DOI: 10.2478/auom-2021-0029
An. Şt. Univ. Ovidius Constanţa **Vol. 29(2),2021, 249–264**

Varma Quantile Entropy Order

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Abstract

We give a stochastic order for Varma residual entropy and study several properties of it, like closure, reversed closure and preservation of this order in some stochastic models.

1 Introduction and Preliminaries

We consider a nonnegative random variable X, which represents a living thing or the lifetime of a device, having absolutely continuous cumulative distribution function F_X, survival function $\overline{F}_X \overset{def}{=} 1 - F_X$ and probability density function f_X. Shannon entropy of X is given via

$$H_X = -\mathbb{E}_Z\left(\log f_X(Z)\right),$$

where "log" designate the natural logarithm function and Z is a nonnegative random variable identically distributed like X.

The concept of Shannon entropy has multiple generalizations (Tsallis entropy, Rényi entropy, Varma entropy etc.) being useful in many technological areas like Physics, Communication Theory, Probability, Statistics, Economics etc. (see [1], [12], [24], [38]). More exactly, there are specific areas where the entropies are used: the income distribution (see [26], [31]), non-coding human DNA (see [22]), earthquakes (see [5]), stock exchanges (see [14]), biostatistics (see [6], [13], [35]), model selection (see [33], [34]) statistical mechanics (see [27], [29], [37]) internet (see [2]) etc.

In this paper we will work with Varma entropy, notion introduced in [39], being now very actual (see [3], [15], [17], [18], [23], [25], [28], [36]). Let $\alpha, \beta \in \mathbb{R}$ such that $\beta \geq 1$ and $\beta - 1 < \alpha < \beta$. We define Varma entropy of X via

Key Words: Varma entropy, Varma residual entropy, Varma quantile entropy.
2010 Mathematics Subject Classification: 60E15, 60K10, 62E10, 62N05, 90B25.
Received: 23.11.2020
Accepted: 30.12.2020

$$H_{\alpha,\beta}^X = \frac{1}{\beta-\alpha}\log\left(\mathbb{E}_Z\left((f_X(Z))^{\alpha+\beta-2}\right)\right).$$

We denote by $1|_A$ the characteristic (indicator) function of the set A.

In [9] and [11], the notion of (Shannon) residual entropy was introduced as a dynamic measure of uncertainty. More exactly, for an absolutely continuous nonnegative random variable X, the residual life of X is $X_t = [X - t \mid X > t]$ and the residual entropy of X at time t is

$$H_X(t) = H_{X_t} = -\mathbb{E}_Z\left(\frac{1}{\overline{F}_X(t)}\log\left(\frac{f_X(Z)}{\overline{F}_X(t)}\right)1|_{[Z>t]}\right) \text{ for any } t \geq 0.$$

In other words, the residual entropy of X measures the uncertainty of the residual life of X_t. Some interesting results concerning the residual entropy can be found in [7], [8], [10], [16], [19], [20], [40] and in many other papers.

We define Varma residual entropy via

$$H_{\alpha,\beta}^X(t) = \frac{1}{\beta-\alpha}\log\left(\mathbb{E}_Z\left(\frac{1}{\overline{F}_X(t)}\left(\frac{f_X(Z)}{\overline{F}_X(t)}\right)^{\alpha+\beta-2}\right)1|_{[Z>t]}\right) \text{ for any } t \geq 0.$$

It is clearly that $H_{\alpha,\beta}^X(0) = H_{\alpha,\beta}^X$.

We consider the quantile function

$$Q_X(u) \overset{def}{=} F_X^{-1}(u) = \inf\{x \in [0,\infty) \mid F_X(x) \geq u\} \text{ for any } u \in [0,1].$$

In some cases the quantile function F_X^{-1} is called the right-continuous inverse function of F_X (or, in short, of X).

We have $F_X(Q_X(u)) = u$. Differentiating both sides of this equality with respect to u, we get $F_X'(Q_X(u))Q_X'(u) = 1$. Denote $q_X(u) = Q_X'(u)$ for any $u \in [0,1]$. Hence $q_X(u)f_X(Q_X(u)) = 1$ for any $u \in [0,1]$.

In [32] was introduced a quantile version of Shannon residual entropy and in [20] and [40] this idea was generalized for Rényi residual entropy. We continue this work for Varma residual entropy, defining

$$\Psi_{\alpha,\beta}^X(u) = H_{\alpha,\beta}^X(Q_X(u)) \text{ for any } u \in [0,1].$$

We have

$$H_{\alpha,\beta}^X(Q_X(u)) = \frac{1}{\beta-\alpha}\log\left(\frac{1}{1-u}\mathbb{E}_Z\left(\left(\frac{f_X(Z)}{1-u}\right)^{\alpha+\beta-2}1|_{[Z>Q_X(u)]}\right)\right) =$$

$$\frac{1}{\beta-\alpha}\log\left(\mathbb{E}_U\left(\left(\frac{\frac{1}{q_X(U)}}{1-u}\right)^{\alpha+\beta-1}q_X(U)1|_{[u<U<1]}\right)\right) =$$

$$\frac{1}{\beta-\alpha}\log\left(\mathbb{E}_U\left(\frac{(q_X(U))^{2-\alpha-\beta}}{(1-u)^{\alpha+\beta-1}}1|_{[u<U<1]}\right)\right)=$$
$$\frac{1}{\beta-\alpha}\log\left(\mathbb{E}_U\left(\frac{(f_X(Q_X(U)))^{\alpha+\beta-2}}{(1-u)^{\alpha+\beta-1}}1|_{[u<U<1]}\right)\right).$$

We use the following lemma.

Lemma 1.1. (see [20]) Let $f:[0,1]\times[0,\infty)\to\mathbb{R}$ with the property that $\mathbb{E}_U\left(f(u,U)1|_{[u<U<1]}\right)\geq 0$ for any $u\in[0,1]$ and $g:[0,\infty)\to[0,\infty)$ be an increasing function. Then

$$\mathbb{E}_U\left(f(u,U)g(U)1|_{[u<U<1]}\right)\geq 0,$$

provided the expectations exist.

2 Results

Let X and Y be two nonnegative absolutely continuous random variables with distribution functions F_X, respectively F_Y, survival functions $\overline{F}_X$, respectively $\overline{F}_Y$ and density functions f_X, respectively f_Y.

From now on, Z will be a nonnegative absolutely continuous random variable identically distributed like another random variables which appear in formulas: X, $X_{1:n}$, $X_{n:n}$ etc. (not necessarily identically distributed like Y, $Y_{1:n}$, $Y_{n:n}$ etc.).

Definition 2.1. We say that X is smaller than Y in *the Varma quantile entropy order* (and denote $X\leq_{VQE}Y$) if $H^X_{\alpha,\beta}(Q_X(u))\leq H^Y_{\alpha,\beta}(Q_Y(u))$ for any $u\in[0,1]$.

The next theorem is very useful in this paper.

Theorem 2.2. *a) The following assertions are equivalent:*

1. $X\leq_{VQE}Y$.

2. $\mathbb{E}_Z\left((f_X(Z))^{\alpha+\beta-2}\left[\left(\frac{f_X(Z)}{f_Y(F_Y^{-1}(F_X(Z)))}\right)^{2-(\alpha+\beta)}-1\right]1|_{[Z>t]}\right)\geq 0$ *for any $t\geq 0$.*

b) The following assertions are equivalent:

1. $Y\leq_{VQE}X$.

2. $\mathbb{E}_Z\left((f_X(Z))^{\alpha+\beta-2}\left[\left(\frac{f_X(Z)}{f_Y(F_Y^{-1}(F_X(Z)))}\right)^{2-(\alpha+\beta)}-1\right]1|_{[Z>t]}\right)\leq 0$ *for any $t\geq 0$.*

Proof. We will prove only *a*), the proof of *b*) being similar.

We have $X \leq_{VQE} Y$ if and only if

$$\mathbb{E}_U\left((f_X(Q_X(U)))^{\alpha+\beta-2}\, 1|_{[u<U<1]}\right) \leq \mathbb{E}_U\left((f_Y(Q_Y(U)))^{\alpha+\beta-2}\, 1|_{[u<U<1]}\right) \text{ for any } u \in [0,1]$$

(see Definition 2.1).

Considering $U = F_X(Z)$ in the above inequality we have the equivalences (for any $u \in [0,1]$):

$$\begin{gathered} X \leq_{VQE} Y \Leftrightarrow \mathbb{E}_Z\left((f_X(Z))^{\alpha+\beta-2}\, 1|_{[Z>F_X^{-1}(u)]}\right) \leq \\ \mathbb{E}_Z\left((f_Y(Q_Y(F_X(Z))))^{\alpha+\beta-2}\, 1|_{[Z>F_X^{-1}(u)]}\right) \Leftrightarrow \\ \mathbb{E}_Z\left((f_X(Z))^{\alpha+\beta-2}\, 1|_{[Z>F_X^{-1}(u)]}\right) \leq \\ \mathbb{E}_Z\left(\left(f_Y(F_Y^{-1}(F_X(Z)))\right)^{\alpha+\beta-2}\, 1|_{[Z>F_X^{-1}(u)]}\right) \Leftrightarrow \\ \mathbb{E}_Z\left((f_X(Z))^{\alpha+\beta-2}\left[\left(\frac{f_X(Z)}{f_Y(F_Y^{-1}(F_X(Z)))}\right)^{2-(\alpha+\beta)} - 1\right] 1|_{[Z>F_X^{-1}(u)]}\right) \geq 0. \end{gathered}$$

Putting $F_X^{-1}(u) = t$ in the preceding equivalences we get the conclusion. □

Definition 2.3. (see [30]) We say that:

1. X is smaller than Y in *the dispersive order* (and write $X \leq_{disp} Y$) if

$$F_X^{-1}(\delta) - F_X^{-1}(\gamma) \leq F_Y^{-1}(\delta) - F_Y^{-1}(\gamma) \text{ for any } 0 < \gamma < \delta < 1,$$

which is equivalent to

$$f_X(x) \geq f_Y(F_Y^{-1}(F_X(x))) \text{ for any } x \geq 0.$$

2. X is smaller than Y in *the convex transform order* (and write $X \leq_c Y$) if the function

$$[0,\infty) \ni x \longrightarrow F_Y^{-1}(F_X(x)) \text{ is convex,}$$

which is equivalent to the fact that the function

$$[0,\infty) \ni x \longrightarrow \frac{f_X(x)}{f_Y(F_Y^{-1}(F_X(x)))} \text{ is nonnegative increasing.}$$

Theorem 2.4. *We assume that $X \leq_{disp} Y$.*

a) If $\alpha + \beta \leq 2$, then $X \leq_{VQE} Y$.

b) If $\alpha + \beta \geq 2$, then $Y \leq_{VQE} X$.

Proof. We will prove only $a)$, the proof of $b)$ being similar.

If $X \leq_{disp} Y$ then $f_X(x) \geq f_Y(F_Y^{-1}(F_X(x)))$ for any $x \geq 0$. We apply Theorem 2.2. $\square$

Theorem 2.5. *We assume that $X \leq_c Y$ and $f_X(0) \geq f_Y(0) > 0$.*

a) If $\alpha + \beta \leq 2$, then $X \leq_{VQE} Y$.

b) If $\alpha + \beta \geq 2$, then $Y \leq_{VQE} X$.

Proof. We will prove only $a)$, the proof of $b)$ being similar.

If $X \leq_c Y$ then the function

$$[0,\infty) \ni x \longrightarrow \frac{f_X(x)}{f_Y(F_Y^{-1}(F_X(x)))} \text{ is nonnegative increasing,}$$

hence

$$\frac{f_X(x)}{f_Y(F_Y^{-1}(F_X(x)))} \geq \frac{f_X(0)}{f_Y(0)} \geq 1.$$

The conclusion follows from Theorem 2.2. $\square$

We consider $X_1, ..., X_n$ and $Y_1, ..., Y_n$ independent and identically distributed (i.i.d.) copies of X, respectively of Y and

$$X_{1:n} = \min\{X_1, ..., X_n\},\ X_{n:n} = \max\{X_1, ..., X_n\},$$

$$Y_{1:n} = \min\{Y_1, ..., Y_n\},\ Y_{n:n} = \max\{Y_1, ..., Y_n\}.$$

We use the same notations as above for distribution functions, survival functions and density functions, namely $F_{X_{1:n}}$, $\overline{F}_{X_{1:n}}$, $f_{X_{1:n}}$ etc.

Theorem 2.6. *Assuming that $X \leq_{VQE} Y$, we have $X_{n:n} \leq_{VQE} Y_{n:n}$.*

Proof. If $X \leq_{VQE} Y$, then, according to Theorem 2.2,

$$\mathbb{E}_Z\left((f_X(Z))^{\alpha+\beta-2}\left[\left(\frac{f_X(Z)}{f_Y(F_Y^{-1}(F_X(Z)))}\right)^{2-(\alpha+\beta)} - 1\right] 1|_{[Z>t]}\right) \geq 0 \text{ for any } t \geq 0.$$

We remark that $f_{X_{n:n}}(x) = n\,(F_X(x))^{n-1} f_X(x)$.

Hence

$$\frac{f_{X_{n:n}}(x)}{f_{Y_{n:n}}(F_{Y_{n:n}}^{-1}(F_{X_{n:n}}(x)))} = \frac{f_X(x)}{f_Y(F_Y^{-1}(F_X(x)))}.$$

Because the function

$$[0,\infty) \ni x \longrightarrow \left(n(F_X(x))^{n-1}\right)^{\alpha+\beta-1} \text{ is nonnegative increasing,}$$

it follows by Lemma 1.1 that

$$\mathbb{E}_Z\left((f_{X_{n:n}}(Z))^{\alpha+\beta-2}\left[\left(\frac{f_{X_{n:n}}(Z)}{f_{Y_{n:n}}(F^{-1}_{Y_{n:n}}(F_{X_{n:n}}(Z)))}\right)^{2-(\alpha+\beta)}-1\right]1|_{[Z>t]}\right)\geq 0$$
for any $t \geq 0$.

Applying Theorem 2.2 we conclude that $X_{n:n} \leq_{VQE} Y_{n:n}$. □

With a similar technique like in Theorem 2.6 we get

Theorem 2.7. *Assuming that $X_{1:n} \leq_{VQE} Y_{1:n}$, we have $X \leq_{VQE} Y$.*

Let $X_1, X_2, ...$ and $Y_1, Y_2, ...$ be sequences of independent and identically distributed copies of X, respectively of Y. Let N be a positive integer random variable having the probability mass function $p_N(n) = P(N = n)$, $n = 1, 2, ...$ We assume that N is independent of $X_i's$ and $Y_i's$. We consider

$$X_{1:N} = \min\{X_1, ..., X_N\},\ X_{N:N} = \max\{X_1, ..., X_N\}$$

and

$$Y_{1:N} = \min\{Y_1, ..., Y_N\},\ Y_{N:N} = \max\{Y_1, ..., Y_N\}.$$

The following two theorems are extensions of Theorems 2.6 and 2.7 from a finite number n to a random variable N. We will prove only Theorem 2.9.

Theorem 2.8. *Assuming that $X \leq_{VQE} Y$, we have $X_{N:N} \leq_{VQE} Y_{N:N}$.*

Theorem 2.9. *Assuming that $X_{1:N} \leq_{VQE} Y_{1:N}$, we have $X \leq_{VQE} Y$.*

Proof. If $X_{1:N} \leq_{VQE} Y_{1:N}$, then

$$\mathbb{E}_Z\left((f_{X_{1:N}}(Z))^{\alpha+\beta-2}\left[\left(\frac{f_{X_{1:N}}(Z)}{f_{Y_{1:N}}(F^{-1}_{Y_{1:N}}(F_{X_{1:N}}(Z)))}\right)^{2-(\alpha+\beta)}-1\right]1|_{[Z>t]}\right)\geq$$
0 for any $t \geq 0$.

It can be easily seen that

$$f_{X_{1:N}}(x) = f_X(x)\sum_{n=1}^{\infty} n\left(\overline{F}_X(x)\right)^{n-1} p_N(n)$$

and

$$f_{Y_{1:N}}(x) = f_Y(x)\sum_{n=1}^{\infty} n\left(\overline{F}_Y(x)\right)^{n-1} p_N(n),$$

where $p_N(n) = P(N = n)$, $n = 1, 2, ...$ is the probability mass function of N.

Also we remark that

$$F^{-1}_{Y_{1:N}}\left(F_{X_{1:N}}(x)\right) = F^{-1}_Y\left(F_X(x)\right).$$

Hence

$$\frac{f_{X_{1:N}}(x)}{f_{Y_{1:N}}(F^{-1}_{Y_{1:N}}(F_{X_{1:N}}(x)))} = \frac{f_X(x)}{f_Y(F^{-1}_Y(F_X(x)))}.$$

Because the function

$$[0,\infty) \ni x \longrightarrow \left(\sum_{n=1}^{\infty} n(\overline{F}_X(x))^{n-1} p_N(n)\right)^{1-(\alpha+\beta)} \text{ is nonnegative increasing,}$$

it follows by Lemma 1.1 that

$$\mathbb{E}_Z\left((f_X(Z))^{\alpha+\beta-2}\left[\left(\frac{f_X(Z)}{f_Y(F^{-1}_Y(F_X(Z)))}\right)^{2-(\alpha+\beta)} - 1\right] 1|_{[Z>t]}\right) \geq 0 \text{ for any } t \geq 0.$$

Applying Theorem 2.2 we conclude that $X \leq_{VQE} Y$. □

Let $\theta > 0$. We consider $X(\theta)$ and $Y(\theta)$ two nonnegative absolutely continuous random variables with survival functions $(\overline{F}_X)^\theta$ and $(\overline{F}_Y)^\theta$ respectively.

Theorem 2.10. *a) If $\theta \geq 1$ and $X \leq_{VQE} Y$, then $X(\theta) \leq_{VQE} Y(\theta)$.*

b) If $0 < \theta \leq 1$ and $X(\theta) \leq_{VQE} Y(\theta)$, then $X \leq_{VQE} Y$.

Proof. For $X(\theta)$ and $Y(\theta)$, we denote the distribution functions by $F_{X(\theta)}$, respectively $F_{Y(\theta)}$, the right continuous inverse functions by $F^{-1}_{X(\theta)}$, respectively $F^{-1}_{Y(\theta)}$ and the density functions, by $f_{X(\theta)}$, respectively $f_{Y(\theta)}$.

For any $x \geq 0$ we have

$$f_{X(\theta)}(x) = \theta\left(\overline{F}_X(x)\right)^{\theta-1} f_X(x),$$

$$F^{-1}_{Y(\theta)}\left(F_{X(\theta)}(x)\right) = F^{-1}_Y\left(F_X(x)\right),$$

$$f_{Y(\theta)}\left(F^{-1}_{Y(\theta)}\left(F_{X(\theta)}(x)\right)\right) = \theta\left(\overline{F}_X(x)\right)^{\theta-1} f_Y\left(F^{-1}_Y\left(F_X(x)\right)\right).$$

Hence

$$\frac{f_{X(\theta)}(x)}{f_{Y(\theta)}\left(F^{-1}_{Y(\theta)}\left(F_{X(\theta)}(x)\right)\right)} = \frac{f_X(x)}{f_Y\left(F^{-1}_Y\left(F_X(x)\right)\right)}. \tag{2.1}$$

According to Theorem 2.2 we have $X(\theta) \leq_{VQE} Y(\theta)$ if and only if

$$\mathbb{E}_Z\left(\left(f_{X(\theta)}(Z)\right)^{\alpha+\beta-2}\left[\left(\frac{f_{X(\theta)}(Z)}{f_{Y(\theta)}(F_{Y(\theta)}^{-1}(F_{X(\theta)}(Z)))}\right)^{2-(\alpha+\beta)}-1\right]1|_{[Z>t]}\right)\geq 0 \text{ for any } t\geq 0$$

and $X \leq_{VQE} Y$ if and only if

$$\mathbb{E}_Z\left((f_X(Z))^{\alpha+\beta-2}\left[\left(\frac{f_X(Z)}{f_Y(F_Y^{-1}(F_X(Z)))}\right)^{2-(\alpha+\beta)}-1\right]1|_{[Z>t]}\right)\geq 0 \text{ for any } t\geq 0.$$

According to (2.1) we have

$$\mathbb{E}_Z\left(\left(f_{X(\theta)}(Z)\right)^{\alpha+\beta-2}\left[\left(\frac{f_{X(\theta)}(Z)}{f_{Y(\theta)}(F_{Y(\theta)}^{-1}(F_{X(\theta)}(Z)))}\right)^{2-(\alpha+\beta)}-1\right]1|_{[Z>t]}\right)=$$
$$\mathbb{E}_Z\left(\left(\theta\left(\overline{F}_X(x)\right)^{\theta-1}\right)^{\alpha+\beta-1}(f_X(Z))^{\alpha+\beta-2}\left[\left(\frac{f_X(Z)}{f_Y(F_Y^{-1}(F_X(Z)))}\right)^{2-(\alpha+\beta)}-1\right]1|_{[Z>t]}\right)$$
$$\text{for any } t\geq 0.$$

a) We assume that $0<\theta\leq 1$ and $X\leq_{VQE} Y$. Hence the function

$$[0,\infty)\ni x\longrightarrow\left(\theta\left(\overline{F}_X(x)\right)^{\theta-1}\right)^{\alpha+\beta-1} \text{ is nonnegative increasing}$$

and

$$\mathbb{E}_Z\left((f_X(Z))^{\alpha+\beta-2}\left[\left(\frac{f_X(Z)}{f_Y(F_Y^{-1}(F_X(Z)))}\right)^{2-(\alpha+\beta)}-1\right]1|_{[Z>t]}\right)\geq 0 \text{ for any } t\geq 0.$$

We apply Lemma 1.1 and conclude that $X(\theta)\leq_{VQE} Y(\theta)$.

b) We assume that $\theta\geq 1$ and $X(\theta)\leq_{VQE} Y(\theta)$. Hence the function

$$[0,\infty)\ni x\longrightarrow\left(\theta\left(\overline{F}_X(x)\right)^{\theta-1}\right)^{1-(\alpha+\beta)} \text{ is nonnegative increasing}$$

and

$$\mathbb{E}_Z\left(\left(f_{X(\theta)}(Z)\right)^{\alpha+\beta-2}\left[\left(\frac{f_{X(\theta)}(Z)}{f_{Y(\theta)}(F_{Y(\theta)}^{-1}(F_{X(\theta)}(Z)))}\right)^{2-(\alpha+\beta)}-1\right]1|_{[Z>t]}\right)\geq 0 \text{ for any } t\geq 0.$$

We apply again Lemma 1.1 and obtain the conclusion. □

In the sequel we consider a proportional hazard model. For any $\theta > 0$ let $X(\theta)$ and $Y(\theta)$ be two nonnegative absolutely continuous random variables with distribution functions $(F_X)^\theta$ and $(F_Y)^\theta$.

Theorem 2.11. *a) If $\theta \geq 1$ and $X \leq_{VQE} Y$, then $X(\theta) \leq_{VQE} Y(\theta)$.*
b) If $0 < \theta \leq 1$ and $X(\theta) \leq_{VQE} Y(\theta)$, then $X \leq_{VQE} Y$.

Proof. The proof is similar with the proof of Theorem 2.10. □

In [21] it was studied the following proportional odds model. Let $\theta > 0$ and X and Y be two nonnegative absolutely continuous random variables having the survival functions $\overline{F}_X$, respectively $\overline{F}_Y$ and density functions f_X, respectively f_Y. The proportional odds random variables X_p and Y_p are defined by the survival functions

$$\overline{F}_{X_p}(x) = \frac{\theta \overline{F}_X(x)}{1-(1-\theta)\overline{F}_X(x)},$$

respectively

$$\overline{F}_{Y_p}(x) = \frac{\theta \overline{F}_Y(x)}{1-(1-\theta)\overline{F}_Y(x)}.$$

Theorem 2.12. *a) If $\theta \geq 1$ and $X \leq_{VQE} Y$, then $X_p \leq_{VQE} Y_p$.*
b) If $0 < \theta \leq 1$ and $X_p \leq_{VQE} Y_p$, then $X \leq_{VQE} Y$.

Proof. For any $\theta > 0$ let $h : [0,1] \to \mathbb{R}$, $h(u) = \dfrac{\theta u}{1-(1-\theta)u}$.

We have:

a) If $\theta \geq 1$, then h is increasing concave on $[0,1]$.

b) If $0 < \theta \leq 1$, then h is increasing convex on $[0,1]$.

We remark that

$$\overline{F}_{X_p}(x) = h\left(\overline{F}_X(x)\right)$$

and

$$\overline{F}_{Y_p}(x) = h\left(\overline{F}_Y(x)\right).$$

Hence

$$f_{X_p}(x) = h'\left(\overline{F}_X(x)\right) f_X(x)$$

and

$$f_{Y_p}(x) = h'\left(\overline{F}_Y(x)\right) f_Y(x).$$

One can prove that $F_{Y_p}^{-1}\left(F_{X_p}(x)\right) = F_Y^{-1}\left(F_X(x)\right)$.

We obtain that

$$\frac{f_{X_p}(x)}{f_{Y_p}\left(F_{Y_p}^{-1}\left(F_{X_p}(x)\right)\right)}=\frac{f_X(x)}{f_Y\left(F_Y^{-1}\left(F_X(x)\right)\right)}.$$

Hence

$$\mathbb{E}_Z\left(\left(f_{X_p}(Z)\right)^{\alpha+\beta-2}\left[\left(\frac{f_{X_p}(Z)}{f_{Y_p}(F_{Y_p}^{-1}(F_{X_p}(Z)))}\right)^{2-(\alpha+\beta)}-1\right]1|_{[Z>t]}\right)=$$
$$\mathbb{E}_Z\left(\left(h'\left(\overline{F}_X(Z)\right)\right)^{\alpha+\beta-1}\left(f_X(Z)\right)^{\alpha+\beta-2}\left[\left(\frac{f_X(Z)}{f_Y(F_Y^{-1}(F_X(Z)))}\right)^{2-(\alpha+\beta)}-1\right]1|_{[Z>t]}\right)$$
for any $t \geq 0$.

According to Theorem 2.2 we have $X \leq_{VQE} Y$ if and only if

$$\mathbb{E}_Z\left(\left(f_X(Z)\right)^{\alpha+\beta-2}\left[\left(\frac{f_X(Z)}{f_Y(F_Y^{-1}(F_X(Z)))}\right)^{2-(\alpha+\beta)}-1\right]1|_{[Z>t]}\right)\geq 0 \text{ for any } t\geq 0$$

and $X_p \leq_{VQE} Y_p$ if and only if

$$\mathbb{E}_Z\left(\left(f_{X_p}(Z)\right)^{\alpha+\beta-2}\left[\left(\frac{f_{X_p}(Z)}{f_{Y_p}(F_{Y_p}^{-1}(F_{X_p}(Z)))}\right)^{2-(\alpha+\beta)}-1\right]1|_{[Z>t]}\right)\geq 0 \text{ for any } t\geq 0.$$

a) If $X \leq_{VQE} Y$ and $\theta \geq 1$, then

$$\mathbb{E}_Z\left(\left(f_X(Z)\right)^{\alpha+\beta-2}\left[\left(\frac{f_X(Z)}{f_Y(F_Y^{-1}(F_X(Z)))}\right)^{2-(\alpha+\beta)}-1\right]1|_{[Z>t]}\right)\geq 0 \text{ for any } t\geq 0$$

and the function

$$[0,\infty)\ni x\longrightarrow\left(h'\left(\overline{F}_X(x)\right)\right)^{\alpha+\beta-1} \text{ is nonnegative increasing,}$$

hence, by Theorem 2.2, we get $X_p \leq_{VQE} Y_p$.

b) If $X_p \leq_{VQE} Y_p$ and $0<\theta\leq 1$, then

$$\mathbb{E}_Z\left(\left(f_{X_p}(Z)\right)^{\alpha+\beta-2}\left[\left(\frac{f_{X_p}(Z)}{f_{Y_p}(F_{Y_p}^{-1}(F_{X_p}(Z)))}\right)^{2-(\alpha+\beta)}-1\right]1|_{[Z>t]}\right)\geq 0 \text{ for any } t\geq 0$$

and the function

$$[0,\infty)\ni x\longrightarrow\frac{1}{\left(h'\left(\overline{F}_X(x)\right)\right)^{\alpha+\beta-1}} \text{ is nonnegative increasing,}$$

hence, again by Theorem 2.2, we get $X \leq_{VQE} Y$. □

In the sequel we are concerned with the preservation of the Varma quantile entropy order in the record values model.

We consider $\{X_i \mid i \geq 1\}$ a sequence of independent and identically distributed random variables from the random variable X with survival function $\overline{F}_X$ and density function f_X. The nth record times are the random variables T_n^X defined via $T_1^X = 1$ and $T_{n+1}^X = \min\{j > T_n^X \mid X_j > X_{T_n^X}\}$, $n \geq 1$.

We denote $X_{T_n^X} \stackrel{def}{=} R_n^X$ and call them the nth record values. For more informations about record values we recommend [4].

Concerning R_n^X we have, for any $x \geq 0$:

$$f_{R_n^X}(x) = \frac{1}{\Gamma(n)} \Lambda_X^{n-1}(x) f_X(x)$$

and

$$\overline{F}_{R_n^X}(x) = \overline{F}_X(x) \sum_{j=1}^{n-1} \frac{(\Lambda_X(x))^j}{j!} = \overline{\Gamma}_n\left(\Lambda_X(x)\right),$$

where $\overline{\Gamma}_n$ is the survival function of a Gamma random variable with the shape parameter n and the scale parameter 1 and $\Lambda_X(x) = -\log \overline{F}_X(x)$ is the cumulative failure rate function of X.

Taking $\{Y_i \mid i \geq 1\}$ a sequence of independent and identically distributed random variables from the random variable Y, we have similar formulas for R_n^Y.

Theorem 2.13. *Let $m, n \in \mathbb{N} \stackrel{def}{=} \{1, 2, ...\}$.*
a) If $X \leq_{VQE} Y$, then $R_n^X \leq_{VQE} R_n^Y$ for any $n \in \mathbb{N}$.
b) If $R_m^X \leq_{VQE} R_m^Y$, then $R_n^X \leq_{VQE} R_n^Y$ for any $n > m \geq 1$.

Proof. $a)$ We assume that $X \leq_{VQE} Y$. Then

$$\mathbb{E}_Z\left((f_X(Z))^{\alpha+\beta-2}\left[\left(\frac{f_X(Z)}{f_Y(F_Y^{-1}(F_X(Z)))}\right)^{2-(\alpha+\beta)} - 1\right] 1|_{[Z>t]}\right) \geq 0 \text{ for any } t \geq 0.$$

We remark that

$$\frac{f_{R_n^X}(x)}{f_{R_n^Y}\left(F_{R_n^Y}^{-1}\left(F_{R_n^X}(x)\right)\right)} = \frac{f_X(x)}{f_Y\left(F_Y^{-1}\left(F_X(x)\right)\right)}.$$

Using the preceding equalities, we have

$$\mathbb{E}_Z\left(\left(f_{R_n^X}(Z)\right)^{\alpha+\beta-2}\left[\left(\frac{f_{R_n^X}(Z)}{f_{R_n^Y}(F_{R_n^Y}^{-1}(F_{R_n^X}(Z)))}\right)^{2-(\alpha+\beta)}-1\right]1|_{[Z>t]}\right)=$$
$$\mathbb{E}_Z\left(\left(\frac{1}{\Gamma(n)}\Lambda_X^{n-1}(Z)\right)^{\alpha+\beta-1}(f_X(Z))^{\alpha+\beta-2}\left[\left(\frac{f_X(Z)}{f_Y(F_Y^{-1}(F_X(Z)))}\right)^{2-(\alpha+\beta)}-1\right]1|_{[Z>t]}\right) \text{ for any } t\geq 0.$$

Because the function

$$[0,\infty)\ni x\longrightarrow\left(\frac{1}{\Gamma(n)}\Lambda_X^{n-1}(x)\right)^{\alpha+\beta-1} \text{ is nonnegative increasing,}$$

we obtain by Lemma 1.1 that

$$\mathbb{E}_Z\left(\left(f_{R_n^X}(Z)\right)^{\alpha+\beta-2}\left[\left(\frac{f_{R_n^X}(Z)}{f_{R_n^Y}(F_{R_n^Y}^{-1}(F_{R_n^X}(Z)))}\right)^{2-(\alpha+\beta)}-1\right]1|_{[Z>t]}\right)\geq 0 \text{ for any } t\geq 0,$$

i.e. $R_n^X\leq_{VQE}R_n^Y$.

b) Let $n>m\geq 1$. If $R_m^X\leq_{VQE}R_m^Y$, then

$$\mathbb{E}_Z\left(\left(f_{R_m^X}(Z)\right)^{\alpha+\beta-2}\left[\left(\frac{f_{R_m^X}(Z)}{f_{R_m^Y}(F_{R_m^Y}^{-1}(F_{R_m^X}(Z)))}\right)^{2-(\alpha+\beta)}-1\right]1|_{[Z>t]}\right)\geq 0 \text{ for any } t\geq 0.$$

Using previous formulas we get

$$\frac{f_{R_m^X}(x)}{f_{R_m^Y}\left(F_{R_m^Y}^{-1}\left(F_{R_m^X}(x)\right)\right)}=\frac{f_{R_n^X}(x)}{f_{R_n^Y}\left(F_{R_n^Y}^{-1}\left(F_{R_n^X}(x)\right)\right)}=\frac{f_X(x)}{f_Y\left(F_Y^{-1}\left(F_X(x)\right)\right)}$$

and

$$\frac{f_{R_n^X}(x)}{f_{R_m^X}(x)}=\frac{\Gamma(m)}{\Gamma(n)}\left(\Lambda_X(x)\right)^{n-m}.$$

Hence

$$\mathbb{E}_Z\left(\left(f_{R_n^X}(Z)\right)^{\alpha+\beta-2}\left[\left(\frac{f_{R_n^X}(Z)}{f_{R_n^Y}(F_{R_n^Y}^{-1}(F_{R_n^X}(Z)))}\right)^{2-(\alpha+\beta)}-1\right]1|_{[Z>t]}\right)=$$
$$\mathbb{E}_Z\left(\left(\frac{\Gamma(m)}{\Gamma(n)}\left(\Lambda_X(Z)\right)^{n-m}\right)^{\alpha+\beta-1}\left(f_{R_m^X}(Z)\right)^{\alpha+\beta-2}\left[\left(\frac{f_{R_m^X}(Z)}{f_{R_m^Y}(F_{R_m^Y}^{-1}(F_{R_m^X}(Z)))}\right)^{2-(\alpha+\beta)}-1\right]1|_{[Z>t]}\right) \text{ for any } t\geq 0.$$

Because the function

$$[0,\infty)\ni x\longrightarrow\left(\frac{\Gamma(m)}{\Gamma(n)}\left(\Lambda_X(x)\right)^{n-m}\right)^{\alpha+\beta-1} \text{ is nonnegative increasing}$$

and

$$\mathbb{E}_Z\left(\left(f_{R_m^X}(Z)\right)^{\alpha+\beta-2}\left[\left(\frac{f_{R_m^X}(Z)}{f_{R_m^Y}(F^{-1}_{R_m^Y}(F_{R_m^X}(Z)))}\right)^{2-(\alpha+\beta)}-1\right]1|_{[Z>t]}\right)\geq 0 \text{ for any } t\geq 0,$$

using Lemma 1.1, we obtain that

$$\mathbb{E}_Z\left(\left(f_{R_n^X}(Z)\right)^{\alpha+\beta-2}\left[\left(\frac{f_{R_n^X}(Z)}{f_{R_n^Y}(F^{-1}_{R_n^Y}(F_{R_n^X}(Z)))}\right)^{2-(\alpha+\beta)}-1\right]1|_{[Z>t]}\right)\geq 0 \text{ for any } t\geq 0,$$

i.e. $R_n^X \leq_{VQE} R_n^Y$. □

Conclusions

The notion of quantile entropy was intensively studied in the last years, mainly for the multiple applications in Physics.

In this paper, we studied closure and reversed closure properties of the Varma quantile entropy order under several reliability transformations. Also we proved the preservation of the Varma quantile entropy order in several stochastic models, like proportional hazard rate model, proportional reversed hazard rate model, proportional odds model and record values model.

We intend to continue this work, considering another reliability transformations and another stochastic models.

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