 5.1, Proposition 5.2 and Example 5.3. Since the zero ideal of R is not a fixed-place ideal, by the above theorem, it follows that $\text{dt}_t(R)$ is infinite. Thus $|M| \neq \text{dt}_t(R)$.

Corollary 5.9. The following statements are equivalent

- (a) $\text{dt}_t(\mathbb{A}\mathbb{G}(R))$ is finite
- (b) $\text{dt}_t(\mathbb{A}\mathbb{G}(R))$ is finite

(c) $\text{Min}(R)$ is finite

Proof. It follows immediately from Theorems 5.5 and 5.7 and the fact that if $\text{Min}(R)$ is finite, then the zero ideal is a fixed-place ideal. \square

Finally in the following proposition, we generalize [27, Theorem 2.3] to the infinite version.

Proposition 5.10. *For each reduced ring R , we have*

$$\text{dt}_t(\Gamma(R)) \leq \text{dt}_t(\mathbb{A}\mathbb{G}(R)).$$

Proof. Suppose that D is a total dominating set of $\mathbb{A}\mathbb{G}(R)$. So for each $I \in P$, there is some $0 \neq a_I \in I$. For every $a \in R$, there is some $I \in D$ such that I is adjacent to Ra in $\mathbb{A}\mathbb{G}(R)$, thus $RaI = \{0\}$, hence $aa_I = 0$, and, therefore a_I is adjacent to a in $\Gamma(R)$. Consequently, $\{a_I : I \in D\}$ is a total dominating set of $\Gamma(R)$ and this implies that $\text{dt}_t(\Gamma(R)) \leq \text{dt}_t(\mathbb{A}\mathbb{G}(R))$. \square

References

- [1] Aalipour Gh, Akbari S, Behboodi M, Nikandish R, Nikmehr M J, Shaveisi F. The classification of the annihilating-ideal graphs of commutative rings. *Algebra Colloq.*, 21(02):249–256, 2014. doi:10.1142/S1005386714000200
- [2] Aalipour Gh, Akbari S, Nikandish R, Nikmehr M J, Shaveisi F. On the coloring of the annihilating-ideal graph of a commutative ring. *Discrete Math.*, 312(17):2620–2626, 2012. doi:10.1016/j.disc.2011.10.020
- [3] Afkhami M, Barati Z, Khashyarmansh K. A graph associated to a lattice. *Ric. Mat.*, 63(1):67–78, 2014. doi:10.1007/s11587-013-0164-6
- [4] Akbari S, Maimani H R, Yassemi S. When a zero-divisor graph is planar or a complete r-partite graph. *Journal of Algebra*, 270(1):169–180, 2003. doi:10.1016/S0021-8693(03)00370-3
- [5] Akbari S, Mohammadian A. On the zero-divisor graph of a commutative ring. *J. Algebra*, 274(2):847–855, 2004. doi:10.1016/S0021-8693(03)00435-6
- [6] Aliabad A R, Badie M. Fixed-place ideals in commutative rings. *Comment. Math. Univ. Carolin.*, 54(1):53–68, 2013.
- [7] Aliabad A R, Badie M. On Bourbaki associated prime divisors of an ideal. *Quaest. Math.*, pages 1–22, 2018. doi:10.2989/16073606.2018.1459924

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- [8] Aliabad A R, Badie M, Nazari S. An extension of z -ideals and z° -ideals. *Hacet. J. Math. Stat.*, 49(1):254–272 2020. doi:10.15672/hujms.455030
- [9] Aliabad A R and Mohamadian R. On sz° -ideals in polynomial rings. *Comm. Algebra*, 39(2):701–717, 2011. doi:10.1080/00927871003591884
- [10] AliniaEIFard F, Behboodi M. Rings whose annihilating-ideal graphs have positive genus. *J. Algebra Appl.*, 11(03):1250049, 2012. doi: 10.1142/S0219498811005774
- [11] Anderson D F, Levy R, Shapiro R. Zero-divisor graphs, von neumann regular rings, and boolean algebras. *J. Pure App. Algebra*, 180(3):221–241, 2003. doi:10.1016/S0022-4049(02)00250-5
- [12] Anderson D F, Livingston P S. The zero-divisor graph of a commutative ring. *J. Algebra*, 217(2):434–447, 1999. doi:10.1006/jabr.1998.7840
- [13] Assari A, Rahimi M. Graphs generated by measures. *J. of Math.*, 2016, 2016. doi:10.1155/2016/1706812
- [14] Atiyah M F, Macdonald I G. Introduction to commutative algebra, volume 2. Addison-Wesley Reading, MA, 1969.
- [15] Azarpanah F and Motamedi M. Zero-divisor graph of $C(X)$. *Acta Math. Hungar.*, 108(1-2):25–36, 2005. doi:10.1007/s10474-005-0205-z
- [16] Badie M. Annihilating-ideal graph of $C(X)$. *J. Algebr. Syst.*, to appear.
- [17] Beck I. Coloring of commutative rings. *J. Algebra*, 116(1):208–226, 1988. doi:10.1016/0021-8693(88)90202-5
- [18] Behboodi M, Rakeei Z. The annihilating-ideal graph of commutative rings I. *J. Algebra Appl.*, 10(04):727–739, 2011. doi:10.1142/S0219498811004896
- [19] Behboodi M, Rakeei Z. The annihilating-ideal graph of commutative rings II. *J. Algebra Appl.*, 10(04):741–753, 2011. doi:10.1142/S0219498811004902
- [20] Bondy J A, Murty U S R. Graph theory with Application. The Macmillan Press, New York, 1976.
- [21] Chelvam T, Selvakumar K. On the connectivity of the annihilating-ideal graphs. *Discuss. Math. Gen. Algebra Appl.*, 35(2):195–204, 2015. doi:10.7151/dmgaa.1241

- [22] Gillman L, Jerison M. Rings of continuous functions. Van. Nostrand Reinhold, New York, 1960.
- [23] Henriksen M, Jerison M. The space of minimal prime ideals of a commutative ring. Trans. Amer. Math. Soc., 115:110–130, 1965. doi:10.2307/1994260
- [24] Levy R, Shapiro J. The zero-divisor graph of von neumann regular rings. 2002. doi:10.1081/AGB-120013178
- [25] Mason G. Prime ideals and quotient rings of reduced rings. Sci. Math. Jpn., 34(6):941–956, 1989.
- [26] Nikandish R, Maimani H R. Dominating sets of the annihilating-ideal graphs. Electron. Notes Discrete Math., 45:17–22, 2014. doi:10.1016/j.endm.2013.11.005
- [27] Nikandish R, Maimani H R, Kiani S. Domination number in the annihilating-ideal graphs of commutative rings. Publ. Inst. Math., 97(111): 225–231, 2015. doi:10.2298/PIM140222001N
- [28] Redmond S. Central sets and radii of the zero-divisor graphs of commutative rings. Comm. Algebra, 34(7):2389–2401, 2006. doi:10.1080/00927870600649103
- [29] Samei K. The zero-divisor graph of a reduced ring. J. Pure App. Algebra, 209(3):813–821, 2007. doi:10.1016/j.jpaa.2006.08.008
- [30] Sharp R Y. Steps in commutative algebra, volume 51. Cambridge university press, 2000.
- [31] Willard S. General Topology. Addison Wesley Publishing Company, New York, 1970.

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