Self dual, reversible and complementary duals constacyclic codes over finite local Frobenius non-chain rings of length 5 and nilpotency index 4.

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Abstract

Over finite local Frobenius non-chain rings of length 5 and nilpotency index 4 and when the length of the code is relatively prime to the characteristic of the residue field of the ring, the structure of the dual of γ-constacyclic codes is established and the algebraic characterization of self-dual, reversible γ-constacyclic codes and γ-constacyclic codes with complementary dual are given.

1 Introduction

After the work of R. Hammons et al. see [7] codes over finite rings have received considerable attention. Parallel to this, constacyclic codes, self dual codes, reversible codes and codes with complementary duals over finite rings have also gained a large popularity among coding theorists. For some of the works in these directions we refer to [1] [2], [3], [4], [6], [8], [9], [14], [17]. The class of γ-constacyclic codes plays a very significant role in the theory of error-correcting codes, these are codes invariant under the mapping $\sigma_\gamma : \mathrm{R}^n \longrightarrow \mathrm{R}^n$ given by $\sigma_\gamma(a_0, \ldots, a_{n-1}) = (\gamma a_{n-1}, a_0, \ldots, a_{n-2})$, γ is a unit of the ring R. Reversible codes are codes invariant under the mapping $\mathfrak{r} : \mathrm{R}^n \longrightarrow \mathrm{R}^n$ given

by $\mathfrak{r}(a_0, \ldots, a_{n-1}) = (a_{n-1}, \ldots, a_0)$, see [10]. These codes are used to obtain DNA codes. Linear complementary dual codes (which is abbreviated to LCD code) are linear codes that trivially intersect with their duals, [11]. Over fields cyclic LCD codes are the reversible codes and LCD codes have a good asymptotically property and meet the asymptotic Gilbert-Varshamov bound, see [13], [16].

Duals of γ-constacyclic codes, self dual, reversible γ-constacyclic codes and γ-constacyclic codes with complementary duals over finite local Frobenius non-chain rings whose maximal ideal has nilpotency index 3 and the length of the code is relatively prime to the characteristic of the residue field of the ring were studied in [3], [4]. In [5], finite local Frobenius non-chain rings of length 5 whose maximal ideal has nilpotency index 4 were determined and the structure of γ-constacyclic codes over these rings was described. Now, it would be interesting to determine the dual of γ-constacyclic codes and the algebraic characterization of self-dual, reversible γ-constacyclic codes and γ-constacyclic codes with complementary duals over finite local Frobenius non-chain rings of length 5 whose maximal ideal has nilpotency index 4.

Throughout this work all codes are over finite local Frobenius non-chain rings of length 5 whose maximal ideal has nilpotency index 4 and the length of the code is relatively prime to the characteristic of the residue field of the ring. The purpose of this paper is to obtain structure Theorems for the dual of a constacyclic code, for self-dual constacyclic codes, for reversible constacyclic codes and for constacyclic codes with complementary duals. The paper is organized as follows: in Section 2 facts on finite commutative local rings are recalled, properties of the reciprocal polynomial over local rings and properties of finite local Frobenius non-chain rings of length 5 whose maximal ideal has nilpotency index 4 are given. In Section 3 the structure of the dual of a γ-constacyclic code is determined, generators for the dual code are obtained from the generators of the original constacyclic code. The algebraic characterization of self-dual γ-constacyclic codes is presented in Section 4. In Section 5 the algebraic characterization of reversible γ-constacyclic codes is presented. The algebraic characterization of complementary γ-constacyclic codes is given in Section 6. Examples are included to illustrate the main results.

2 Preliminaries

Throughout this work $\mathrm{GF}(q)$ denotes the finite field with $q = p^d$ elements, p a prime, all rings are assumed to be finite, commutative with unit element and all modules are finitely generated.

2.1   Rings and local rings

Let R and B be rings, I an ideal of R and M an R-module. The *length* of M, denoted by $\ell_R(M)$, is the length of a composition series for M. The *annihilator ideal* of M in R is defined as $\text{ann}_R(M) := \{a \in R : am = 0, \forall\, m \in M\}$. If I is contained in $\text{ann}_R(M)$ there is a natural structure of (R/I)-module on M, the scalar multiplication is given by $(a + I, m) \mapsto am$, $a \in R, m \in M$, and the lattice of R-submodules and (R/I)-submodules of M are the same. If R has only one maximal ideal $\mathfrak{m}$ then it is called *local*, $k = R/\mathfrak{m}$ its residue field and it will be denoted by the triple $(R, \mathfrak{m}, GF(q))$. R is called a chain ring if the lattice of its ideals is a chain under set-theoretic inclusion. A finite local ring $(R, \mathfrak{m}, GF(q))$ is Frobenius if $\text{ann}_R(\mathfrak{m})$ is the unique minimal ideal of R, see [15]. $\mathfrak{F}_5^4$ denotes the family of finite local Frobenius non-chain rings of length 5 whose maximal ideal has nilpotency index 4, see [5].

If $(R, \mathfrak{m}, GF(q))$ is a finite local ring, then $|M| = |GF(q)|^{\ell_R(M)}$, see [2], and there is an integer $t \geq 1$ such that $\mathfrak{m}^t = \langle 0 \rangle$ and $\mathfrak{m}^{t-1} \neq \langle 0 \rangle$, called the *nilpotency index* of $\mathfrak{m}$. A subset G of M generates M if and only if its image $\overline{G}$ in $M/\mathfrak{m}M$ generates $M/\mathfrak{m}M$ as a GF(q)-vector space. A set of generators for M obtained from lifting a basis of the GF(q)-vector space $M/\mathfrak{m}M$ is called a *minimal R-generating set* for M and $v_R(M)$ denotes the number of elements in a minimal R-generating set for M, see [12]. Note that $v_R(M) = \dim_{GF(q)}(M/\mathfrak{m}M) = \ell_R(M/\mathfrak{m}M)$.

Let $(R, \mathfrak{m}, GF(q))$ be a finite local ring, $f, g \in R[T]$, γ a unit of R and $n \in \mathbb{N}$ with $(n, q) = 1$. $^{-} : R[T] \to GF(q)[T]$ is the natural ring homomorphism given by $a_0 + a_1T + \ldots + a_mT^m \mapsto (a_0 + \mathfrak{m}) + (a_1 + \mathfrak{m})T + \ldots + (a_m + \mathfrak{m})T^m$, the polynomial $f \in R[T]$ is called *basic irreducible* if $\bar{f}$ is irreducible in $GF(q)[T]$, f and g are called *coprime* if $\langle f \rangle + \langle g \rangle = R[T]$. Hensel's Lemma guarantees that factorization into a product of pairwise coprime polynomials in $GF(q)[T]$ lifts to such a factorization over R[T], see [12]. In fact there is a unique family of monic basic irreducible pairwise coprime polynomials such that $T^n - \gamma = f_1 \cdots f_r$, see [3].

Let $f \in R[T]$ be a monic basic irreducible polynomial, $\deg(\bar{f}) = s$, $\mathbb{T} \subset R$ be a set of representatives of GF(q) and I an ideal of R. From [2], [3] and [5] we have: (1) $B = R[T]/\langle f \rangle = \{a_0 + a_1T + \cdots + a_{s-1}T^{s-1} : a_i \in R\}$ is a local separable extension of R with maximal ideal $\mathfrak{m}B$ and residue field $GF(q^s)$; (2) the set $\mathbb{T}_s := \{a_0 + a_1T + \cdots + a_{s-1}T^{s-1} : a_i \in \mathbb{T}\} \subset B$ is a set of representatives of $GF(q^s)$; (3) the nilpotency index of $\mathfrak{m}$ and $\mathfrak{m}B$ is the same; (4) $\ell_R(I) = \ell_B(IB)$; (5) $v_R(I) = v_B(IB)$; (6) A minimal R-generating set for I is a minimal B-generating set for IB; (7) $(\text{ann}_R(I))B = \text{ann}_B(IB)$; (8) R is a chain ring if and only if B is a chain ring; (9) R is Frobenius if and only if B is Frobenius, in this case $\text{ann}_B(\mathfrak{m}B) = \mathfrak{m}^{t-1}B$, where t is the nilpotency index

of $\mathfrak{m}$.

If R is a finite local Frobenius ring, (10) $\mathrm{ann_R}(\mathrm{ann_R}(I)) = I$; (11) $\ell_R(\mathrm{ann_R}(I)) + \ell_R(I) = \ell_R(R)$; (12) $\mathfrak{m}^{i-1}\mathrm{ann_R}(\mathfrak{m}^i) = \mathfrak{m}^{t-1}$, $2 \leq i \leq t-1$; (13) $\mathrm{v_R}(\mathrm{ann_R}(\mathfrak{m}^2)) = \mathrm{v_R}(\mathfrak{m})$; (14) $(\mathfrak{m}B)^{i-1}\mathrm{ann_B}((\mathfrak{m}B)^i) = \mathfrak{m}^{i-1}\mathrm{ann_B}(\mathfrak{m}^iB) = \mathfrak{m}^{i-1}\mathrm{ann_R}(\mathfrak{m}^i)B = \mathfrak{m}^{t-1}B = (\mathfrak{m}B)^{t-1}$, $2 \leq i \leq t-1$; (15) $\mathrm{v_B}(\mathfrak{m}B) = \mathrm{v_R}(\mathfrak{m}) = \mathrm{v_R}(\mathrm{ann_R}(\mathfrak{m}^2)) = \mathrm{v_B}(\mathrm{ann_R}(\mathfrak{m}^2)B) = \mathrm{v_B}(\mathrm{ann_B}(\mathfrak{m}^2B)) = \mathrm{v_B}(\mathrm{ann_B}((\mathfrak{m}B)^2))$.

Let $(R, \mathfrak{m}, \mathrm{GF}(q))$ be a finite local Frobenius ring, $f \in R[T]$ be a basic irreducible polynomial, $s = \deg(\bar{f})$, $(B = R[T]/\langle f \rangle, \mathfrak{m}B, \mathrm{GF}(q^s))$ the separable extension of R determined by f, $\mathbb{T} \subset R$ a set of representatives of $\mathrm{GF}(q)$ with $0, 1 \in \mathbb{T}$, M a B-module, $\tilde{\theta} = \{\theta_1, \ldots, \theta_l\}$ a sequence of elements of M and $H = (h_{ij})$ a $(k \times l)$ matrix over $\mathrm{GF}(q^s)$. For the remainder of this work the following notation will be used:

(1) The number $\mathrm{v_R}(\mathfrak{m}) = \mathrm{v_R}(\mathrm{ann_R}(\mathfrak{m}^2)) = \mathrm{v_B}(\mathfrak{m}B) = \mathrm{v_B}(\mathrm{ann_B}((\mathfrak{m}B)^2))$ is denoted by χ;

(2) $\tilde{\alpha} = \{\alpha_1, \ldots, \alpha_\chi\}$ will be a minimal R-generating set for $\mathfrak{m}$;

(3) $\tilde{\beta} = \{\beta_1, \ldots, \beta_\chi\}$ will be a minimal R-generating set for $\mathrm{ann_R}(\mathfrak{m}^2)$;

(4) $\mathbb{T}_s = \{a_0 + a_1T + \cdots + a_{s-1}T^{s-1} : a_i \in \mathbb{T}\} \subset B$ will be the set of representatives of $B/\mathfrak{m}B = \mathrm{GF}(q^s)$;

(5) For $a \in \mathrm{GF}(q^s)$, $a^{\mathbb{T}_s}$ will denote the unique representative of a in $\mathbb{T}_s$;

(6) The B-submodule of M, $\langle h_{11}^{\mathbb{T}_s}\theta_1 + \ldots + h_{1l}^{\mathbb{T}_s}\theta_l, \ldots, h_{k1}^{\mathbb{T}_s}\theta_1 + \ldots + h_{kl}^{\mathbb{T}_s}\theta_l \rangle$, will be denoted by $H_{\tilde{\theta}}^{\mathbb{T}_s}$.

A $(k \times n)$ matrix over $\mathrm{GF}(q)$ is said to be in reduced row echelon form, (rre)-form, if in each row, $i = 1, \ldots, k$, the first nonzero entry is equal to 1, the index of the column in which the 1 occurs, called a *pivotal column*, strictly increases with i, and the k pivotal columns are, in order, the columns of the $(k \times k)$ identity matrix. Since each matrix is row equivalent to a unique reduced row echelon form matrix, the k-dimensional vector subspaces of V are in one-to-one correspondence with the $(k \times n)$ matrices over $\mathrm{GF}(q)$ in (rre)-form. The $(k \times n)$ matrix (a_{ij}) over the field in (rre)-form corresponds to the subspace $\langle \sum_{i=1}^n a_{1i}\alpha_i, \ldots, \sum_{i=1}^n a_{ki}\alpha_i \rangle$.

Lemma 1. *([2], Lemma 1) Let $(R, \mathfrak{m}, \mathrm{GF}(q))$ be a finite local ring, $\mathbb{T} \subseteq R$ a set of representatives of $\mathrm{GF}(q)$, M an R-module and $\tilde{\theta} = \{\theta_1, \ldots, \theta_l\}$ be a minimal R-generating set for M. Then the B-submodules of M between M and $\mathfrak{m}M$ of length $k + \ell_R(\mathfrak{m}M)$, where $0 < k < l = \dim_{\mathrm{GF}(q)}(M/\mathfrak{m}M) = \mathrm{v_R}(M)$, are in one to one correspondence with the $(k \times l)$ matrices over $\mathrm{GF}(q)$ in (rre)-form. The matrix $H = (\overline{a_{ij}})$ corresponds to the submodule $\langle \sum_{i=1}^n a_{1i}\theta_i, \ldots, \sum_{i=1}^n a_{ki}\theta_i \rangle + \mathfrak{m}M$.*

Observation 1. *Let* $(R, \mathfrak{m}, GF(q))$ *be a finite local Frobenius ring,* f, s *and* $(B, \mathfrak{m}B, GF(q^s))$ *be as above. Let* $\mathfrak{F} = \{J$ *ideal of* $B : \mathrm{ann}_B(\mathfrak{m}B) \subset J \subset \mathrm{ann}_B((\mathfrak{m}B)^2)\}$, $J \in \mathfrak{F}$, $\mathfrak{G} = \{I$ *ideal of* $B : (\mathfrak{m}B)^2 \subset I \subset \mathfrak{m}B\}$ *and* $I \in \mathfrak{G}$, *then:*

(1) *From relation* $\mathrm{ann}_B(\mathrm{ann}_B(*)) = *$, *there is a one to one correspondence between* $\mathfrak{F}$ *and* $\mathfrak{G}$.

(2) *J has length* $k + 1$, $k \in \{1, \ldots, \chi - 1\}$, *and* $\mathrm{ann}_B(J)$ *has length* $\ell_B(B) - k - 1 = \chi - k + \ell_R(\mathfrak{m}^2)$.

(3) *From Lemma 1, relations* $\mathrm{ann}_B((\mathfrak{m}B)^2) = \mathrm{ann}_R(\mathfrak{m}^2)B$ *and* $\mathrm{ann}_R(\mathfrak{m})B = \mathrm{ann}_B(\mathfrak{m}B) = (\mathfrak{m}B)\mathrm{ann}_B((\mathfrak{m}B)^2)$, *J corresponds to a unique* $k \times \chi$ *matrix in (rre)-form over* $GF(q^s)$ *and* $\mathrm{ann}_B(J)$ *corresponds to a unique* $(\chi - k) \times \chi$ *matrix in (rre)-form over* $GF(q^s)$.

(4) *If* $H_J = (a_{ij})$ *is the matrix that corresponds to J, the matrix that corresponds to* $\mathrm{ann}_B(J)$ *will be denoted by* $H_J^\perp = (b_{ij})$.
Let $\pi \in \{1, \ldots, k\}$ *and* $\varpi \in \{1, \ldots, \chi - k\}$, *we have the following relations:*

   (a) $J = (H_J)_{\tilde{\beta}}^{\mathbb{T}_s}$,

   (b) $\mathrm{ann}_B(J) = (H_J^\perp)_{\tilde{\alpha}}^{\mathbb{T}_s} + \mathfrak{m}^2 B$,

   (c) $(a_{\pi 1}^{\mathbb{T}_s} \beta_1 + \ldots + a_{\pi \chi}^{\mathbb{T}_s} \beta_\chi)(b_{\varpi 1}^{\mathbb{T}_s} \alpha_1 + \ldots + b_{\varpi \chi}^{\mathbb{T}_s} \alpha_\chi) = 0$.

(5) *I has length* $k_1 + \ell_R(\mathfrak{m}^2)$, $k_1 \in \{1, \ldots, \chi - 1\}$, *and* $\mathrm{ann}_B(I)$ *has length* $\ell_B(B) - k_1 - \ell_R(\mathfrak{m}^2) = \chi - k_1 + 1$.

(6) *From Lemma 1, I corresponds to a unique* $k_1 \times \chi$ *matrix in (rre)-form over* $GF(q^s)$ *and* $\mathrm{ann}_B(I)$ *corresponds to a unique* $(\chi - k_1) \times \chi$ *matrix in (rre)-form, over* $GF(q^s)$.

(7) *If* $H_I = (c_{ij})$ *is the matrix that corresponds to I, the matrix that corresponds to* $\mathrm{ann}_B(I)$ *will be denoted by* $H_I^\perp = (d_{ij})$.
Let $\pi \in \{1, \ldots, k_1\}$ *and* $\varpi \in \{1, \ldots, \chi - k_1\}$, *we have the following relations:*

   (a) $I = (H_I)_{\tilde{\alpha}}^{\mathbb{T}_s} + \mathfrak{m}^2 B$,

   (b) $\mathrm{ann}_B(I) = (H_I^\perp)_{\tilde{\beta}}^{\mathbb{T}_s}$,

   (c) $(c_{\pi 1}^{\mathbb{T}_s} \alpha_1 + \ldots + c_{\pi \chi}^{\mathbb{T}_s} \alpha_\chi)(d_{\varpi 1}^{\mathbb{T}_s} \beta_1 + \ldots + d_{\varpi \chi}^{\mathbb{T}_s} \beta_\chi) = 0$.

2.2 The reciprocal polynomial and the reciprocal matrix

Let $(R, \mathfrak{m}, GF(q))$ be a finite local ring, $f, g \in R[T]$, γ be a unit of R, $n \in \mathbb{N}$, with $(n, q) = 1$. Let $f_1, \ldots, f_r$ the unique family of monic basic irreducible pairwise coprime polynomials such that $T^n - \gamma = f_1 \cdots f_r$. If a_0 is the constant coefficient of f and a_0 is a unit, the polynomial $a_0^{-1} f$ is denoted by f°. If f is a factor of $T^n - \gamma$ or $T^n - \gamma^{-1}$, we will just write $\widehat{f} = \frac{T^n - \gamma}{f}$ or $\widehat{f} = \frac{T^n - \gamma^{-1}}{f}$, according to the case. The reciprocal of the polynomial f is defined as $f^* = T^{\deg(f)} f(\frac{1}{T})$. The polynomial f is called self-reciprocal if f^* is an associate of f. From [3] and [4] we have the following: (1) $T^{\deg(f) - \deg(f^*)}(f^*)^* = f$; (2) If the constant coefficient of f is not zero, then $\deg(f) = \deg(f^*)$ and $(f^*)^* = f$; (3) If the leader coefficient of f is a unit, then $(fg)^* = f^* g^*$ and $\overline{f^*} = (\overline{f})^*$; (4) If f is a monic basic irreducible polynomial, then f^* is basic irreducible; (5) If $\deg(f) > \deg(g)$, then $(f + g)^* = f^* + T^{\deg(f) - \deg(g)} g^*$; (6) If $\deg(f) = \deg(g)$, then $T^{\deg(f) - \deg(f+g)}(f + g)^* = f^* + g^*$; (7) If the leader coefficients of f and g are units and f and g are coprime, then f^* and g^* are coprime; (8) If $\delta_1, \ldots, \delta_k \in R[T]$, then $(\delta_1 + \ldots + \delta_k)^* = T^{u_1} \delta_1^* + \ldots + T^{u_k} \delta_k^*$, for some $u_1, \ldots, u_k$ non negative integers; (9) Each polynomial f_i^* is basic irreducible; (10) Over fields, self reciprocal polynomials have even degree with the only exception of $T + 1$; (11) For $1 \le i \le r$, the constant coefficient of f_i is a unit, $\deg(f_i) = \deg(f_i^*)$, $(f_i^*)^* = f_i$ and $(\widehat{f_i}^*)^* = \widehat{f_i}$; (12) $(f_1^\circ)^*, \ldots, (f_r^\circ)^*$ are the unique monic basic irreducible pairwise coprime polynomials such that $T^n - \gamma^{-1} = (f_1^\circ)^* \cdots (f_r^\circ)^*$; (13) If U is a subset of $\{1, \ldots, r\}$ and $F = \prod_{u \in U} f_i$, then $T^n - \gamma = F\widehat{F}$, $T^n - \gamma^{-1} = -\gamma^{-1} F^* \widehat{F^*}$ and $\widehat{F^*} = -\gamma^{-1} \widehat{\prod_{u \in U} f_i}^*$, in particular $\widehat{f_i^*} = -\gamma^{-1} \widehat{f_i}^*$, $1 \le i \le r$, note that $\widehat{}$ in the left side is respect to $T^n - \gamma^{-1}$ and $\widehat{}$ in the right side is respect to $T^n - \gamma$; (14) If $\gamma = \gamma^{-1}$, then f_i^* is an associate of f_j if and only if $\overline{f_i^*}$ is an associate of $\overline{f_j}$, in particular f_i is self-reciprocal if and only if $\overline{f_i}$ is self-reciprocal; (15) If $\gamma = \gamma^{-1}$, f_i is a self reciprocal and $f_i \ne T - 1$, then f_i has even degree; (16) If $\gamma = \gamma^{-1}$, after renumbering there are non negative integers r_1, r_2 such that $r = r_1 + 2r_2$, f_{2i-1}^* is an associate of f_{2i}, $1 \le i \le r_1$, and f_{2r_1+i} is a self-reciprocal polynomial, $1 \le i \le r_2$.
For the remainder of the manuscript the following notation will be used:
(1) When $\gamma = \gamma^{-1}$, the order of the sequence $f_1, \ldots, f_r$ is fixed according to the last assertion; (2) For $u \in \{1, \ldots, r\}$, u^* denotes the index of the polynomial which is associate of f_u, observe that $(u^*)^* = u$; (3) For $U \subseteq \{1, \ldots, r\}$, let $U^* = \{u^* : u \in U\}$. U is called self-reciprocal if $U^* = U$ and special if $U \cap U^* = \emptyset$. Observe that U is a self-reciprocal set if and only if $\{1, \ldots, r\} \setminus U$ is a self-reciprocal set; if U is a special set then U^* is a special set and $U \subseteq \{1, \ldots, 2r_1\}$; $\emptyset$ is considered a self-reciprocal and special set.

The reciprocal matrix is helpful to determine reversible constacyclic codes and the dual of a constacyclic code, over finite local rings, see [4]. Let $f, g, h \in GF(q)[T]$, whit f irreducible and $gT \equiv 1 \mod\langle f \rangle$, observe that $h(g) \equiv g^{\deg(h)}h^* \mod\langle f \rangle$ and $\psi : GF(q)[T]/\langle f^* \rangle \to GF(q)[T]/\langle f \rangle$ given by $\psi(h + \langle f^* \rangle) = h(g) + \langle f \rangle = g^{\deg(h)}h^* + \langle f \rangle = \frac{h^* + \langle f \rangle}{T^{\deg(h)} + \langle f \rangle}$ is an isomorphism over $GF(q)$.

Definition 1. *Let $(R, \mathfrak{m}, GF(q))$ be a finite local ring, $f \in R[T]$ a monic basic irreducible polynomial, $n \geq \deg(f)$, $H = (a_{ij})$ a matrix over $GF(q^{\deg(f)}) = GF(q)[T]/\langle \overline{f^*} \rangle$ and $\psi : GF(q)[T]/\langle \overline{f^*} \rangle \to GF(q)[T]/\langle \overline{f} \rangle$ as above. The matrix $(\overline{T}^n \psi(a_{ij}))$ over $GF(q^{\deg(f)}) = GF(q)[T]/\langle \overline{f} \rangle$ is called the n-th reciprocal matrix of H with respect f and is denoted by $H^*_{(n,f)}$.*

Under the notation as in Definition 1, from [4], we have the following properties: (1) $(uH)^*_{(n,f)} = uH^*_{(n,f)}$, $u \in GF(q)$; (2) $H^*_{(n+r,f)} = \overline{T}^r H^*_{(n,f)}$; (3) $(H^*_{(n,f)})^*_{(n,f^*)} = H$. Suppose γ is a unit of R and f is a divisor of $T^n - \gamma$, then: (4) if $\deg(f) = 1$, $\overline{\gamma^{-1}}H^*_{(n,f)} = H$; (5) $\overline{\gamma^{-1}}H^*_{(n,f)} = \overline{\gamma^{-1}}(\overline{T}^n \psi(a_{ij})) = (\psi(a_{ij}))$; (6) if H is in (rre)-form, the matrix $\overline{\gamma^{-1}}H^*_{(n,f)}$ is in (rre)-form.

Observation 2. *Let $(R, \mathfrak{m}, GF(q))$, f, n, $H = (a_{ij})$, ψ as in Definition 1 and γ a unit of R. Suppose f is self reciprocal, $\deg(f) = 2s$ and $f|T^n - \gamma$.*

(1) *ψ is an automorphism of $GF(q^{2s})$ and $\psi^2(h + \langle f \rangle) = \psi[\frac{h^* + \langle f \rangle}{T^{\deg(h)} + \langle f \rangle}] =$*
$$\frac{\frac{(h^*)^* + \langle f \rangle}{T^{\deg(h^*)} + \langle f \rangle}}{\frac{(T^{\deg(h)})^* + \langle f \rangle}{T^{\deg(h)} + \langle f \rangle}} = T^{\deg(h) - \deg(h^*)}(h^*)^* + \langle f \rangle = h + \langle f \rangle.$$

(2) *Let $\alpha \in GF(q^{2s}) = GF(q)[T]/\langle f \rangle$ be a root of an irreducible polynomial over $GF(q)[T]$ of degree s. Since $GF(q^{2s})$ has only one automorphism of degree 2 over $GF(q)$, then this automorphism is ψ and the fixed field of this automorphism is $GF(q^s) = \{h(\alpha) : h(T) \in GF(q)[T], \deg(h) < s\}$*

(3) *$\overline{\gamma^{-1}}H^*_{(n,f)} = H \Leftrightarrow H$ is a matrix over $GF(q^s)$.*

The following result will be useful later; claims are treated in Corollary 1 and Lemma 2 of [4].

Lemma 2. *Let $(R, \mathfrak{m}, GF(q))$ be a finite local ring, $f \in R[T]$ a monic basic irreducible, K an ideal of R and $\tilde{\theta} = \{\theta_1, \ldots, \theta_l\}$ a minimal R-generating set for K. Let $\vec{\rho} = (\rho_1, \ldots, \rho_l), \vec{\nu} = (\nu_1, \ldots, \nu_l) \in (R[T])^l$, where $\vec{\rho}$ is in (rre)-form, the leader coefficient of each ρ_i is a unit, $n \geq \max\{\deg(\rho_i) : 1 \leq i \leq l\}$ and $(\nu_1 + \langle f \rangle, \ldots, \nu_l + \langle f \rangle) = (\rho_1 + \langle f^* \rangle, \ldots, \rho_l + \langle f^* \rangle)^*_{(n,f)}$. There are u_1, u_2 non negative integers such that:*

(1) $((\rho_1\theta_1 + \rho_2\theta_2 + \ldots + \rho_l\theta_l)^*)^* = \rho_1\theta_1 + \rho_2\theta_2 + \ldots + \rho_l\theta_l;$

(2) $\mathrm{T}^{u_1}(\rho_1\theta_1 + \ldots + \rho_l\theta_l)^* \equiv \nu_1\theta_1 + \ldots + \nu_l\theta_l \mod \langle \mathfrak{m}\mathrm{K}, \mathrm{f} \rangle;$

(3) $\rho_1\theta_1 + \ldots + \rho_l\theta_l \equiv \mathrm{T}^{u_2}(\nu_1\theta_1 + \ldots + \nu_l\theta_l)^* \mod \langle \mathfrak{m}\mathrm{K}, \mathrm{f}^* \rangle.$

Lemma 3. *Let* $(\mathrm{R}, \mathfrak{m}, \mathrm{GF}(q))$, f, $\deg(\mathrm{f}) = s$, $\mathbb{T}$, $\mathbb{T}_s$, χ, $\tilde{\alpha} = \{\alpha_1, \ldots, \alpha_\chi\}$, $\tilde{\beta} = \{\beta_1, \ldots, \beta_\chi\}$, $\mathfrak{F}$, $\mathrm{J} \in \mathfrak{F}$, $\ell_\mathrm{B}(\mathrm{J}) = k+1$, *where* $k \in \{1, \ldots, \chi-1\}$, $\mathrm{H_J} = (a_{ij})$, $\mathrm{H_J^\perp} = (b_{ij})$, $(\mathrm{H_J})^*_{(n,\mathrm{f}^*)} = (\nu_{ij})$, $(\mathrm{H_J^\perp})^*_{(n,\mathrm{f}^*)} = (\mu_{ij})$, *as in Observation 1. Let* n *an integer with* $n \geq \deg(\mathrm{f})$, $(\mathrm{B} = \mathrm{R[T]}/\langle\mathrm{f}\rangle, \mathfrak{m}\mathrm{B}, \mathrm{GF}(q^s))$, $(\mathrm{B}^* = \mathrm{R[T]}/\langle\mathrm{f}^*\rangle, \mathfrak{m}\mathrm{B}^*, \mathrm{GF}(q^s))$ *the separable extensions of* R *determined by* f *and* f^*, *respectively. We have the following:*
(1) Let $\pi \in \{1, \ldots, k\}$ *and* $\varpi \in \{1, \ldots, \chi-k\}$, *the following relations hold in the ring* B:

$$(\nu_{\pi 1}^{\mathbb{T}_s}\beta_1 + \ldots + \nu_{\pi\chi}^{\mathbb{T}_s}\beta_\chi)^*(b_{\varpi 1}^{\mathbb{T}_s}\alpha_1 + \ldots + b_{\varpi\chi}^{\mathbb{T}_s}\alpha_\chi) = 0,$$

$$(a_{\pi 1}^{\mathbb{T}_s}\beta_1 + \ldots + a_{\pi\chi}^{\mathbb{T}_s}\beta_\chi)(\mu_{\varpi 1}^{\mathbb{T}_s}\alpha_1 + \ldots + \mu_{\varpi\chi}^{\mathbb{T}_s}\alpha_\chi)^* = 0.$$

(2) The following relation holds in the ring B^*:

$$\langle(a_{11}^{\mathbb{T}_s}\beta_1 + \ldots + a_{1\chi}^{\mathbb{T}_s}\beta_\chi)^*, \ldots, (a_{k1}^{\mathbb{T}_s}\beta_1 + \ldots + a_{k\chi}^{\mathbb{T}_s}\beta_\chi)^*\rangle + \mathrm{ann}_\mathrm{R}(\mathfrak{m})\mathrm{B}^* =$$

$$\langle\nu_{11}^{\mathbb{T}_s}\beta_1 + \ldots + \nu_{1\chi}^{\mathbb{T}_s}\beta_\chi, \ldots, \nu_{k1}^{\mathbb{T}_s}\beta_1 + \ldots + \nu_{k\chi}^{\mathbb{T}_s}\beta_\chi\rangle + \mathrm{ann}_\mathrm{R}(\mathfrak{m})\mathrm{B}^*.$$

(3) Let $\phi \in \mathrm{R[T]}$ *such that* $\phi + \langle\mathrm{f}\rangle \in \mathrm{J} = (\mathrm{H_J})_{\tilde{\beta}}^{\mathbb{T}_s}$, *then there exist* $\mathrm{h}_1, \ldots, \mathrm{h}_k \in \mathrm{R[T]}$ *such that:*

$$\phi^* \equiv \mathrm{h}_1(\nu_{11}^{\mathbb{T}_s}\beta_1 + \ldots + \nu_{1\chi}^{\mathbb{T}_s}\beta_\chi) + \ldots + \mathrm{h}_k(\nu_{k1}^{\mathbb{T}_s}\beta_1 + \ldots + \nu_{k\chi}^{\mathbb{T}_s}\beta_\chi) \mod \langle\mathrm{ann}_\mathrm{R}(\mathfrak{m}), \mathrm{f}^*\rangle.$$

In particular $\phi^* + \langle\mathrm{f}^*\rangle \in ((\mathrm{H_J})^*_{(n,\mathrm{f}^*)})_{\tilde{\beta}}^{\mathbb{T}_s}$, *in the ring* B^*.

Proof: Observe that T is a unit of the rings B and B^*.
(1) By (3) of Lemma 2, there are non negative integer u_1, u_2 such that the following relations hold in B:
(a) $a_{\pi 1}^{\mathbb{T}_s}\beta_1 + \ldots + a_{\pi\chi}^{\mathbb{T}_s}\beta_\chi \equiv \mathrm{T}^{u_1}(\nu_{\pi 1}^{\mathbb{T}_s}\beta_1 + \ldots + \nu_{\pi\chi}^{\mathbb{T}_s}\beta_\chi)^* \mod \mathfrak{m}\mathrm{ann}_\mathrm{R}(\mathfrak{m}^2)\mathrm{B} = \mathrm{ann}_\mathrm{R}(\mathfrak{m})\mathrm{B}$,
(b) $b_{\varpi 1}^{\mathbb{T}_s}\alpha_1 + \ldots + b_{\varpi\chi}^{\mathbb{T}_s}\alpha_\chi \equiv \mathrm{T}^{u_2}(\mu_{\varpi 1}^{\mathbb{T}_s}\alpha_1 + \ldots + \mu_{\varpi\chi}^{\mathbb{T}_s}\alpha_\chi)^* \mod \mathfrak{m}^2\mathrm{B}$,
From (4c) of Observation 1 and (a) and (b) we have

$$0 = (a_{\pi 1}^{\mathbb{T}_s}\beta_1 + \ldots + a_{\pi\chi}^{\mathbb{T}_s}\beta_\chi)(b_{\varpi 1}^{\mathbb{T}_s}\alpha_1 + \ldots + b_{\varpi\chi}^{\mathbb{T}_s}\alpha_\chi) =$$

$$\mathrm{T}^{u_1}(\nu_{\pi 1}^{\mathbb{T}_s}\beta_1 + \ldots + \nu_{\pi\chi}^{\mathbb{T}_s}\beta_\chi)^*(b_{\varpi 1}^{\mathbb{T}_s}\alpha_1 + \ldots + b_{\varpi\chi}^{\mathbb{T}_s}\alpha_\chi) =$$

$$\mathrm{T}^{u_2}(a_{\pi 1}^{\mathbb{T}_s}\beta_1 + \ldots + a_{\pi\chi}^{\mathbb{T}_s}\beta_\chi)(\mu_{\varpi 1}^{\mathbb{T}_s}\alpha_1 + \ldots + \mu_{\varpi\chi}^{\mathbb{T}_s}\alpha_\chi)^*.$$

(2) From (2) of Lemma 2, there are units of the ring B^*, $\sigma_1, \ldots, \sigma_k$, such that

$$(a_{11}^{\mathbb{T}_s}\beta_1 + \ldots + a_{1\chi}^{\mathbb{T}_s}\beta_\chi)^* \equiv \sigma_1(\nu_{11}^{\mathbb{T}_s}\beta_1 + \ldots + \nu_{1\chi}^{\mathbb{T}_s}\beta_\chi) \mod \langle \mathrm{ann}_R(\mathfrak{m}), f^* \rangle,$$

$$\vdots$$

$$(a_{k1}^{\mathbb{T}_s}\beta_1 + \ldots + a_{k\chi}^{\mathbb{T}_s}\beta_\chi)^* \equiv \sigma_k(\nu_{k1}^{\mathbb{T}_s}\beta_1 + \ldots + \nu_{k\chi}^{\mathbb{T}_s}\beta_\chi) \mod \langle \mathrm{ann}_R(\mathfrak{m}), f^* \rangle,$$

the assertion follows.

(3) There are $\phi_1, \ldots, \phi_k \in R[T]$ such that

$$\phi \equiv \phi_1(a_{11}^{\mathbb{T}_s}\beta_1 + \ldots + a_{1\chi}^{\mathbb{T}_s}\beta_\chi) + \ldots + \phi_k(a_{k1}^{\mathbb{T}_s}\beta_1 + \ldots + a_{k\chi}^{\mathbb{T}_s}\beta_\chi) \mod \langle \mathrm{ann}_R(\mathfrak{m}), f \rangle,$$

then there are $v_1, \ldots, v_k$ non negative integers such that

$$\phi_1^* T^{v_1}(a_{11}^{\mathbb{T}_s}\beta_1 + \ldots + a_{1\chi}^{\mathbb{T}_s}\beta_\chi)^* + \ldots + \phi_k^* T^{v_k}(a_{k1}^{\mathbb{T}_s}\beta_1 + \ldots + a_{k\chi}^{\mathbb{T}_s}\beta_\chi)^*$$
$$\equiv \quad \phi_1^*\sigma_1 T^{v_1}(\nu_{11}^{\mathbb{T}_s}\beta_1 + \ldots + \nu_{1\chi}^{\mathbb{T}_s}\beta_\chi) + \ldots + \phi_k^*\sigma_k T^{v_k}(\nu_{k1}^{\mathbb{T}_s}\beta_1 + \ldots + \nu_{k\chi}^{\mathbb{T}_s}\beta_\chi)$$
$$\equiv \quad \phi^* \mod \langle \mathrm{ann}_R(\mathfrak{m}), f^* \rangle.$$

Lemma 4. *Let $(R, \mathfrak{m}, GF(q))$, f, B, B^*, $\mathbb{T}$, $\mathbb{T}_s$, χ, $\tilde{\alpha} = \{\alpha_1, \ldots, \alpha_\chi\}$, $\tilde{\beta} = \{\beta_1, \ldots, \beta_\chi\}$ and n as in Lemma 3. Let $\mathfrak{G}$ as in Observation 1, $I \in \mathfrak{G}$ with $\ell_B(I) = k + \ell_R(\mathfrak{m}^2)$, $k \in \{1, \ldots, \chi - 1\}$, $H_I = (c_{ij})$, $H_I^\perp = (d_{ij})$, $(H_I)^*_{(n, f^*)} = (\varsigma_{ij})$ and $(H_I^\perp)^*_{(n, f^*)} = (\epsilon_{ij})$. We have the following:*

(1) Let $\pi \in \{1, \ldots, k\}$ and $\varpi \in \{1, \ldots, \chi - k\}$, the following relations hold in the ring B:

$$(\varsigma_{\pi 1}^{\mathbb{T}_s}\alpha_1 + \ldots + \varsigma_{\pi\chi}^{\mathbb{T}_s}\alpha_\chi)^*(d_{\varpi 1}^{\mathbb{T}_s}\beta_1 + \ldots + d_{\varpi\chi}^{\mathbb{T}_s}\beta_\chi) = 0,$$

$$(c_{\pi 1}^{\mathbb{T}_s}\alpha_1 + \ldots + c_{\pi\chi}^{\mathbb{T}_s}\alpha_\chi)(\epsilon_{\varpi 1}^{\mathbb{T}_s}\beta_1 + \ldots + \epsilon_{\varpi\chi}^{\mathbb{T}_s}\beta_\chi)^* = 0.$$

(2) The following relation holds in the ring B^*:

$$\langle (c_{11}^{\mathbb{T}_s}\alpha_1 + \ldots + c_{1\chi}^{\mathbb{T}_s}\alpha_\chi)^*, \ldots, (c_{k1}^{\mathbb{T}_s}\alpha_1 + \ldots + c_{k\chi}^{\mathbb{T}_s}\alpha_\chi)^* \rangle + \mathfrak{m}^2 B^* =$$

$$\langle \varsigma_{11}^{\mathbb{T}_s}\alpha_1 + \ldots + \varsigma_{1\chi}^{\mathbb{T}_s}\alpha_\chi, \ldots, \varsigma_{k1}^{\mathbb{T}_s}\alpha_1 + \ldots + \varsigma_{k\chi}^{\mathbb{T}_s}\alpha_\chi \rangle + \mathfrak{m}^2 B^*.$$

(3) Let $\phi \in R[T]$ such that $\phi + \langle f \rangle \in I = (H_I)_{\tilde{\alpha}}^{\mathbb{T}_s} + \mathfrak{m}^2 B$, then there exist $h_1, \ldots, h_k \in R[T]$ such that

$$\phi^* \equiv h_1(\varsigma_{11}^{\mathbb{T}_s}\alpha_1 + \ldots + \varsigma_{1\chi}^{\mathbb{T}_s}\alpha_\chi) + \ldots + h_k(\varsigma_{k1}^{\mathbb{T}_s}\alpha_1 + \ldots + \varsigma_{k\chi}^{\mathbb{T}_s}\alpha_\chi) \mod \langle \mathfrak{m}^2, f^* \rangle.$$

In particular $\phi^* + \langle f^* \rangle \in ((H_I)^*_{(n, f^*)})_{\tilde{\alpha}}^{\mathbb{T}_s} + \mathfrak{m}^2 B^*$, in the ring B^*.

Proof: Observe that T is a unit of the rings B and B^*.

(1) By (3) of Lemma 2, there are non negative integer u_1, u_2 such that

(a) $c_{\pi1}^{\mathbb{T}_s}\alpha_1 + \ldots + c_{\pi\chi}^{\mathbb{T}_s}\alpha_\chi \equiv T^{u_1}(\varsigma_{\pi1}^{\mathbb{T}_s}\alpha_1 + \ldots + \varsigma_{\pi\chi}^{\mathbb{T}_s}\alpha_\chi)^* \bmod \mathfrak{m}^2 B$,

(b) $d_{\varpi1}^{\mathbb{T}_s}\beta_1 + \ldots + d_{\varpi\chi}^{\mathbb{T}_s}\beta_\chi \equiv T^{u_2}(\epsilon_{\varpi1}^{\mathbb{T}_s}\beta_1 + \ldots + \epsilon_{\varpi\chi}^{\mathbb{T}_s}\beta_\chi)^* \bmod \mathfrak{m}\mathrm{ann}_R(\mathfrak{m}^2)B = \mathrm{ann}_R(\mathfrak{m})B$.

From (7) of Observation 1 and (a) and (b) we have

$$0 = (c_{\pi1}^{\mathbb{T}_s}\alpha_1 + \ldots + c_{\pi\chi}^{\mathbb{T}_s}\alpha_\chi)(d_{\varpi1}^{\mathbb{T}_s}\beta_1 + \ldots + d_{\varpi\chi}^{\mathbb{T}_s}\beta_\chi) =$$

$$T^{u_1}(\varsigma_{\pi1}^{\mathbb{T}_s}\alpha_1 + \ldots + \varsigma_{\pi\chi}^{\mathbb{T}_s}\alpha_\chi)^*(d_{\varpi1}^{\mathbb{T}_s}\beta_1 + \ldots + d_{\varpi\chi}^{\mathbb{T}_s}\beta_\chi) =$$

$$T^{u_2}(c_{\pi1}^{\mathbb{T}_s}\alpha_1 + \ldots + c_{\pi\chi}^{\mathbb{T}_s}\alpha_\chi)(\epsilon_{\varpi1}^{\mathbb{T}_s}\beta_1 + \ldots + \epsilon_{\varpi\chi}^{\mathbb{T}_s}\beta_\chi)^*,$$

the assertion follows.

(2) From (2) of Lemma 2, there are units of the ring B^*, $\sigma_1, \ldots, \sigma_k$, such that

$$(c_{11}^{\mathbb{T}_s}\alpha_1 + \ldots + c_{1\chi}^{\mathbb{T}_s}\alpha_\chi)^* \equiv \sigma_1(\varsigma_{11}^{\mathbb{T}_s}\alpha_1 + \ldots + \varsigma_{1\chi}^{\mathbb{T}_s}\alpha_\chi) \bmod \langle \mathfrak{m}^2, \mathrm{f}^* \rangle$$

$$\vdots$$

$$(c_{k1}^{\mathbb{T}_s}\alpha_1 + \ldots + c_{k\chi}^{\mathbb{T}_s}\alpha_\chi)^* \equiv \sigma_k(\varsigma_{k1}^{\mathbb{T}_s}\alpha_1 + \ldots + \varsigma_{k\chi}^{\mathbb{T}_s}\alpha_\chi) \bmod \langle \mathfrak{m}^2, \mathrm{f}^* \rangle$$

the assertion follows.

(3) There are $\phi_1, \ldots, \phi_k \in R[T]$ such that

$$\phi \equiv \phi_1(c_{11}^{\mathbb{T}_s}\alpha_1 + \ldots + c_{1\chi}^{\mathbb{T}_s}\alpha_\chi) + \ldots + \phi_k(c_{k1}^{\mathbb{T}_s}\alpha_1 + \ldots + c_{k\chi}^{\mathbb{T}_s}\alpha_\chi) \bmod \langle \mathfrak{m}^2, \mathrm{f} \rangle,$$

then there are $v_1, \ldots, v_k$ non negative integers such that

$$\phi_1^* T^{v_1}(c_{11}^{\mathbb{T}_s}\alpha_1 + \ldots + c_{1\chi}^{\mathbb{T}_s}\alpha_\chi)^* + \ldots + \phi_k^* T^{v_k}(c_{k1}^{\mathbb{T}_s}\alpha_1 + \ldots + c_{k\chi}^{\mathbb{T}_s}\alpha_\chi)^*$$

$$\equiv \quad \phi_1^*\sigma_1 T^{v_1}(\varsigma_{11}^{\mathbb{T}_s}\alpha_1 + \ldots + \varsigma_{1\chi}^{\mathbb{T}_s}\alpha_\chi) + \ldots + \phi_k^*\sigma_k T^{v_k}(\varsigma_{k1}^{\mathbb{T}_s}\alpha_1 + \ldots + \varsigma_{k\chi}^{\mathbb{T}_s}\alpha_\chi)$$

$$\equiv \quad \phi^* \bmod \langle \mathfrak{m}^2, \mathrm{f}^* \rangle.$$

2.3   The family $\mathfrak{F}_5^4$

The family of finite local Frobenius non-chain rings of length 5 whose maximal ideal has nilpotency index 4 is denoted by $\mathfrak{F}_5^4$. Recently, the rings in the family $\mathfrak{F}_5^4$ and the lattice of ideals of a separable extension of a ring in this family were determined in [5] and for completeness we recall the results here.

Theorem 1. *Let* $(R, \mathfrak{m}, GF(p^d)) \in \mathfrak{F}_5^4$, *then* R *is isomorphic to one of the following rings:*
If p *is odd*

(i) $\mathrm{char(R)} = p^2$ and $\{0, 1, \zeta, \ldots, \zeta^{2^d-2}\}$ is the Teichmüller set of the Galois ring $\mathrm{GR}(p^2, d)$.

(a) $\mathfrak{m}^2 = \langle p \rangle = \langle \mathrm{y}^2 \rangle$ and $\mathfrak{m}^3 = \langle \mathrm{y}^3 \rangle = \langle \mathrm{x}^2 \rangle$.
(1) $\mathrm{GR}(p^2, d)[\mathrm{X}, \mathrm{Y}]/\langle \mathrm{Y}^2 - p, \mathrm{X}^2 - \mathrm{Y}^3, \mathrm{XY} \rangle$,
(2) $\mathrm{GR}(p^2, d)[\mathrm{X}, \mathrm{Y}]/\langle \zeta \mathrm{Y}^2 - p, \mathrm{X}^2 - \mathrm{Y}^3, \mathrm{XY} \rangle$,
(3) $\mathrm{GR}(p^2, d)[\mathrm{X}, \mathrm{Y}]/\langle \mathrm{Y}^2 - p, \mathrm{X}^2 - \zeta \mathrm{Y}^3, \mathrm{XY} \rangle$,
(4) $\mathrm{GR}(p^2, d)[\mathrm{X}, \mathrm{Y}]/\langle \zeta \mathrm{Y}^2 - p, \mathrm{X}^2 - \zeta \mathrm{Y}^3, \mathrm{XY} \rangle$.

(b) $\mathfrak{m}^2 = \langle \mathrm{y}^2 \rangle$, $\mathfrak{m}^3 = \langle p \rangle = \langle \mathrm{y}^3 \rangle = \langle \mathrm{x}^2 \rangle$ and $3 | p^d - 1$.
(5) $\mathrm{GR}(p^2, d)[\mathrm{X}, \mathrm{Y}]/\langle \mathrm{Y}^3 - p, \mathrm{X}^2 - \mathrm{Y}^3, \mathrm{XY} \rangle$,
(6) $\mathrm{GR}(p^2, d)[\mathrm{X}, \mathrm{Y}]/\langle \zeta \mathrm{Y}^3 - p, \mathrm{X}^2 - \mathrm{Y}^3, \mathrm{XY} \rangle$,
(7) $\mathrm{GR}(p^2, d)[\mathrm{X}, \mathrm{Y}]/\langle \zeta^2 \mathrm{Y}^3 - p, \mathrm{X}^2 - \mathrm{Y}^3, \mathrm{XY} \rangle$,
(8) $\mathrm{GR}(p^2, d)[\mathrm{X}, \mathrm{Y}]/\langle \mathrm{Y}^3 - p, \mathrm{X}^2 - \zeta \mathrm{Y}^3, \mathrm{XY} \rangle$,
(9) $\mathrm{GR}(p^2, d)[\mathrm{X}, \mathrm{Y}]/\langle \zeta \mathrm{Y}^3 - p, \mathrm{X}^2 - \zeta \mathrm{Y}^3, \mathrm{XY} \rangle$,
(10) $\mathrm{GR}(p^2, d)[\mathrm{X}, \mathrm{Y}]/\langle \zeta^2 \mathrm{Y}^3 - p, \mathrm{X}^2 - \zeta \mathrm{Y}^3, \mathrm{XY} \rangle$.

(c) $\mathfrak{m}^2 = \langle \mathrm{y}^2 \rangle$, $\mathfrak{m}^3 = \langle p \rangle = \langle \mathrm{y}^3 \rangle = \langle \mathrm{x}^2 \rangle$ and $3 \nmid p^d - 1$.
(11) $\mathrm{GR}(p^2, d)[\mathrm{X}, \mathrm{Y}]/\langle \mathrm{Y}^3 - p, \mathrm{X}^2 - \mathrm{Y}^3, \mathrm{XY} \rangle$,
(12) $\mathrm{GR}(p^2, d)[\mathrm{X}, \mathrm{Y}]/\langle \mathrm{Y}^3 - p, \mathrm{X}^2 - \zeta \mathrm{Y}^3, \mathrm{XY} \rangle$.

(ii) $\mathrm{char(R)} = p$

(a) $\mathfrak{m}^2 = \langle \mathrm{y}^2 \rangle$ and $\mathfrak{m}^3 = \langle \mathrm{y}^3 \rangle = \langle \mathrm{x}^2 \rangle$.
(13) $\mathrm{GF}(p^d)[\mathrm{X}, \mathrm{Y}]/\langle \mathrm{X}^2 - \mathrm{Y}^3, \mathrm{XY} \rangle$.

(iii) $\mathrm{char(R)} = p^4$ and $\{0, 1, \zeta, \ldots, \zeta^{p^d-2}\}$ is the Teichmüller set of the Galois ring $\mathrm{GR}(p^4, d)$.

(a) $p \notin \mathfrak{m}^2$, $\mathfrak{m}^2 = \langle p^2 \rangle$ and $\mathfrak{m}^3 = \langle p^3 \rangle = \langle \mathrm{x}^2 \rangle$.
(14) $\mathrm{GR}(p^4, d)[\mathrm{X}]/\langle \mathrm{X}^2 - p^3, p\mathrm{X} \rangle$,
(15) $\mathrm{GR}(p^4, d)[\mathrm{X}]/\langle \mathrm{X}^2 - \zeta p^3, p\mathrm{X} \rangle$.

(iv) $\mathrm{char(R)} = p^3$ and $\{0, 1, \zeta, \ldots, \zeta^{p^d-2}\}$ is the Teichmüller set of the Galois ring $\mathrm{GR}(p^3, d)$.

(a) $p \notin \mathfrak{m}^2$, $\mathfrak{m}^2 = \langle \mathrm{x}^2 \rangle$, $\mathfrak{m}^3 = \langle p^2 \rangle = \langle \mathrm{x}^3 \rangle$ and $3 \nmid p^d - 1$.
(16) $\mathrm{GR}(p^3, d)[\mathrm{X}]/\langle p^2 - \mathrm{X}^3, p\mathrm{X} \rangle$.

(b) $p \notin \mathfrak{m}^2$, $\mathfrak{m}^2 = \langle \mathrm{x}^2 \rangle$, $\mathfrak{m}^3 = \langle p^2 \rangle = \langle \mathrm{x}^3 \rangle$ and $3 | p^d - 1$,
(17) $\mathrm{GR}(p^3, d)[\mathrm{X}]/\langle p^2 - \mathrm{X}^3, p\mathrm{X} \rangle$,
(18) $\mathrm{GR}(p^3, d)[\mathrm{X}]/\langle p^2 - \zeta \mathrm{X}^3, p\mathrm{X} \rangle$,
(19) $\mathrm{GR}(p^3, d)[\mathrm{X}]/\langle p^2 - \zeta^2 \mathrm{X}^3, p\mathrm{X} \rangle$.

If $p = 2$

(i) $\operatorname{char}(R) = 2^2$ and $\{0, 1, \zeta, \ldots, \zeta^{2^d-2}\}$ is the Teichmüller set of the Galois ring $\operatorname{GR}(2^2, d)$.

    (a) $\mathfrak{m}^2 = \langle 2 \rangle = \langle y^2 \rangle$ and $\mathfrak{m}^3 = \langle x^2 \rangle = \langle y^3 \rangle$.
       (20) $\operatorname{GR}(2^2, d)[X, Y]/\langle Y^2 - 2, X^2 - Y^3, XY \rangle$.

    (b) $\mathfrak{m}^2 = \langle y^2 \rangle$, $\mathfrak{m}^3 = \langle 2 \rangle = \langle x^2 \rangle = \langle y^3 \rangle$ and $3 | 2^d - 1$.
       (21) $\operatorname{GR}(2^2, d)[X, Y]/\langle Y^3 - 2, X^2 - Y^3, XY \rangle$,
       (22) $\operatorname{GR}(2^2, d)[X, Y]/\langle \zeta Y^3 - 2, X^2 - Y^3, XY \rangle$,
       (23) $\operatorname{GR}(2^2, d)[X, Y]/\langle \zeta^2 Y^3 - 2, X^2 - Y^3, XY \rangle$.

    (c) $\mathfrak{m}^2 = \langle y^2 \rangle$, $\mathfrak{m}^3 = \langle 2 \rangle = \langle x^2 \rangle = \langle y^3 \rangle$ and $3 \nmid 2^d - 1$.
       (24) $\operatorname{GR}(2^2, d)[X, Y]/\langle Y^3 - 2, X^2 - Y^3, XY \rangle$.

(ii) $\operatorname{char}(R) = 2$

    (a) $\mathfrak{m}^2 = \langle y^2 \rangle$ and $\mathfrak{m}^3 = \langle x^2 \rangle = \langle y^3 \rangle$.
       (25) $\operatorname{GF}(2^d)[X, Y]/\langle X^2 - Y^3, XY \rangle$.

(iii) $\operatorname{char}(R) = 2^4$.

    (a) $\mathfrak{m}^2 = \langle 2^2 \rangle$, $\mathfrak{m}^3 = \langle 2^3 \rangle = \langle x^2 \rangle$ and $2 \notin \mathfrak{m}^2$.
       (26) $\operatorname{GR}(2^4, d)[X]/\langle X^2 - 2^3, 2X \rangle$.

(iv) $\operatorname{char}(R) = 2^3$ and $\{0, 1, \zeta, \ldots, \zeta^{2^d-2}\}$ is the Teichmüller set of the Galois ring $\operatorname{GR}(2^3, d)$.

    (a) $2 \notin \mathfrak{m}^2$, $\mathfrak{m}^2 = \langle x^2 \rangle$, $\mathfrak{m}^3 = \langle 2^2 \rangle = \langle x^3 \rangle$ and $3 \nmid 2^d - 1$.
       (27) $\operatorname{GR}(2^3, d)[X]/\langle 2^2 - X^3, 2X \rangle$.

    (b) $2 \notin \mathfrak{m}^2$, $\mathfrak{m}^2 = \langle x^2 \rangle$, $\mathfrak{m}^3 = \langle 2^2 \rangle = \langle x^3 \rangle$ and $3 | 2^d - 1$.
       (28) $\operatorname{GR}(2^3, d)[X]/\langle 2^2 - X^3, 2X \rangle$,
       (29) $\operatorname{GR}(2^3, d)[X]/\langle 2^2 - \zeta X^3, 2X \rangle$,
       (30) $\operatorname{GR}(2^3, d)[X]/\langle 2^2 - \zeta^2 X^3, 2X \rangle$.

For the remainder of the manuscript the following notation will be used. Let $(R, \mathfrak{m}, \operatorname{GF}(q)) \in \mathfrak{F}_5^4$, $\mathbb{T}$ and $\mathbb{T}_s$ be as above, $\tilde{\alpha} = \{\alpha_1, \alpha_2\}$ a minimal R-generating set for $\mathfrak{m}$, $f \in R[T]$ a monic basic irreducible polynomial of degree s and $(B = R[T]/\langle f \rangle, \mathfrak{m}B, \operatorname{GF}(q^s))$ the separable extension of R determined by f.

(1) A fixed minimal R-generating set $\{\alpha_1, \alpha_2\}$ of $\mathfrak{m}$ will be considered.
    If the ring R is one of the rings $(1) - (13)$ or $(20) - (25)$ mentioned in Theorem 1, $\alpha_1 = x$ and $\alpha_2 = y$.
    If the ring R is one of the rings $(14), (15)$ mentioned in Theorem 1, $\alpha_1 = x$ and $\alpha_2 = p$.

If the ring R is one of the rings $(16) - (19)$ mentioned in Theorem 1, $\alpha_1 = p$ and $\alpha_2 = $ x.

If the ring R is one of the rings $(27) - (30)$ mentioned in Theorem 1, $\alpha_1 = 2$ and $\alpha_2 = $ x.

If the ring R is the ring (26) mentioned in Theorem 1, $\alpha_1 = $ x and $\alpha_2 = 2$.

When we take a minimal R-generating set for $\mathfrak{m}$ we understand that $\{\alpha_1, \alpha_2\}$ is the ordered minimal R-generating set for $\mathfrak{m}$.

(2) Since $\text{ann}_R(\mathfrak{m}^2) = \langle \alpha_1, \alpha_2^2 \rangle$, $\{\alpha_1, \alpha_2^2\}$ will be the fixed minimal R-generating set for $\text{ann}_R(\mathfrak{m}^2)$.

When we take a minimal R-generating set for $\text{ann}_R(\mathfrak{m}^2)$ we understand that $\{\beta_1, \beta_2\}$ is the ordered minimal R-generating set for $\text{ann}_R(\mathfrak{m}^2)$.

(3) We write $\tilde{\alpha}$ for $\{\alpha_1, \alpha_2\}$ and $\tilde{\beta}$ for $\{\alpha_1, \alpha_2^2\}$.

Lemma 5. *Let* $(R, \mathfrak{m}, GF(q)) \in \mathfrak{F}_5^4$, $\mathbb{T}$ *and* $\mathbb{T}_s$ *be as above,* $\tilde{\alpha} = \{\alpha_1, \alpha_2\}$ *a minimal R-generating set for* $\mathfrak{m}$, $f \in R[T]$ *a monic basic irreducible polynomial of degree* s, $(B = R[T]/\langle f \rangle, \mathfrak{m}B, GF(q^s))$ *the separable extension of* R *determined by* f, *then:*

(1) *The ideals of length* $0, 1, 4, 5$ *of* B *are* $\langle 0 \rangle$, $\mathfrak{m}^3$, $\mathfrak{m}$, B, *respectively.*

(2) *Ideals of length* 2 *of* B *are between* $\text{ann}_R(\mathfrak{m}^2)B$ *and* $\mathfrak{m}^3 B$. *These ideals are:*

$$\mathfrak{m}^2 = \langle \alpha_2^2 \rangle B, \langle \alpha_1 + \lambda_1^{\mathbb{T}_s} \alpha_2^2 \rangle B, \ldots, \langle \alpha_1 + \lambda_{q^s}^{\mathbb{T}_s} \alpha_2^2 \rangle B, \quad \lambda_i \in GF(q^s).$$

(3) *Ideals of length* 3 *of* B *are between* $\mathfrak{m}B$ *and* $\mathfrak{m}^2 B$. *These ideals are:*

$$\langle \alpha_1, \alpha_2^2 \rangle B, \langle \alpha_2 \rangle B, \langle \alpha_1 + \lambda_1^{\mathbb{T}_s} \alpha_2 \rangle B, \ldots, \langle \alpha_1 + \lambda_{q^s-1}^{\mathbb{T}_s} \alpha_2 \rangle B, \ \lambda_i \in GF(q^s) \setminus \{0\}.$$

Observation 3. *With the notation as in Lemma 5. Observe the following:*

(1) $\text{ann}_B(\mathfrak{m}B) = \mathfrak{m}^3 B = \langle \alpha_2^3 \rangle B = \langle \alpha_1^2 \rangle B$.

(2) $\mathfrak{m}B = \langle \alpha_1, \alpha_2 \rangle B$ *and* $\text{ann}_R(\mathfrak{m}^2)B = \langle \alpha_1, \alpha_2^2 \rangle B$ *are the only non principal ideals of* B.

(3) *The ideals* $(0,1)_{\tilde{\alpha}}^{\mathbb{T}_s} = \langle \alpha_2 \rangle B$, $(1, \lambda_i)_{\tilde{\alpha}}^{\mathbb{T}_s} = \langle \alpha_1 + \lambda_i^{\mathbb{T}_s} \alpha_2 \rangle B$, $\lambda_i \in GF(q) \setminus \{0\}$, *contain* $\mathfrak{m}^2 B$.

(4) *The ideal* $\text{ann}_R(\mathfrak{m}^2)B = \langle \alpha_1, \alpha_2^2 \rangle B = (1,0)_{\tilde{\alpha}}^{\mathbb{T}_s} + \mathfrak{m}^2 B$ *is simply denoted by* $(1,0)_{\tilde{\alpha}}^{\mathbb{T}_s}$.

(5) *From* (1) *of Observation 1 and relation* $\ell_{\mathrm{B}}(\mathrm{I}) + \ell_{\mathrm{B}}(\mathrm{ann}_{\mathrm{B}}(\mathrm{I})) = \ell_{\mathrm{B}}(\mathrm{B})$, *where* I *is an ideal of* B, *the annihilator ideal over the ideals of* B *of length* 2 *and* 3 *induces the permutations:*

$$(*)^{\perp_1}, (*)^{\perp_2} = [(*)^{\perp_1}]^{-1} : \{(0,1), (1,\lambda) : \lambda_i \in \mathrm{GF}(q^s)\}$$

$$\rightarrow \{(0,1), (1,\lambda) : \lambda_i \in \mathrm{GF}(q^s)\}$$

$$\vec{u} \mapsto \vec{u}^{\perp_1} := \vec{v} \ if \ \mathrm{ann}_{\mathrm{B}}(\vec{u}_{\tilde{\alpha}}^{\mathbb{T}_s}) = \vec{v}_{\tilde{\beta}}^{\mathbb{T}_s},$$

$$\vec{u} \mapsto \vec{u}^{\perp_2} := \vec{v} \ if \ \mathrm{ann}_{\mathrm{B}}(\vec{u}_{\tilde{\beta}}^{\mathbb{T}_s}) = \vec{v}_{\tilde{\alpha}}^{\mathbb{T}_s}.$$

Lemma 6. *Let* $(\mathrm{R}, \mathfrak{m}, \mathrm{GF}(q)) \in \mathfrak{F}_5^4$ *be any of the rings* (1), (2), (5) − (7), (11), (13), (14), (16), (17), (20) − (28) *in Theorem 1. Let* f, $\deg(\mathrm{f}) = s$, (B, $\mathfrak{m}$B, $\mathrm{GF}(q^s)$), $\mathbb{T}$, $\mathbb{T}_s$, $\tilde{\alpha} = \{\alpha_1, \alpha_2\}$ *and* $\tilde{\beta} = \{\alpha_1, \alpha_2^2\}$ *be as above. Then* $(1,0)^{\perp_1} = (0,1)$, $(0,1)^{\perp_1} = (1,0)$, $(1,\lambda_1)^{\perp_1} = (1,\lambda_2)$, $\lambda_1, \lambda_2 \in \mathbb{T}$, $\bar{\lambda}_1 \bar{\lambda}_2 = -1$.

Proof: Assertions follow from the following relations: $\alpha_1^2 = \alpha_2^3$, $\alpha_2 \alpha_1 = 0$, $(0,1)_{\tilde{\alpha}}^{\mathbb{T}_s} = \alpha_2 \mathrm{B}$, $(1,0)_{\tilde{\beta}}^{\mathbb{T}_s} = \alpha_1 \mathrm{B}$, $(0,1)_{\tilde{\beta}}^{\mathbb{T}_s} = \mathfrak{m}^2 \mathrm{B}$, $(1,0)_{\tilde{\alpha}}^{\mathbb{T}_s} = \mathrm{ann}_{\mathrm{B}}(\mathfrak{m}^2 \mathrm{B})$ and $(1,\lambda_1)_{\tilde{\alpha}}^{\mathbb{T}_s}(1,\lambda_2)_{\tilde{\beta}}^{\mathbb{T}_s} = \langle (1 + \lambda_1 \lambda_2) \alpha_2^3 \rangle$.

Lemma 7. *Let* $(\mathrm{R}, \mathfrak{m}, \mathrm{GF}(q)) \in \mathfrak{F}_5^4$ *be any of the rings* (3), (4), (8) − (10), (12), (15), (18), (29) *in Theorem 1. Let* f, $\deg(\mathrm{f}) = s$, (B = R[T]/$\langle$f$\rangle$, $\mathfrak{m}$B, $\mathrm{GF}(q^s)$), $\mathbb{T}$, $\mathbb{T}_s$, $\tilde{\alpha} = \{\alpha_1, \alpha_2\}$ *and* $\tilde{\beta} = \{\alpha_1, \alpha_2^2\}$ *be as above. Then* $(1,0)^{\perp_1} = (0,1)$, $(0,1)^{\perp_1} = (1,0)$, $(1,\lambda_1)^{\perp_1} = (1,\lambda_2)$, $\lambda_1, \lambda_2 \in \mathbb{T}$, $\bar{\lambda}_1 \bar{\lambda}_2 = -\bar{\zeta}$.

Proof: Assertions follow from relations $\alpha_1^2 = \zeta \alpha_2^3$ and $(1,\lambda_1)_{\tilde{\alpha}}^{\mathbb{T}_s}(1,\lambda_2)_{\tilde{\beta}}^{\mathbb{T}_s} = \langle (\zeta + \lambda_1 \lambda_2) \alpha_2^3 \rangle$, and the same kind of arguments as in the proof of Lemma 6.

Lemma 8. *Let* $(\mathrm{R}, \mathfrak{m}, \mathrm{GF}(q)) \in \mathfrak{F}_5^4$ *be any of the rings* (19), (30) *in Theorem 1. Let* f, $\deg(\mathrm{f}) = s$, (B = R[T]/$\langle$f$\rangle$, $\mathfrak{m}$B, $\mathrm{GF}(q^s)$), $\mathbb{T}$, $\mathbb{T}_s$, $\tilde{\alpha} = \{\alpha_1, \alpha_2\}$ *and* $\tilde{\beta} = \{\alpha_1, \alpha_2^2\}$ *be as above. Then* $(1,0)^{\perp_1} = (0,1)$, $(0,1)^{\perp_1} = (1,0)$, $(1,\lambda_1)^{\perp_1} = (1,\lambda_2)$, $\lambda_1, \lambda_2 \in \mathbb{T}$, $\bar{\lambda}_1 \bar{\lambda}_2 = -\bar{\zeta}^2$.

Proof: Assertions follow from relations $\alpha_1^2 = \zeta^2 \alpha_2^3$ and $(1,\lambda_1)_{\tilde{\alpha}}^{\mathbb{T}_s}(1,\lambda_2)_{\tilde{\beta}}^{\mathbb{T}_s} = \langle (\zeta^2 + \lambda_1 \lambda_2) \alpha_2^3 \rangle$, and the same kind of arguments as in the proof of Lemma 6.

3 Duals of constacyclic codes over rings in $\mathfrak{F}_5^4$

Let R be a ring and γ be a unit of R. A γ-constacyclic code over R is a submodule of R^n invariant under the permutation $\sigma_\gamma : \mathrm{R}^n \rightarrow \mathrm{R}^n$ given by $(a_0, \ldots, a_{n-1}) \mapsto (\gamma a_{n-1}, a_0, \ldots, a_{n-2})$. These codes can be thought as ideals in the ring R[T]/$\langle \mathrm{T}^n - \gamma \rangle$ via the γ-polynomial representation of R^n, $r_\gamma : \mathrm{R}^n \rightarrow$

$R[T]/\langle T^n - \gamma \rangle$ given by $(a_0, \ldots, a_{n-1}) \mapsto a_0 + a_1 T + \ldots + a_{n-1} T^{n-1} + \langle T^n - \gamma \rangle$. The dual of a linear code C over R of length n is defined as $C^\perp = \{\vec{a} \in R^n : \vec{a} \cdot \vec{b} = 0, \ \forall \ \vec{b} \in C\}$, where $\cdot$ is the usual scalar product on R^n. Let $f = \sum_{i=0}^{n-1} a_i T^i + \langle T^n - \gamma \rangle$, $g = \sum_{i=0}^{n-1} b_i T^i + \langle T^n - \gamma \rangle$ in $R[T]/\langle T^n - \gamma \rangle$, then $fg^* = 0$ in $R[T]/\langle T^n - \gamma \rangle$ if and only if $(b_0, \ldots, b_{n-1})$ is orthogonal to $(a_0, \ldots, a_{n-1})$ and all of its γ-constacyclic shifts, if and only if $(a_0, \ldots, a_{n-1})$ is orthogonal to $(b_0, \ldots, b_{n-1})$ and all of its γ^{-1}-constacyclic shifts.

For the rest of this manuscript we write $\sum_{i=0}^{n-1} a_i T^i$ for the corresponding coset $\sum_{i=0}^{n-1} a_i T^i + \langle T^n - \gamma \rangle$ in the ring $R[T]/\langle T^n - \gamma \rangle$, n will be the length of codes, $(n, p) = 1$, so that, $T^n - \gamma$ is a product of unique monic basic irreducible pairwise coprime polynomials in $R[T]$.

The following is the result on the structure of γ-constacyclic codes over a ring in $\mathfrak{F}_5^4$, given in [5].

Theorem 2. *Let* $(R, \mathfrak{m}, GF(q)) \in \mathfrak{F}_5^4$, γ *a unit of* R, $\tilde{\alpha} = \{\alpha_1, \alpha_2\}$, $\tilde{\beta} = \{\alpha_1, \alpha_2^2\}$, $\mathbb{T}$ *and* $\mathbb{T}_s$ *as above and* $f_1, \ldots, f_r$ *the unique monic basic irreducible pairwise coprime polynomials such that* $T^n - \gamma = f_1 \cdots f_r$, $s_i = \deg(\bar{f}_i)$. *Let* C *a* γ-*constacyclic code of length* n *over* R. *Then*

(1) *There exists a unique partition of* $[1, r]$, $U_0, U_1, U_2, U_3, U_4, U_5$.

(2) *For each* $i \in \{2, 3\}$ *and each* $u \in U_i$, *there is a unique* $\vec{v}_u \in \{(0, 1), (1, \lambda) : \lambda \in GF(q^{s_u})\}$ *such that the corresponding ideal, in* $R[T]/\langle T^n - \gamma \rangle$, *of* C *is*

$$\langle \mathfrak{m}^3 \widehat{\prod_{u \in U_1} f_u}, \mathfrak{m} \widehat{\prod_{u \in U_4} f_u}, \widehat{\prod_{u \in U_5} f_u}, (\vec{v}_u)_{\tilde{\beta}}^{\mathbb{T}_{s_u}} \widehat{f_u}, (\vec{v}_w)_{\tilde{\alpha}}^{\mathbb{T}_{s_w}} \widehat{f_w} : u \in U_2, w \in U_3 \rangle$$

and

$$|C| = q^{5 \sum_{u \in U_5} s_u + 4 \sum_{u \in U_4} s_u + 3 \sum_{\nu \in U_3} s_u + 2 \sum_{u \in U_2} s_u + \sum_{u \in U_1} s_u}.$$

The next result gives the structure of the dual of a constacyclic code over a ring of the family $\mathfrak{F}_5^4$. Recall that if C is a linear code of length n over a finite Frobenius ring R, then $|C||C^\perp| = |R|^n$, see [15].

Theorem 3. *Let* $(R, \mathfrak{m}, GF(q)) \in \mathfrak{F}_5^4$, γ, $\tilde{\alpha} = \{\alpha_1, \alpha_2\}$, $\tilde{\beta} = \{\alpha_1, \alpha_2^2\}$, $\mathbb{T}$, $\mathbb{T}_s$, $f_1, \ldots, f_r$, $s_i = \deg(\bar{f}_i)$, C *as in Theorem 2 and*

(1) $U_0, U_1, U_2, U_3, U_4, U_5$ *the unique partition of* $\{1, \ldots, r\}$ *associated to* C,

(2) $\{\vec{v}_u : u \in U_2 \cup U_3\}$ *the vectors such that the corresponding ideal, in* $R[T]/\langle T^n - \gamma \rangle$, *of* C *is*

$$\langle \mathfrak{m}^3 \widehat{\prod_{u\in U_1} f_u}, \mathfrak{m}\widehat{\prod_{u\in U_4} f_u}, \widehat{\prod_{u\in U_5} f_u}, (\vec{v}_u)_{\tilde{\beta}}^{\mathbb{T}_{s_u}}\widehat{f_u}, (\vec{v}_w)_{\tilde{\alpha}}^{\mathbb{T}_{s_w}}\widehat{f_w} : u\in U_2, w\in U_3\rangle.$$

Then

(1) $f_1^{\circ *},\ldots,f_r^{\circ *}$ are the unique polynomials such that $T^n - \gamma^{-1} = f_1^{\circ *}\cdots f_r^{\circ *}$,

(2) The unique partition of $\{1,\ldots,r\}$, $V_0, V_1, V_2, V_3, V_4, V_5$, associated to the γ^{-1} constacyclic code $C^\perp$ is $V_0 = U_5$, $V_1 = U_4$, $V_2 = U_3$, $V_3 = U_2$, $V_4 = U_1$, $V_5 = U_0$,

(3) For $u \in V_2$, the unique element in $\{(0,1),(1,\lambda)\}$ over $GF(q^{s_u})$ associated to $C^\perp$ is $\overline{\gamma^{-1}}(\vec{v}_u^{\perp_1})^*_{(n,f_u^*)}$, $\perp_1$ is taken on the ring $R[T]/\langle f_u\rangle$,

(4) For $w \in V_3$, the unique element in $\{(0,1),(1,\lambda)\}$ over $GF(q^{s_w})$ associated to $C^\perp$ is $\overline{\gamma^{-1}}(\vec{v}_w^{\perp_2})^*_{(n,f_w^*)}$, $\perp_2$ is taken on the ring $R[T]/\langle f_w\rangle$,

(5) The ideal in $R[T]/\langle T^n - \gamma^{-1}\rangle$, associated to the γ^{-1}-constacyclic code $C^\perp$ is:

$$\langle \mathfrak{m}^3 \widehat{\prod_{u\in U_4} f_u^{\circ *}}, \mathfrak{m}\widehat{\prod_{u\in U_1} f_u^{\circ *}}, \widehat{\prod_{u\in U_0} f_u^{\circ *}}, (\overline{\gamma^{-1}}(\vec{v}_w^{\perp_1})^*_{(n,f_w^*)})_{\tilde{\beta}}^{\mathbb{T}_{s_w *}}\widehat{f_w^{\circ *}},$$

$$(\overline{\gamma^{-1}}(\vec{v}_u^{\perp_2})^*_{(n,f_u^*)})_{\tilde{\alpha}}^{\mathbb{T}_{s_u *}}\widehat{f_u^{\circ *}} : u\in U_2, w\in U_3\rangle =$$

$$\langle \mathfrak{m}^3 \widehat{\prod_{u\in U_4} f_{u^*}}, \mathfrak{m}\widehat{\prod_{u\in U_1} f_{u^*}}, \widehat{\prod_{u\in U_0} f_{u^*}}, (\overline{\gamma^{-1}}(\vec{v}_w^{\perp_1})^*_{(n,f_{w^*})})_{\tilde{\beta}}^{\mathbb{T}_{s_w *}}\widehat{f_{w^*}},$$

$$(\overline{\gamma^{-1}}(\vec{v}_u^{\perp_2})^*_{(n,f_{u^*})})_{\tilde{\alpha}}^{\mathbb{T}_{s_u *}}\widehat{f_{u^*}} : u\in U_2, w\in U_3\rangle =$$

$$\langle \mathfrak{m}^3 \widehat{\prod_{u\in U_4^*} f_u}, \mathfrak{m}\widehat{\prod_{u\in U_1^*} f_u}, \widehat{\prod_{u\in U_0^*} f_u}, (\overline{\gamma^{-1}}(\vec{v}_{w^*}^{\perp_1})^*_{(n,f_w)})_{\tilde{\beta}}^{\mathbb{T}_{s_w}}\widehat{f_w},$$

$$(\overline{\gamma^{-1}}(\vec{v}_{u^*}^{\perp_2})^*_{(n,f_u)})_{\tilde{\alpha}}^{\mathbb{T}_{s_u}}\widehat{f_u} : u\in U_2^*, w\in U_3^*\rangle$$

and

$$|C^\perp| = q^{5\sum_{u\in U_0} s_u + 4\sum_{u\in U_1} s_u + 3\sum_{u\in U_2} s_u + 2\sum_{u\in U_3} s_u + \sum_{u\in U_4} s_u}.$$

Proof: The relation $|M| = |GF(q)|^{\ell_R(M)}$ implies $|R| = q^5$, see Section 2. First, we prove the relation on $|C^\perp|$. By Theorem 2 and since $n = \sum_{u\in U_0} s_u + \sum_{u\in U_1} s_u + \sum_{u\in U_2} s_u + \sum_{u\in U_3} s_u + \sum_{u\in U_4} s_u + \sum_{u\in U_5} s_u$, then

$$|C^\perp| = \frac{q^{5n}}{|C|} = q^{5\sum_{u\in U_0} s_u + 4\sum_{u\in U_1} s_u + 3\sum_{u\in U_2} s_u + 2\sum_{u\in U_3} s_u + \sum_{u\in U_4} s_u}.$$

The equality about the ideals is easy to prove. Let D the γ^{-1}-constacyclic code given by

$$\langle \mathfrak{m}^3 \widehat{\prod_{u\in U_4} f_u^{\circ *}}, \mathfrak{m}\widehat{\prod_{u\in U_1} f_u^{\circ *}}, \widehat{\prod_{u\in U_0} f_u^{\circ *}}, (\overline{\gamma^{-1}}(\vec{v}_w^{\perp_1})^*_{(n,f_w^*)})_{\tilde{\beta}}^{\mathbb{T}_{s_w *}}\widehat{f_w^{\circ *}},$$

$$\overline{(\overline{\gamma^{-1}}(\vec{v}_u^{\perp 2})_{(n,f_u^*)}^*)}_{\tilde{\alpha}}^{\mathbb{T}_{s_u*}} \widehat{f_u^{\circ *}} : u \in U_2, w \in U_3 \rangle.$$

Since $\deg(f_u^{\circ *}) = \deg(f_u)$, using Theorem 2, the number of elements of D is $|D| = q^{5\sum_{u\in U_0} s_u + 4\sum_{u\in U_1} s_u + 3\sum_{u\in U_2} s_u + 2\sum_{u\in U_3} s_u + \sum_{u\in U_4} s_u} = |C^\perp|$. From (1) of Lemma 3, (1) of Lemma 4 and relations: $\mathfrak{m}^4 = \langle 0 \rangle$, $(f^*)^* = f$, when f is a polynomial with nonzero constant coefficient, $\widehat{f_i}\widehat{f_j} = 0$, in the ring $R[T]/\langle T^n - \gamma \rangle$, for $i \neq j$, we have $D \subseteq C^\perp$ and the assertion follows.

Let $R = GF(p^d)[X, Y]/\langle X^2 - Y^3, XY \rangle$, be the ring of cases (13) or (25) in Theorem 1. Every element of R can be uniquely written as $a_0 + a_1 x + a_2 y + a_3 y^2 + a_4 y^3$, where $a_i \in GF(p^d)$, $\tilde{\alpha} = \{x, y\}$, $\tilde{\beta} = \{x, y^2\}$, $\mathfrak{m}^2 = \langle y^2 \rangle$, $ann_R(\mathfrak{m}^2) = \langle x, y^2 \rangle$, $\mathfrak{m}^3 = \langle y^3 \rangle$ and $\mathbb{T} = GF(p^d)$ is a set of representatives for the residue field of R.

Example 1. *Let* $R = GF(2)[X, Y]/\langle X^2 - Y^3, XY \rangle$, $\tilde{\alpha} = \{x, y\}$, $\tilde{\beta} = \{x, y^2\}$ *be as above and* γ *be a unit of* R. *By Hensel's Lemma,* $T^{15} - \gamma = f_1 f_2 f_3 f_4 f_5$, *where* $\overline{f_1} = T^4 + T + 1$, $\overline{f_2} = T^4 + T^3 + 1$, $\overline{f_3} = T^4 + T^3 + T^2 + T + 1$, $\overline{f_4} = T^2 + T + 1$, $\overline{f_5} = T + 1 \in GF(2)[T]$, *then* $1^* = 2$, $2^* = 1$, $3^* = 3$, $4^* = 4$, $5^* = 5$.
If $U_0 = \{1\}$, $U_1 = \{2\}$, $U_2 = \{3\}$, $U_3 = \{4\}$, $U_4 = \{5\}$, $U_5 = \emptyset$, *let* $\vec{v}_3 = (1, 1 + T + T^3)$ *over* $GF(2^4) = R[T]/\langle f_3 \rangle$, $\vec{v}_4 = (1, 1 + T)$ *over* $GF(2^2) = R[T]/\langle f_4 \rangle$. *The associated code is:*

$$\langle \mathfrak{m}^3 \widehat{\prod_{u\in U_1} f_u}, \mathfrak{m} \widehat{\prod_{u\in U_4} f_u}, \widehat{\prod_{u\in U_5} f_u}, (\vec{v}_u)_{\tilde{\beta}}^{\mathbb{T}_{s_u}} \widehat{f_u}, (\vec{v}_w)_{\tilde{\alpha}}^{\mathbb{T}_{s_w}} \widehat{f_w} : u \in U_2, w \in U_3 \rangle =$$

$$\langle y^3 \widehat{f_2}, x\widehat{f_5}, y\widehat{f_5}, [x + (1 + T + T^3)y^2]\widehat{f_3}, [x + (1 + T)y]\widehat{f_4} \rangle =$$

$$\langle y^3 f_1 f_3 f_4 f_5, xf_1 f_2 f_3 f_4, yf_1 f_2 f_3 f_4, [x + (1 + T + T^3)y^2]f_1 f_2 f_4 f_5, [x + (1 + T)y]f_1 f_2 f_3 f_5 \rangle.$$

By Lemma 6 and because $(1 + T + T^3)^{-1} = T + T^2 + T^3$, *in* $R[T]/\langle f_3 \rangle$, $(1 + T)^{-1} = T$, *in* $R[T]/\langle f_4 \rangle$, *we have* $\vec{v}_3^{\perp 2} = (1, T + T^2 + T^3)$, $\vec{v}_4^{\perp 1} = (1, T)$. *Then* $\overline{\gamma^{-1}}(\vec{v}_3^\perp)_{(15, f_3)}^* = (1, 1 + T)$, $\overline{\gamma^{-1}}(\vec{v}_4^\perp)_{(15, f_4)}^* = (1, 1 + T)$ *and the dual code,* $C^\perp$, *is:*

$$\langle \mathfrak{m}^3 \widehat{\prod_{u\in U_4} f_u^*}, \mathfrak{m} \widehat{\prod_{u\in U_1} f_u^*}, \widehat{\prod_{u\in U_0} f_u^*}, (\overline{\gamma^{-1}}(\vec{v}_w^{\perp 1})_{(15, f_w^*)}^*)_{\tilde{\beta}}^{\mathbb{T}_{s_w*}} \widehat{f_w^*},$$

$$(\overline{\gamma^{-1}}(\vec{v}_u^{\perp 2})_{(15, f_u^*)}^*)_{\tilde{\alpha}}^{\mathbb{T}_{s_u*}} \widehat{f_u^*} : u \in U_2, w \in U_3 \rangle =$$

$$\langle y^3 \widehat{f_5}, x\widehat{f_1}, y\widehat{f_1}, \widehat{f_2}, [x + (1 + T)y^2]\widehat{f_4}, [x + (1 + T)y]\widehat{f_3} \rangle = \langle y^3 f_1 f_2 f_3 f_4,$$

$$xf_2 f_3 f_4 f_5, yf_2 f_3 f_4 f_5, f_1 f_3 f_4 f_5, [x + (1 + T)y^2]f_1 f_2 f_3 f_5, [x + (1 + T)y]f_1 f_2 f_4 f_5 \rangle.$$

4 Self-dual constacyclic codes over rings in $\mathfrak{F}_5^4$

The code C is called *self-dual* if $C = C^\perp$. In this Section the algebraic charac-
terization of self-dual γ-constacyclic codes over rings in the family $\mathfrak{F}_5^4$ is given,
when $\gamma = \gamma^{-1}$.

Let $(R, \mathfrak{m}, GF(q))$ be a finite local ring, $f_1, \ldots, f_r$ the unique monic basic
irreducible pairwise coprime polynomials such that $T^n - \gamma = f_1 \cdots f_r$, $u \in$
$\{1, \ldots, r\}$ and $U \subseteq \{1, \ldots, r\}$. Recall that, U^* is the set of indexes of the
polynomials which are associates of the polynomials in $\{f_u : u \in U\}$, u^* denotes
the index of the polynomial which is associate of f_u, $(u^*)^* = u$, $f_{u^*} = f_u^{\circ *}$,
$\prod_{u \in U} f_u^{\circ *} = \prod_{u \in U} f_{u^*} = \prod_{u \in U^*} f_u$ and there are r_1, r_2 non negative integers
such that f_{2i-1} is associate of f_{2i}, $1 \le i \le r_1$, and f_{2r_1+i} is self-reciprocal
polynomial, $1 \le i \le r_2$. U is called self-reciprocal if $U^* = U$ and special if
$U \cap U^* = \emptyset$.

Theorem 4. *Let* $(R, \mathfrak{m}, GF(q)) \in \mathfrak{F}_5^4$, γ, $\tilde{\alpha} = \{\alpha_1, \alpha_2\}$, $\tilde{\beta} = \{\alpha_1, \alpha_2^2\}$, $\mathbb{T}$, $\mathbb{T}_s$,
$f_1, \ldots, f_r$, $s_i = \deg(\bar{f}_i)$, C *as in Theorem 2,* r_1, r_2 *as above and*

(1) $U_0, U_1, U_2, U_3, U_4, U_5$ *the unique partition of* $\{1, \ldots, r\}$, *associated to* C,

(2) $\{\vec{v}_u : u \in U_2 \cup U_3\}$ *the vectors such that the corresponding ideal, in*
$R[T]/\langle T^n - \gamma \rangle$, *of* C *is*

$$\langle \mathfrak{m}^3 \widehat{\prod_{u \in U_1} f_u}, \mathfrak{m} \widehat{\prod_{u \in U_4} f_u}, \widehat{\prod_{u \in U_5} f_u}, (\vec{v}_u)_{\tilde{\beta}}^{\mathbb{T}_{su}} \widehat{f_u}, (\vec{v}_w)_{\tilde{\alpha}}^{\mathbb{T}_{sw}} \widehat{f_w} : u \in U_2, w \in U_3 \rangle.$$

The following conditions are equivalent:

(1) C *is self-dual code;*

(2) $U_i^* = U_{5-i}$, *for* $i \in \{0, \ldots, 5\}$, $\vec{v}_u = \overline{\gamma}(\vec{v}_{u^*}^{\perp_1})_{(n, f_u)}^*$, *for* $u \in U_2$ *and*
$\vec{v}_w = \overline{\gamma}(\vec{v}_{w^*}^{\perp_2})_{(n, f_w)}^*$, *for* $w \in U_3$.

(3) $U_i^* \subseteq U_{5-i}$, *for* $i \in \{0, \ldots, 5\}$, $\vec{v}_u = \overline{\gamma}(\vec{v}_{u^*}^{\perp_1})_{(n, f_u)}^*$, *for* $u \in U_2$ *and*
$\vec{v}_w = \overline{\gamma}(\vec{v}_{w^*}^{\perp_2})_{(n, f_w)}^*$, *for* $w \in U_3$.

Proof: $(2) \Leftrightarrow (3)$ is easy to prove.
$(1) \Leftrightarrow (2)$ From uniqueness in Theorem 2, by Theorem 3 and since $\{U_0^*, U_1^*,$
$U_2^*, U_3^*, U_4^*, U_5^*\}$ is a partition of $\{1, \ldots, r\}$,
$C = C^\perp \Leftrightarrow$

$$\langle \mathfrak{m}^3 \widehat{\prod_{u \in U_1} f_u}, \mathfrak{m} \widehat{\prod_{u \in U_4} f_u}, \widehat{\prod_{u \in U_5} f_u}, (\vec{v}_u)_{\tilde{\beta}}^{\mathbb{T}_{su}} \widehat{f_u}, (\vec{v}_w)_{\tilde{\alpha}}^{\mathbb{T}_{sw}} \widehat{f_w} : u \in U_2, w \in U_3 \rangle =$$

$$\langle \mathfrak{m}^3 \widehat{\prod_{u \in U_4^*} f_u}, \mathfrak{m} \widehat{\prod_{u \in U_1^*} f_u}, \widehat{\prod_{u \in U_0^*} f_u}, (\overline{\gamma}(\vec{v}_{w^*}^{\perp_1})^*_{(n,f_w)})^{\mathbb{T}_{s_w}}_{\tilde{\beta}} \widehat{f_w}, (\overline{\gamma}(\vec{v}_{u^*}^{\perp_2})^*_{(n,f_u)})^{\mathbb{T}_{s_u}}_{\tilde{\alpha}} \widehat{f_u} :$$

$$u \in U_2^*, w \in U_3^* \rangle,$$

$\Leftrightarrow U_1 = U_4^*$, $U_5 = U_0^*$, $U_2 = U_3^*$, $\langle (\vec{v}_u)^{\mathbb{T}_{s_u}}_{\tilde{\beta}} \widehat{f_u} : u \in U_2 \rangle = \langle (\overline{\gamma}(\vec{v}_{u^*}^{\perp_1})^*_{(n,f_u)})^{\mathbb{T}_{s_u}}_{\tilde{\beta}} \widehat{f_u} :$

$u \in U_3^* \rangle$ and $\langle (\vec{v}_u)^{\mathbb{T}_{s_u}}_{\tilde{\alpha}} \widehat{f_u} : u \in U_3 \rangle = (\overline{\gamma}(\vec{v}_{u^*}^{\perp_2})^*_{(n,f_u)})^{\mathbb{T}_{s_u}}_{\tilde{\alpha}} \widehat{f_u} : u \in U_2^* \rangle$,

$\Leftrightarrow U_1 = U_4^*$, $U_5 = U_0^*$, $U_2 = U_3^*$, for each $u \in U_2$ there is $\varepsilon \in U_3$, for each
$w \in U_3$ there is $\epsilon \in U_2$ such that $\langle (\vec{v}_u)^{\mathbb{T}_{s_u}}_{\tilde{\beta}} \widehat{f_u} \rangle = \langle (\overline{\gamma}(\vec{v}_{\varepsilon}^{\perp_1})^*_{(n,f_{\varepsilon^*})})^{\mathbb{T}_{s_{\varepsilon^*}}}_{\tilde{\beta}} \widehat{f_{\varepsilon^*}} \rangle$ and

$\langle (\vec{v}_w)^{\mathbb{T}_{s_w}}_{\tilde{\alpha}} \widehat{f_w} \rangle = \langle (\overline{\gamma}(\vec{v}_{\epsilon}^{\perp_2})^*_{(n,f_{\epsilon^*})})^{\mathbb{T}_{s_{\epsilon^*}}}_{\tilde{\alpha}} \widehat{f_{\epsilon^*}} \rangle$,

$\Leftrightarrow U_1 = U_4^*$, $U_5 = U_0^*$, $U_2 = U_3^*$, for each $u \in U_2$ there is $\varepsilon \in U_3$, for
each $w \in U_3$ there is $\epsilon \in U_2$ such that $f_u = f_{\varepsilon^*}$, $f_w = f_{\epsilon^*}$, $u = \varepsilon^*, w = \epsilon^*$,
$\vec{v}_u = \overline{\gamma}(\vec{v}_{u^*}^{\perp_1})^*_{(n,f_u)}$ and $\vec{v}_w = \overline{\gamma}(\vec{v}_{w^*}^{\perp_2})^*_{(n,f_w)}$, by Lemma 5.

For the next result recall that if $\overline{\gamma} = 1$, the binomial $T^n - \gamma$ always has
a basic irreducible self reciprocal factor in $R[T]$ and if $\overline{\gamma} = -1$, the binomial
$T^n - \gamma$ might not have a basic irreducible self reciprocal factor in $R[T]$.

Corollary 1. *With the notation as in Theorem 4, if* C *is a self-dual code there
is not* $u \in \{1, \ldots, r\}$ *such that* f_u *is self reciprocal.*
In particular if $\overline{\gamma} = 1$ *there is not any self dual* γ-*constacyclic code over* R.

Proof: Since $U_1 = U_4^*$, $U_5 = U_0^*$ and $U_2 = U_3^*$, then each U_i is special and
the assertion follows.

In the following lines some examples are given illustrating the above results.

Example 2. *Let* $(R, \mathfrak{m}, GF(2^d)) \in \mathfrak{F}_5^4$ *and* γ *a unit of* R. *From Corollary 1,
there is not any self dual* γ-*constacyclic code over* R.

Example 3. *Let* $R = GF(3)[X, Y]/\langle X^2 - Y^3, XY \rangle$ *be the ring of case* (13) *in
Theorem 1,* $\tilde{\alpha} = \{x, y\}$, $\tilde{\beta} = \{x, y^2\}$ *as above and* $\gamma = 2$. *By Hensel's Lemma,*
$T^4 - 2 = T^4 + 1 = f_1 f_2$, *where* $f_1 = T^2 + 2T + 2$, $f_2 = T^2 + T + 2 \in R[T]$. *We
have* $1^* = 2$.
Let $U_0 = U_1 = U_4 = U_5 = \emptyset$, $U_2 = \{1\}$, $U_3 = \{2\}$, $\vec{v}_1 = (1, 1 + T)$ *over*
$GF(2^2) = R[T]/\langle f_1 \rangle$, $\vec{v}_1^{\perp_2} = (1, 1 + T)$, *by Lemma 6,* $\vec{v}_2 = 2(\vec{v}_1^{\perp_2})^*_{(4,f_2)} = (1, 2 + T)$, *over* $GF(2^2) = R[T]/\langle f_2 \rangle$.
Then the γ-*constacyclic code*

$$C = \langle \mathfrak{m}^3 \widehat{\prod_{u \in U_1} f_u}, \mathfrak{m} \widehat{\prod_{u \in U_4} f_u}, \widehat{\prod_{u \in U_5} f_u}, (\vec{v}_u)^{\mathbb{T}_{s_u}}_{\tilde{\beta}} \widehat{f_u}, (\vec{v}_w)^{\mathbb{T}_{s_w}}_{\tilde{\alpha}} \widehat{f_w} : u \in U_2, w \in U_3 \rangle =$$

$$\langle [x + (1 + T)y^2]\widehat{f_1}, [x + (2 + T)y]\widehat{f_2} \rangle = \langle [x + (1 + T)y^2]f_2, [x + (2 + T)y]f_1 \rangle$$

is self-dual.

5 Reversible constacyclic codes over rings in $\mathfrak{F}_5^4$

A linear code C of length n over the ring R is reversible if it is invariant under the reversible permutation $\mathfrak{r} : \mathrm{R}^n \to \mathrm{R}^n$ given by $(a_0, a_1, \ldots, a_{n-1}) \mapsto (a_{n-1}, \ldots, a_1, a_0)$. In this Section the algebraic characterization of reversible γ-constacyclic codes over rings in the family $\mathfrak{F}_5^4$ is given, when $\gamma = \gamma^{-1}$.

The following result will be useful; claim (2) was treated in [4].

Lemma 9. *Let* R *be a finite ring,* γ *a unit of* R, $\mathfrak{r}$, σ_γ *and* r_γ *the reversible permutation, the* γ*-constacyclic permutation and the* γ*-polynomial representation of* R^n, *respectively, then:*

(1) $\mathfrak{r} \circ \sigma_\gamma^{n-1}(\gamma \vec{a}) = \sigma_\gamma \circ \mathfrak{r}(\vec{a})$, *for all* $\vec{a} \in \mathrm{R}^n$.

(2) $\gamma \mathrm{f}^* = r_\gamma \circ \sigma_\gamma^{k+1} \circ \mathfrak{r} \circ r_\gamma^{-1}(\mathrm{f})$, *for* $\mathrm{f} = a_0 + \ldots + a_{n-1}\mathrm{T}^{n-1} \in \mathrm{R}[\mathrm{T}]/\langle \mathrm{T}^n - \gamma \rangle$ *with* $\deg(\mathrm{f}) = k$.

Proof: (1) Let $\vec{a} = (a_0, a_1, \ldots, a_{n-1})$, then

$$\mathfrak{r} \circ \sigma_\gamma \circ \mathfrak{r}(\vec{a}) = \mathfrak{r} \circ \sigma_\gamma(a_{n-1}, \ldots, a_1, a_0) = \mathfrak{r}(\gamma a_0, a_{n-1}, \ldots, a_1) =$$

$$(a_1, \ldots, a_{n-1}, \gamma a_0) = \sigma_\gamma^{n-1}(\gamma(a_0, a_1, \ldots, a_{n-1})) = \sigma_\gamma^{n-1}(\gamma(\vec{a})).$$

Observation 4. *Under the notation as in Lemma 9, let* C *be a* γ*-constacyclic code.*

(1) *From relation* (1) *of Lemma 9,* $\sigma_\gamma(\mathfrak{r}(\mathrm{C})) = \mathfrak{r} \circ \sigma_\gamma^{n-1}(\gamma \mathrm{C}) = \mathfrak{r} \circ \sigma_\gamma^{n-1}(\mathrm{C}) = \mathfrak{r}(\mathrm{C})$ *and* $\mathfrak{r}(\mathrm{C})$ *is a* γ*-constacyclic code.*

(2) *Let* $\vec{a} \in \mathrm{R}^n$, *relation* $\sigma_\gamma^n(\vec{a}) = \gamma\vec{a}$ *and* (2) *of Lemma 9, imply* $\sigma_\gamma^{n-k-1} \circ r_\gamma^{-1}[(r_\gamma(\vec{a}))^*] = \mathfrak{r}(\vec{a})$, *where* $k = \deg(r_\gamma(\vec{a}))$. *This relation means that the reversible permutation of* $\vec{a}$ *can be obtained from the* γ*- constacyclic permutation and the reciprocal operation of* $r_\gamma(\vec{a})$. *This implies that the ideal corresponding to* $\mathfrak{r}(\mathrm{C})$ *is the ideal generated by* $\{\mathrm{f}^* : \mathrm{f} \in r_\gamma(\mathrm{C})\}$.

For the next result recall that if $\delta_1, \ldots, \delta_k, \mathrm{h}_1, \ldots, \mathrm{h}_k \in \mathrm{R}[\mathrm{T}]$ and the leader coefficient of each δ_i is a unit, then $(\delta_1 \mathrm{h}_1 + \ldots + \delta_k \mathrm{h}_k)^* = \mathrm{T}^{u_1}\delta_1^* \mathrm{h}_1^* + \ldots + \mathrm{T}^{u_k}\delta_k^* \mathrm{h}_1^*$, for some $u_1, \ldots, u_k$ non negative integers, see Section 2. This implies if $\psi \in \langle \delta_1, \ldots, \delta_k \rangle$ then $\psi^* \in \langle \delta_1^*, \ldots, \delta_k^* \rangle$.

Lemma 10. *Let* $(\mathrm{R}, \mathfrak{m}, \mathrm{GF}(q)) \in \mathfrak{F}_5^4$, γ, $\tilde{\alpha} = \{\alpha_1, \alpha_2\}$, $\tilde{\beta} = \{\alpha_1, \alpha_2^2\}$, $\mathbb{T}$, $\mathbb{T}_s$, $\mathrm{f}_1, \ldots, \mathrm{f}_r$, $s_i = \deg(\bar{\mathrm{f}}_i)$, C *as in Theorem 2,* $\mathfrak{r}$ *the reversible permutation on* R^n *and*

(1) $\mathrm{U}_0, \mathrm{U}_1, \mathrm{U}_2, \mathrm{U}_3, \mathrm{U}_4, \mathrm{U}_5$ *the unique partition of* $\{1, \ldots, r\}$, *associated to* C,

(2) $\{\vec{v}_u : u \in U_2 \cup U_3\}$ *the vectors such that the corresponding ideal, in* $R[T]/\langle T^n - \gamma \rangle$, *of* C *is*

$$\langle \mathfrak{m}^3 \widehat{\prod_{u \in U_1} f_u}, \mathfrak{m} \widehat{\prod_{u \in U_4} f_u}, \widehat{\prod_{u \in U_5} f_u}, (\vec{v}_u)^{\mathbb{T}_{su}}_{\tilde{\beta}} \widehat{f_u}, (\vec{v}_w)^{\mathbb{T}_{sw}}_{\tilde{\alpha}} \widehat{f_w} : u \in U_2, w \in U_3 \rangle.$$

Then

(1) $V_0 = U_0^*, \ V_1 = U_1^*, \ V_2 = U_2^*, \ V_3 = U_3^*, \ V_4 = U_4^*, \ V_5 = U_5^*$ *is the unique partition of* $\{1, \ldots, r\}$, *associated to* $\mathfrak{r}(C)$,

(2) $\{\overline{\gamma}(\vec{v}_{u^*})^*_{(n,f_u)}, \overline{\gamma}(\vec{v}_{w^*})^*_{(n,f_w)} : u \in V_2 = U_2^*, w \in V_3 = U_3^*\}$ *are the unique vectors such that the corresponding ideal, in* $R[T]/\langle T^n - \gamma \rangle$, *of* $\mathfrak{r}(C)$ *is*

$$\langle \mathfrak{m}^3 \widehat{\prod_{u \in U_1^*} f_u}, \mathfrak{m} \widehat{\prod_{u \in U_4^*} f_u}, \widehat{\prod_{u \in U_5^*} f_u}, (\overline{\gamma}(\vec{v}_{u^*})^*_{(n,f_u)})^{\mathbb{T}_{su}}_{\tilde{\beta}} \widehat{f_u}, (\overline{\gamma}(\vec{v}_{w^*})^*_{(n,f_w)})^{\mathbb{T}_{sw}}_{\tilde{\alpha}} \widehat{f_w} :$$

$$u \in U_2^*, w \in U_3^* \rangle.$$

Proof: Let D the γ-constacyclic code given by

$$\langle \mathfrak{m}^3 \widehat{\prod_{u \in U_1} f_u}^*, \mathfrak{m} \widehat{\prod_{u \in U_4} f_u}^*, \widehat{\prod_{u \in U_5} f_u}^*, ((\vec{v}_u)^*_{(n,f_u^*)})^{\mathbb{T}_{su^*}}_{\tilde{\beta}} \widehat{f_u^*}, ((\vec{v}_w)^*_{(n,f_w^*)})^{\mathbb{T}_{sw^*}}_{\tilde{\alpha}} \widehat{f_w^*} :$$

$$u \in U_2, w \in U_3 \rangle =$$

$$\langle \mathfrak{m}^3 \widehat{\prod_{u \in U_1^*} f_u}, \mathfrak{m} \widehat{\prod_{u \in U_4^*} f_u}, \widehat{\prod_{u \in U_5^*} f_u}, (\overline{\gamma}(\vec{v}_{u^*})^*_{(n,f_{u^*}^*)})^{\mathbb{T}_{s(u^*)^*}}_{\tilde{\beta}} \widehat{f_{u^*}^*},$$

$$(\overline{\gamma}(\vec{v}_{w^*})^*_{(n,f_{w^*}^*)})^{\mathbb{T}_{s(w^*)^*}}_{\tilde{\alpha}} \widehat{f_{w^*}^*} : u \in U_2^*, w \in U_3^* \rangle =$$

$$\langle \mathfrak{m}^3 \widehat{\prod_{u \in U_1^*} f_u}, \mathfrak{m} \widehat{\prod_{u \in U_4^*} f_u}, \widehat{\prod_{u \in U_5^*} f_u}, (\overline{\gamma}(\vec{v}_{u^*})^*_{(n,f_u)})^{\mathbb{T}_{su}}_{\tilde{\beta}} \widehat{f_u}, (\overline{\gamma}(\vec{v}_{w^*})^*_{(n,f_w)})^{\mathbb{T}_{sw}}_{\tilde{\alpha}} \widehat{f_w} :$$

$$u \in U_2^*, w \in U_3^* \rangle.$$

By (2) and (3) of Lemma 3, (2) and (3) of Lemma 4, and Observation 4, the corresponding ideal of $\mathfrak{r}(C)$ is contained in D. Since $\deg(f_i) = \deg(f_i^*)$, using Theorem 2, the number of elements in D and in C is the same. The assertion follows.

Theorem 5. *Let* $(R, \mathfrak{m}, GF(q)) \in \mathfrak{F}_5^4$, γ, $\tilde{\alpha} = \{\alpha_1, \alpha_2\}$, $\tilde{\beta} = \{\alpha_1, \alpha_2^2\}$, $\mathbb{T}$, $\mathbb{T}_s$, $f_1, \ldots, f_r$ *and* $s_i = \deg(\overline{f}_i)$, C *as in Theorem 2 and*

(1) $U_0, U_1, U_2, U_3, U_4, U_5$ *the unique partition of* $\{1, \ldots, r\}$, *associated to* C,

(2) $\{\vec{v}_u : u \in U_2 \cup U_3\}$ *the vectors such that the corresponding ideal, in* $R[T]/\langle T^n - \gamma \rangle$, *of* C *is*

$$\langle \mathfrak{m}^3 \widehat{\prod_{u \in U_1} f_u}, \mathfrak{m} \widehat{\prod_{u \in U_4} f_u}, \widehat{\prod_{u \in U_5} f_u}, (\vec{v}_u)_{\tilde{\beta}}^{\mathbb{T}_{s_u}} \widehat{f_u}, (\vec{v}_w)_{\tilde{\alpha}}^{\mathbb{T}_{s_w}} \widehat{f_w} : u \in U_2, w \in U_3 \rangle.$$

The following conditions are equivalent:

(1) C *is a reversible code;*

(2) U_i *is self-reciprocal, for* $i \in \{1, \ldots, 5\}$, *and* $\vec{v}_u = \overline{\gamma}(\vec{v}_{u^*})^*_{(n, f_u)}$, *for* $u \in U_2 \cup U_3$;

(3) U_i *is self-reciprocal, for* $i \in \{0, \ldots, 5\}$, *and* $\vec{v}_u = \overline{\gamma}(\vec{v}_{u^*})^*_{(n, f_u)}$, *for* $u \in U_2 \cup U_3$.

Proof: $(1) \Leftrightarrow (2)$ C is reversible $\Leftrightarrow$
$U_1 = U_1^*$, $U_4 = U_4^*$, $U_5 = U_5^*$, for each $u \in U_2$ there exists $\varepsilon \in U_2^*$, for each
$w \in U_3$ there exists $\epsilon \in U_3^*$ such that $\langle (\vec{v}_u)_{\tilde{\beta}}^{\mathbb{T}_{s_u}} \widehat{f_u} \rangle = \langle (\overline{\gamma}(\vec{v}_{\varepsilon^*})^*_{(n, f_\varepsilon)})_{\tilde{\beta}}^{\mathbb{T}_{s_\varepsilon}} \widehat{f_\varepsilon} \rangle$ and
$\langle (\vec{v}_w)_{\tilde{\alpha}}^{\mathbb{T}_{s_w}} \widehat{f_w} \rangle = \langle (\overline{\gamma}(\vec{v}_{\epsilon^*})^*_{(n, f_\epsilon)})_{\tilde{\alpha}}^{\mathbb{T}_{s_\epsilon}} \widehat{f_\epsilon} \rangle$, by Lemma 10, uniqueness in Theorem 2
and because U_0^*, U_1^*, U_2^*, U_3^*, U_4^*, U_5^* is a partition of $\{1, \ldots, r\}$,
$\Leftrightarrow U_1 = U_1^*$, $U_4 = U_4^*$, $U_5 = U_5^*$, for each $u \in U_2$ there exists $\varepsilon \in U_2^*$, for each
$w \in U_3$ there exists $\epsilon \in U_3^*$ such that $\langle (\vec{v}_u)_{\tilde{\beta}}^{\mathbb{T}_{s_u}} \widehat{f_u} \rangle = \langle (\overline{\gamma}(\vec{v}_\varepsilon)^*_{(n, f_{\varepsilon^*})})_{\tilde{\beta}}^{\mathbb{T}_{s_{\varepsilon^*}}} \widehat{f_{\varepsilon^*}} \rangle$ and
$\langle (\vec{v}_w)_{\tilde{\alpha}}^{\mathbb{T}_{s_w}} \widehat{f_w} \rangle = \langle (\overline{\gamma}(\vec{v}_\epsilon)^*_{(n, f_{\epsilon^*})})_{\tilde{\alpha}}^{\mathbb{T}_{s_{\epsilon^*}}} \widehat{f_{\epsilon^*}} \rangle$,
$\Leftrightarrow U_1$, U_2, U_3, U_4, U_5 are self-reciprocal, for each $u \in U_2$ there exists $\varepsilon \in U_2$
such that $f_u = f_\varepsilon^{\circ *}$ and $\vec{v}_u = \overline{\gamma}(\vec{v}_\varepsilon)^*_{(n, f_{\varepsilon^*})}$, for each $w \in U_3$ there exists $\epsilon \in U_3$
such that $f_w = f_\epsilon^{\circ *}$ and $\vec{v}_w = \overline{\gamma}(\vec{v}_\epsilon)^*_{(n, f_{\epsilon^*})}$, by Lemma 5,
$\Leftrightarrow U_1$, U_2, U_3, U_4, U_5 are self-reciprocal, for each $u \in U_2$, $\vec{v}_u = \overline{\gamma}(\vec{v}_{u^*})^*_{(n, f_u)}$,
and for each $w \in U_3$, $\vec{v}_w = \overline{\gamma}(\vec{v}_{w^*})^*_{(n, f_w)}$,
$\Leftrightarrow U_1$, U_2, U_3, U_4, U_5 are self-reciprocal and for each $u \in U_2 \cup U_3$, $\vec{v}_u = \overline{\gamma}(\vec{v}_{u^*})^*_{(n, f_u)}$.
$(2) \Leftrightarrow (3)$ It is easy to check.

Example 4. *Let* $R = GF(2)[X, Y]/\langle X^2 - Y^3, XY \rangle$, γ, $\tilde{\alpha} = \{x, y\}$, f_1, f_2, f_3,
f_4, f_5 *as in Example 1, then* $\overline{\gamma} = 1$, $1^* = 2$, $2^* = 1$, $3^* = 3$, $4^* = 4$, $5^* = 5$.
Let $U_1 = \{4\}$, $U_2 = \{1, 2\}$, $U_3 = \{3\}$, $U_4 = \{5\}$, $U_0 = U_5 = \emptyset$, $\vec{v}_1 = (1, T)$, *over* $GF(2^4) = GF(2)[T]/\langle f_1 \rangle$, $\vec{v}_2 = \overline{\gamma}(\vec{v}_1)^*_{(15, f_2)} = (1, T^3 + T^2)$, *over*
$GF(2)[T]/\langle f_2 \rangle$.
Since $T^3 + T^2 \in GF(2)[T]/\langle f_3 \rangle$ *is a root of the irreducible polynomial* $T^2 + T + 1 \in GF(2)[T]$ *then* $\vec{v}_3 = (1, T^3 + T^2 + 1)$ *over* $GF(2)[T]/\langle f_3 \rangle$ *has the property*

$\vec{v}_3 = \overline{\gamma}(\vec{v}_3)^*_{(15,f_3)}$, *by Observation 2. Hence the γ-constacyclic code:*

$$C = \langle \mathfrak{m}^3 \widehat{\prod_{u \in U_1} f_u}, \mathfrak{m} \widehat{\prod_{u \in U_4} f_u}, \widehat{\prod_{u \in U_5} f_u}, (\vec{v}_u)^{\mathbb{T}_s u}_{\tilde{\beta}} \widehat{f_u}, (\vec{v}_w)^{\mathbb{T}_s w}_{\tilde{\alpha}} \widehat{f_w} : u \in U_2, w \in U_3 \rangle =$$

$$\langle \widehat{f_4}, \mathfrak{m}\widehat{f_5}, [x + Ty]\widehat{f_1}, [x + (T^3 + T^2)y]\widehat{f_2}, [x + (T^3 + T^2 + 1)y]\widehat{f_3} \rangle =$$

$$\langle f_1 f_2 f_3 f_5, \mathfrak{m}f_1 f_2 f_3 f_4, [x + Ty]f_2 f_3 f_4 f_5, [x + (T^3 + T^2)y]f_1 f_3 f_4 f_5,$$

$$[x + (T^3 + T^2 + 1)y]f_1 f_2 f_4 f_5 \rangle$$

is reversible.

6   Constacyclic codes with Complementary Duals over rings in $\mathfrak{F}_5^4$

A linear code with complementary dual is defined to be a linear code C whose dual code satisfies $C \cap C^\perp = \langle 0 \rangle$. In this Section the algebraic characterization of complementary dual γ-constacyclic codes over rings in the family $\mathfrak{F}_5^4$ is given, when $\gamma = \gamma^{-1}$.

Theorem 6. *Let* $(R, \mathfrak{m}, GF(q)) \in \mathfrak{F}_5^4$, γ, $\tilde{\alpha} = \{\alpha_1, \alpha_2\}$, $\tilde{\beta} = \{\alpha_1, \alpha_2^2\}$, $\mathbb{T}$, $\mathbb{T}_s$, $f_1, \ldots, f_r$, $s_i = \deg(\bar{f}_i)$, C *as in Theorem 2 and*

(1) $U_0, U_1, U_2, U_3, U_4, U_5$ *the unique partition of* $\{1, \ldots, r\}$, *associated to* C,

(2) $\{\vec{v}_u : u \in U_2 \cup U_3\}$ *the vectors such that the corresponding ideal, in* $R[T]/\langle T^n - \gamma \rangle$, *of* C *is*

$$\langle \mathfrak{m}^3 \widehat{\prod_{u \in U_1} f_u}, \mathfrak{m} \widehat{\prod_{u \in U_4} f_u}, \widehat{\prod_{u \in U_5} f_u}, (\vec{v}_u)^{\mathbb{T}_s u}_{\tilde{\beta}} \widehat{f_u}, (\vec{v}_w)^{\mathbb{T}_s w}_{\tilde{\alpha}} \widehat{f_w} : u \in U_2, w \in U_3 \rangle.$$

The following conditions are equivalent:

(1) C *is a Complementary dual code;*

(2) U_0 *and* U_5 *are self-reciprocal and* $U_i = \emptyset$, *for* $i \in \{1, 2, 3, 4\}$;

(3) U_5 *is self-reciprocal and* $U_i = \emptyset$, *for* $i \in \{1, 2, 3, 4\}$.

In this case $C = \langle \widehat{\prod_{u \in U_5} f_u} \rangle$ *and* $C^\perp = \langle \widehat{\prod_{u \in U_0} f_u} \rangle$.

Proof: First recall the following facts: The intersection of two non zero ideals of R is not zero; if $U = \{u_1, \ldots, u_k\} \subseteq \{1, r\}$, the ideal generated by $\widehat{\prod_{u \in U} f_u}$ in $R[T]/\langle T^n - \gamma \rangle$ is identified with $\langle \vec{0} \rangle \oplus R[T]/\langle f_{u_1} \rangle \oplus \langle \vec{0} \rangle \oplus \ldots \oplus \langle \vec{0} \rangle \oplus R[T]/\langle f_{u_k} \rangle \oplus \langle \vec{0} \rangle$; the rings $R[T]/\langle f_i \rangle$ are Frobenius, $i \in \{1, \ldots, r\}$.
Now, from Theorem 3,

$$C^\perp = \langle \mathfrak{m}^3 \widehat{\prod_{u \in U_4^*} f_u}, \mathfrak{m} \widehat{\prod_{u \in U_1^*} f_u}, \widehat{\prod_{u \in U_0^*} f_u}, (\overline{\gamma}(\vec{v}_{w^*}^{\perp_1})_{(n, f_w)}^*)_{\tilde{\beta}}^{\mathbb{T}_{s_w}} \widehat{f_w},$$

$$(\overline{\gamma}(\vec{v}_{u^*}^{\perp_2})_{(n, f_u)}^*)_{\tilde{\alpha}}^{\mathbb{T}_{s_u}} \widehat{f_u} : u \in U_2^*, w \in U_3^* \rangle.$$

Then $C \cap C^\perp = \langle 0 \rangle$
$\Leftrightarrow \left[U_0^* \cup U_1^* \cup U_2^* \cup U_3^* \cup U_4^* \right] \cap \left[U_1 \cup U_2 \cup U_3 \cup U_4 \cup U_5 \right] = \emptyset$
$\Leftrightarrow U_0^* \cup U_1^* \cup U_2^* \cup U_3^* \cup U_4^* = \{1, \ldots, r\} \setminus U_5^* \subseteq U_0$
$\Leftrightarrow U_0^* = U_0, U_1^* = U_2^* = U_3^* = U_4^* = \emptyset$, because $|U_0^*| = |U_0|$
$\Leftrightarrow U_0$ and U_5 are self-reciprocal and $U_1^* = U_2^* = U_3^* = U_4^* = \emptyset$
$\Leftrightarrow U_5$ is self-reciprocal and $U_1^* = U_2^* = U_3^* = U_4^* = \emptyset$.

Example 5. *Let* $R = GF(2)[X, Y]/\langle X^2 - Y^3, XY \rangle$, γ, $\tilde{\alpha} = \{x, y\}$, f_1, f_2, f_3, f_4, f_5 *as in Example 1, then* $1^* = 2$, $2^* = 1$, $3^* = 3$, $4^* = 4$, $5^* = 5$ *and the self-reciprocal subsets of* $\{1, 2, 3, 4, 5\}$ *are* $\{1, 2\}$, $\{1, 2, 3\}$, $\{1, 2, 3, 4\}$, $\{1, 2, 3, 4, 5\}$, $\{1, 2, 4\}$, $\{1, 2, 4, 5\}$, $\{1, 2, 5\}$, $\{3\}$, $\{3, 4\}$, $\{3, 4, 5\}$ $\{4\}$, $\{4, 5\}$, $\{5\}$ *and* $\emptyset$.
The Complementary γ-*constacyclic code over* R *of length* 15 *and their duals are the following:*

(1) $C = \langle \widehat{f_1 f_2} \rangle = \langle f_3 f_4 f_5 \rangle$ *and its dual is* $C^\perp = \langle \widehat{f_3 f_4 f_5} \rangle = \langle f_1 f_2 \rangle$;

(2) $C = \langle \widehat{f_1 f_2 f_3} \rangle = \langle f_4 f_5 \rangle$ *and its dual is* $C^\perp = \langle \widehat{f_4 f_5} \rangle = \langle f_1 f_2 f_3 \rangle$;

(3) $C = \langle \widehat{f_1 f_2 f_3 f_4} \rangle = \langle f_5 \rangle$ *and its dual is* $C^\perp = \langle \widehat{f_5} \rangle = \langle f_1 f_2 f_3 f_4 \rangle$;

(4) $C = \langle \widehat{f_1 f_2 f_3 f_4 f_5} \rangle = \langle 1 \rangle$ *and its dual is* $C^\perp = \langle 0 \rangle$;

(5) $C = \langle \widehat{f_1 f_2 f_4} \rangle = \langle f_3 f_5 \rangle$ *and its dual is* $C^\perp = \langle \widehat{f_3 f_5} \rangle = \langle f_1 f_2 f_4 \rangle$;

(6) $C = \langle \widehat{f_1 f_2 f_4 f_5} \rangle = \langle f_3 \rangle$ *and its dual is* $C^\perp = \langle \widehat{f_3} \rangle = \langle f_1 f_2 f_4 f_5 \rangle$;

(7) $C = \langle \widehat{f_1 f_2 f_5} \rangle = \langle f_3 f_4 \rangle$ *and its dual is* $C^\perp = \langle \widehat{f_3 f_4} \rangle = \langle f_1 f_2 f_5 \rangle$;

(8) $C = \langle \widehat{f_3} \rangle = \langle f_1 f_2 f_4 f_5 \rangle$ *and its dual is* $C^\perp = \langle \widehat{f_1 f_2 f_4 f_5} \rangle = \langle f_3 \rangle$;

(9) $C = \langle \widehat{f_3 f_4} \rangle = \langle f_1 f_2 f_5 \rangle$ *and its dual is* $C^\perp = \langle \widehat{f_1 f_2 f_5} \rangle = \langle f_3 f_4 \rangle$;

(10) $C = \langle \widehat{f_3 f_4 f_5} \rangle = \langle f_1 f_2 \rangle$ *and its dual is* $C^\perp = \langle \widehat{f_1 f_2} \rangle = \langle f_3 f_4 f_5 \rangle$;

(11) $C = \langle \widehat{f_4} \rangle = \langle f_1 f_2 f_3 f_5 \rangle$ *and its dual is* $C^\perp = \langle \widehat{f_1 f_2 f_3 f_5} \rangle = \langle f_4 \rangle;$

(12) $C = \langle \widehat{f_4 f_5} \rangle = \langle f_1 f_2 f_3 \rangle$ *and its dual is* $C^\perp = \langle \widehat{f_1 f_2 f_3} \rangle = \langle f_4 f_5 \rangle;$

(13) $C = \langle \widehat{f_5} \rangle = \langle f_1 f_2 f_3 f_4 \rangle$ *and its dual is* $C^\perp = \langle \widehat{f_1 f_2 f_3 f_4} \rangle = \langle f_5 \rangle;$

(14) $C = \langle 0 \rangle$ *and its dual is* $C^\perp = \langle 1 \rangle.$

Acknowledgements

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