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Strong convergence to a solution of the inclusion problem for a finite family of monotone operators in Hadamard spaces

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Abstract

In this paper, in the setting of Hadamard spaces, a iterative scheme is proposed for approximating a solution of the inclusion problem for a finite family of monotone operators which is a unique solution of a variational inequality. Some applications in convex minimization and fixed point theory are also presented to support the main result.

1 Introduction

A valuable tool in the study of problems associated to optimization, equilibrium point, variational inequality is the concept of monotonicity. Finding solutions to inclusion problems (i.e. zeros of monotone operators) is one of the most fundamental issues in monotone operators theory. The design of algorithms to approximate the zeros of monotone operators has always been of interest to many authors. Rockefellar, in a seminal work [27], defined the proximal point algorithm for monotone operators by means of the following iterative scheme:

$$0 \in A(x_{n+1}) + \lambda_n(x_{n+1} - x_n), \quad n = 0, 1, 2, ... \tag{1.1}$$

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where $A : H \longrightarrow 2^H$ is a monotone operator on Hilbert space H, (λ_n) is a sequence of real positive numbers and x_0 is an initial point. In fact, Rockafellar [27] proved that the sequence generated by (1.1) is weakly convergent to a solution of the inclusion problem

$$\text{Finding} \quad x \in H \quad \text{such that} \quad 0 \in Ax, \tag{1.2}$$

provided $\lambda_n \geq \lambda > 0, \quad \forall n \geq 1$. Kamimura and Takahashi [18], by using Halpern regularization, proved a strong convergence theorem to approximate a zero of the monotone operator A, which is a solution of the inclusion problem (1.2). The study of the convergence problem of the proximal point algorithm has been very fruitful and has engaged researchers from different areas, such as variational inequalities, optimization and metric fixed-point theory. In the case of Hilbert spaces, the reader can consult [6, 8, 12, 18, 19, 24, 27].

The extensions to convex abstract spaces and CAT(0) spaces of the concepts and techniques that fit in Euclidean spaces are natural and non-trivial. Actually, in recent years some algorithms defined to solve nonlinear equations, variational inequalities and minimization problems, which involve monotone operators, have been extended from the Hilbert space framework to the more general setting of CAT(0) spaces . In particular, Bačák [3] extended the work of Martinet [24] to complete CAT(0) spaces. Recently, Khatibzadeh and the author [20] considered some properties of a monotone operator and its resolvent operator and upgraded the proximal point method (1.1) to complete CAT(0) spaces, (Also, see [25]). Then, the author and Khatibzadeh [26] extended the work of Kamimura and Takahashi [18] to complete CAT(0) spaces. Very recently, Heydari, Khadem and the author [15] proved the Δ-convergence (see, Theorem 2.3) of a modified proximal point algorithm to solve the inclusion problem

$$\text{Finding} \quad x \in X \quad \text{such that} \quad 0 \in A_i x \quad (\text{for} \ \ i = 1, 2, ..., m,) \tag{1.3}$$

for approximating a common zero of monotone operators $A_1, A_2, ..., A_m$, on complete CAT(0) spaces X.

In this paper, we propose the Halpern regularization method of proximal point algorithm for a finite family of monotone operators to establish a strong convergence theorem for getting to a solution of the inclusion problem (1.3) which is a common zero of finite family $A_1, A_2, ..., A_m$ of monotone operators on a complete CAT(0) space. This solution is, also, a unique solution to a variational inequality. Some applications of the main result in convex minimization problems and fixed point theory are also presented. Our results extend and improve the related results in the literature.

The paper has been organized as follows.
In Section 2, we give a brief introduction of CAT(0) spaces and some lemmas

that we need to prove the main result. In Section 3, we propose the Halpern regularization method of proximal point algorithm for a finite family of monotone operators and prove, under suitable conditions, strong convergence of the proposed sequence to a solution of the inclusion problem (1.3) which is a common zero of a finite family of monotone operators and a unique solution of a variational inequality in complete CAT(0) spaces. Section 4 and Section 5 are devoted to the applications of the main result in convex minimization problems and fixed point theory.

2 Preliminaries

Let (X, d) be a metric space. A continuous mapping from the interval $[0, 1]$ to X is called a path. Given a pair of points $x, y \in X$, we say that a path $c : [0, 1] \longrightarrow X$ joins x and y if $c(0) = x, c(d(x, y)) = y$. A path $c : [0, 1] \longrightarrow X$ is called a geodesic if $d(c(s), c(t)) = d(c(0), c(1))|s - t|$ for every $s, t \in [0, 1]$, that is, if it parametrized proportionally to the arc length. In particular, a geodesic is an injection unless it is trivial, that is, unless $c(0) = c(1)$. The metric space (X, d) is a geodesic space if every two points $x, y \in X$ are connected by a geodesic. If moreover every two points of (X, d) are connected by a unique geodesic, the space (X, d) is called uniquely geodesic. In this case, such a geodesic is denoted by $[x, y]$, but one must remember that such a geodesic is not uniquely determined by its endpoints in general. For a point $z \in [x, y]$, the notation $z = (1 - t)x \oplus ty$ is used, where $t = \dfrac{d(x, z)}{d(x, y)}$ and we say z is a convex combination of x and y. A subset C of X is called convex if $[x, y] \subseteq C$ for all $x, y \in C$.

A non-positive curvature metric space or a CAT(0) space (in honour of E. **C**artan, AD. **A**lexandrov and V.A. **T**oponogov) is a geodesic space (X, d) which satisfies the following condition.

CN−inequality: If $x, y_0, y_1, y_2 \in X$ such that $d(y_0, y_1) = d(y_0, y_2) = \frac{1}{2}d(y_1, y_2)$, then

$$d^2(x, y_0) \leq \frac{1}{2}d^2(x, y_1) + \frac{1}{2}d^2(x, y_2) - \frac{1}{4}d^2(y_1, y_2).$$

It is known that a CAT(0) space is a uniquely geodesic space. A complete CAT(0) space is called a *Hadamard* space. For other equivalent definitions and basic properties, we refer the reader to the standard texts such as [4, 7, 9, 14, 17]. The following are the prominent examples of Hadamard spaces:

Hilbert spaces, Hadamard manifolds (i.e. simply connected complete Riemannian manifolds with non-positive sectional curvature which can be of infinite dimension), R-trees as well as examples that have been built out of given

Hadamard spaces such as closed convex subsets, direct products, warped products, L^2-spaces, direct limits and Reshetnyak's gluing (see [29], Section 3). Berg and Nikolaev [5] have introduced the concept of *quasilinearization* for the CAT(0) space X. They denote a pair $(a, b) \in X \times X$ by $\overrightarrow{ab}$ and called it a *vector*. Then the quasilinearization map $\langle . \rangle : (X \times X) \times (X \times X) \to \mathbb{R}$ is defined by

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \tfrac{1}{2}(d^2(a,d) + d^2(b,c) - d^2(a,c) - d^2(b,d)), \qquad (a,b,c,d \in X).$$

It can be easily verified that $\langle \overrightarrow{ab}, \overrightarrow{ab} \rangle = d^2(a,b)$, $\langle \overrightarrow{ba}, \overrightarrow{cd} \rangle = -\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle$ and $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \langle \overrightarrow{ae}, \overrightarrow{cd} \rangle + \langle \overrightarrow{eb}, \overrightarrow{cd} \rangle$ are satisfied for all $a, b, c, d, e \in X$. Also, we can formally add compatible vectors, more precisely $\overrightarrow{ac} + \overrightarrow{cb} = \overrightarrow{ab}$, for all $a, b, c \in X$. We say that X satisfies the Cauchy-Schwarz inequality if

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle \leq d(a,b)d(c,d), \qquad (a,b,c,d \in X).$$

Berg and Nikolaev have then proved the following result.

Theorem 2.1. *[5, Corollary 3] A geodesically connected metric space is a CAT(0) space if and only if it satisfies the Cauchy-Schwarz inequality.*

Ahmadi Kakavandi and Amini [2] have introduced the concept of dual space of a Hadamard space X, based on a work of Berg and Nikolaev [5], as follows.
Consider the map $\Theta : \mathbb{R} \times X \times X \to C(X, \mathbb{R})$ defined by

$$\Theta(t,a,b)(x) = t\langle \overrightarrow{ab}, \overrightarrow{ax} \rangle, \qquad (t \in \mathbb{R},\ a,b,x \in X),$$

where $C(X, \mathbb{R})$ is the space of all continuous real-valued functions on X. Then the Cauchy-Schwarz inequality implies that $\Theta(t,a,b)$ is a Lipschitz function with Lipschitz semi-norm $L(\Theta(t,a,b)) = |t|d(a,b)$ $(t \in \mathbb{R}, \quad a,b \in X)$, where $L(\varphi) = sup\{\frac{\varphi(x)-\varphi(y)}{d(x,y)} : x, y \in X, x \neq y\}$ is the Lipschitz semi-norm for any function $\varphi : X \to \mathbb{R}$. A pseudometric D on $\mathbb{R} \times X \times X$ is defined by

$$D((t,a,b),(s,c,d)) = L(\Theta(t,a,b) - \Theta(s,c,d)), \qquad (t,s \in \mathbb{R},\ a,b,c,d \in X).$$

For a Hadamard space (X, d), the pseudometric space $(\mathbb{R} \times X \times X, D)$ can be considered as a subspace of the pseudometric space of all real-valued Lipschitz functions $(Lip(X, \mathbb{R}), L)$.

Lemma 2.2. *[2, Lemma 2.1]* $D((t,a,b),(s,c,d)) = 0$ *if and only if* $t\langle \overrightarrow{ab}, \overrightarrow{xy} \rangle = s\langle \overrightarrow{cd}, \overrightarrow{xy} \rangle$, *for all* $x, y \in X$.

By Lemma 2.2, D imposes an equivalent relation on $\mathbb{R} \times X \times X$, where the equivalence class of (t,a,b) is

$$[t\overrightarrow{ab}] = \{s\overrightarrow{cd} : D((t,a,b),(s,c,d)) = 0\}.$$

The set $X^* = \{[t\overrightarrow{ab}] : (t,a,b) \in \mathbb{R} \times X \times X\}$ is a metric space with metric $D([t\overrightarrow{ab}],[s\overrightarrow{cd}]) := D((t,a,b),(s,c,d))$, which is called the dual space of (X,d). It is clear that $[\overrightarrow{aa}] = [\overrightarrow{bb}]$ for all $a,b \in X$. Fix $o \in X$, we write $\mathbf{0} = [\overrightarrow{oo}]$ as the zero of the dual space. In [2], it is shown that the dual of a closed and convex subset of Hilbert space H with nonempty interior is H and $t(b-a) \equiv [t\overrightarrow{ab}]$ for all $t \in \mathbb{R},\ a,b \in H$. Note that X^* acts on $X \times X$ by

$$\langle x^*, \overrightarrow{xy}\rangle = t\langle \overrightarrow{ab}, \overrightarrow{xy}\rangle, \qquad (x^* = [t\overrightarrow{ab}] \in X^*, x,y \in X).$$

Also, we use the following notation:

$$\langle \alpha x^* + \beta y^*, \overrightarrow{xy}\rangle := \alpha\langle x^*, \overrightarrow{xy}\rangle + \beta\langle y^*, \overrightarrow{xy}\rangle, \qquad (\alpha,\beta \in \mathbb{R},\ x,y \in X,\ x^*,y^* \in X^*).$$

A notion of convergence in Hadamard spaces, Δ-convergence introduced by Lim [23], has been studied by many authors (e.g. [11, 13]).

A. Kakavandi [1] proved the following characterization for Δ-convergence.

Theorem 2.3. *[1, Theorem 2.6] Let (X,d) be a Hadamard space, (x_n) be a sequence in X and $x \in X$. Then (x_n) Δ-converges to x if and only if*

$$\limsup_{n\to\infty} \langle \overrightarrow{xx_n}, \overrightarrow{xy}\rangle \le 0, \quad \textit{for all} \quad y \in X.$$

It is well-known that in all CAT(0) spaces, every bounded sequence has a Δ-convergent subsequence.

In the following, we present some properties of the resolvent operator of a monotone operator in CAT(0) space which verified in [20], and we need them in the sequel.

Definition 2.4. [16] Let X be a Hadamard space with dual space X^*. The multi-valued operator $A : X \to 2^{X^*}$ with domain $\mathbb{D}(A) := \{x \in X : A(x) \neq \emptyset\}$, is monotone if and only if

$$\langle x^* - y^*, \overrightarrow{yx}\rangle \ge 0,$$

for all $x,y \in \mathbb{D}(A), x \neq y,\ x^* \in Ax, y^* \in Ay$.

Definition 2.5. [20] Let $\lambda > 0$ and $A : X \to 2^{X^*}$ be a set-valued operator. The resolvent of A of order λ is the set-valued mappings $J_\lambda : X \to 2^X$ defined by $J_\lambda(x) := \{z \in X \mid \ [\frac{1}{\lambda}\overrightarrow{zx}] \in Az\}$.

Definition 2.6. [20] Let X be a Hadamard space and $T : C \subset X \to X$ be a mapping. We say that T is firmly nonexpansive if $\quad d^2(Tx,Ty) \le \langle \overrightarrow{TxTy}, \overrightarrow{xy}\rangle$, for any $x,y \in C$.

By the definition and Cauchy-Schwarz inequality, it is clear that any firmly nonexpansive mapping T is nonexpansive.
We denote $F(T) := \{x \in X \text{ such that } Tx = x\}$.

Remark 2.7. Let C be a nonempty convex subset of a Hadamard space X. The map $P : X \longrightarrow C$ with $u = P_C x$ is called the projection mapping where $d(x,u) \leq d(x,y), \ \forall\, y \in C$, (see, [10]). Dehghan and Rooin [10] proved that $\langle \overrightarrow{xu}, \overrightarrow{yu} \rangle \leq 0, \ \forall\, y \in C$. [20, Corollary 3.8.] shows that the projection mapping is a firmly nonexpansive mapping.

Theorem 2.8. *[20] Let X be a CAT(0) space and J_λ is resolvent of the operator A of order λ. We have;*
(i) For any $\lambda > 0$, $\mathbb{R}(J_\lambda) \subset \mathbb{D}(A)$, $F(J_\lambda) = A^{-1}(\mathbf{0})$.
(ii) If A is monotone, then J_λ is a single-valued and firmly nonexpansive mapping.
(iii) If A is monotone and $\lambda \leq \mu$ then $d(x, J_\lambda x) \leq 2d(x, J_\mu x)$.

Remark 2.9. [20] It is well-known that if T is a nonexpansive mapping on subset C of CAT(0) space X then $F(T)$ is closed and convex. Thus, if A is a monotone operator on CAT(0) space X then, by parts (i) and (ii) of Theorem 2.8, $A^{-1}(\mathbf{0})$ is closed and convex.

The following lemmas are needed to prove the main result.

Lemma 2.10. *[11] Let (X,d) be a CAT(0) space. Then, for all $x, y, z \in X$ and all $t \in [0,1]$:*
(1) $\quad d^2(tx \oplus (1-t)y, z) \leq td^2(x,z) + (1-t)d^2(y,z) - t(1-t)d^2(x,y)$,
(2) $\quad d(tx \oplus (1-t)y, z) \leq td(x,z) + (1-t)d(y,z)$,
In addition, by using (1), we have

$$d(tx \oplus (1-t)y, tx \oplus (1-t)z) \leq (1-t)d(y,z).$$

Lemma 2.11. *[30] Let X be a Hadamard space, C be a nonempty closed and convex subset of X and $T : C \longrightarrow C$ be a nonexpansive mapping. For any contraction $f : C \longrightarrow C$ and $t \in (0,1)$, let $x_t \in C$ be the unique fixed point of the contraction $x \longrightarrow tf(x) \oplus (1-t)Tx$, i.e., $x_t = tf(x_t) \oplus (1-t)Tx_t$. Then (x_t) converges strongly as $t \longrightarrow 0$ to a point x^* such that $x^* = P_{F(T)} f(x^*)$, where $P_{F(T)}$ is the metric projection from $X \longrightarrow F(T)$, which is also the unique solution to the following variational inequality:*

$$\langle \overrightarrow{x^* f(x^*)}, \overrightarrow{xx^*} \rangle \geq 0, \quad \forall\, x \in F(T).$$

Lemma 2.12. *[28] Let $\{s_n\}$ be a sequence of nonnegative real numbers, $\{\alpha_n\}$ be a sequence of real numbers in $(0,1)$ with $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\{t_n\}$ be a* *sequence of real numbers. Suppose that*

$$s_{n+1} \leq (1-\alpha_n)s_n + \alpha_n t_n \quad \text{for all} \quad n \geq 1.$$

If $\limsup_{k\to\infty} t_{m_k} \leq 0$ *for every subsequence* $\{s_{m_k}\}$ *of* $\{s_n\}$ *satisfying* $\liminf_k (s_{m_k+1} - s_{m_k}) \geq 0$, *then* $\lim_{n\to\infty} s_n = 0$.

3 Strong convergence to a common zero of monotone operators

Let X be a Hadamard space with dual X^*. We say that the operator $A : X \to 2^{X^*}$ satisfies the *range condition* if for every $\lambda > 0$, $D(J_\lambda^A) = X$ (see [20]). It is known that if A is a maximal monotone operator on the Hilbert space H then $R(I + \lambda A) = H, \quad \forall \lambda > 0$, where I is the identity operator. Thus, every maximal monotone operator A on a Hilbert space satisfies the *range condition.* Also as it has shown in [22], if A is a maximal monotone operator on a Hadamard manifold, then A satisfies the range condition. For presenting some examples of monotone operators that satisfy the range condition in CAT(0) spaces, refer to [20, Sections 5 and 6].

Let $A_1, A_2, ..., A_m : X \to 2^{X^*}$ be multi-valued monotone operators on the Hadamard space X with dual X^* that satisfy the *range condition* and $(\lambda_{(n,i)})$ for $i = 1, 2, ..., m$ be some sequences of nonnegative real numbers. For strong convergence to a solution of the inclusion problem (1.3) which is a common zero of the finite family $A_1, A_2, ..., A_m$ of monotone operators in Hadamard spaces, we propose Halpern regularization method of proximal point algorithm which is the sequence generated by:

$$\begin{cases} x_1 \in X, \\ z_n^i = J_{\lambda_{(n,i)}}^{A_i} z_n^{i+1}, & \text{for } i \in \{1, 2, ..., m\}, \\ z_n^{m+1} = x_n, & \text{for all } n \in \mathbb{N}, \\ x_{n+1} = \alpha_n f(x_n) \oplus (1-\alpha_n) z_n^1, & \text{for all } n \in \mathbb{N}, \end{cases} \tag{3.1}$$

where (α_n) is a sequence in $[0, 1]$ and $f : X \longrightarrow X$ is a contractive mapping with coefficient κ, (i.e. there exists $\kappa \in (0, 1)$, such that $d(f(x), f(y)) \leq \kappa d(x, y) \ \ \forall\, x, y \in X$).

In the following, under suitable assumptions, we prove that the sequence (x_n) generated by (3.1) is convergent strongly to the element p of $\cap_{i=1}^m A_i^{-1}(\mathbf{0})$ which is a solution of the inclusion problem (1.3) and a unique solution of the variational inequality:

$$\langle \overrightarrow{pf(p)}, \overrightarrow{xp} \rangle \geq 0, \quad \forall\, x \in \cap_{i=1}^m A_i^{-1}(\mathbf{0}).$$

Lemma 3.1. *Let X be a Hadamard space with dual X^* and $A_1, A_2, ..., A_m : X \to 2^{X^*}$ be multi-valued monotone operators such that satisfy the range condition and $\cap_{i=1}^m A_i^{-1}(\mathbf{0}) \neq \emptyset$. Suppose $(\lambda_{(n,i)})$ for $i = 1, 2, ..., m$ are some sequences of nonnegative real numbers. Then*

$$U_{\lambda_n} := J^{A_1}_{\lambda_{(n,1)}} o J^{A_2}_{\lambda_{(n,2)}} o J^{A_3}_{\lambda_{(n,3)}} o...o J^{A_m}_{\lambda_{(n,m)}}$$

is a nonexpansive mapping and

$$F(U_{\lambda_n}) = F(J^{A_1}_{\lambda_{(n,1)}} o J^{A_2}_{\lambda_{(n,2)}} o J^{A_3}_{\lambda_{(n,3)}} o...o J^{A_m}_{\lambda_{(n,m)}}) = \cap_{i=1}^k A_i^{-1}(\mathbf{0}).$$

Proof. It is clear that U_{λ_n} is nonexpansive by part (ii) of Theorem 2.8. We prove $F(U_{\lambda_n}) = \cap_{i=1}^m A_i^{-1}(\mathbf{0})$. The inclusion $\cap_{i=1}^m A_i^{-1}(\mathbf{0}) \subset F(U_{\lambda_n})$ is obvious. We show $F(U_{\lambda_n}) \subset \cap_{i=1}^m A_i^{-1}(\mathbf{0})$. If $\cap_{i=1}^m A_i^{-1}(\mathbf{0}) = X$, then the proof is complete, otherwise, suppose that $x \notin \cap_{i=1}^m A_i^{-1}(\mathbf{0})$ and $p \in \cap_{i=1}^m A_i^{-1}(\mathbf{0})$. For $i = 1, 2, 3, ..., m$, $J^{A_i}_{\lambda_{(n,i)}}$ is firmly nonexpansive. Thus, by definition 2.6, for $i = 1, 2, 3, ..., m$, we obtain

$$d^2(J^{A_i}_{\lambda_{(n,i)}} x, J^{A_i}_{\lambda_{(n,i)}} p) \leq \langle \overrightarrow{(J^{A_i}_{\lambda_{(n,i)}} x)(J^{A_i}_{\lambda_{(n,i)}} p)}, \overrightarrow{xp} \rangle,$$

which by $p \in \cap_{i=1}^m A_i^{-1}(\mathbf{0})$, follows

$$d^2(J^{A_i}_{\lambda_{(n,i)}} x, p) \leq \langle \overrightarrow{(J^{A_i}_{\lambda_{(n,i)}} x)p}, \overrightarrow{xp} \rangle,$$

which implies

$$d^2(J^{A_i}_{\lambda_{(n,i)}} x, p) \leq d^2(x, p) - d^2(x, J^{A_i}_{\lambda_{(n,i)}} x) \rangle,$$

thus by $x \notin \cap_{i=1}^m A_i^{-1}(\mathbf{0})$, for $i = 1, 2, 3, ..., m$, we have

$$d(J^{A_i}_{\lambda_{(n,i)}} x, p) < d(x, p).$$

This together with nonexpansiveness of U_{λ_n} imply

$$\begin{aligned} d(U_{\lambda_n} x, p) &= d(J^{A_1}_{\lambda_{(n,1)}} o J^{A_2}_{\lambda_{(n,2)}} o J^{A_3}_{\lambda_{(n,3)}} o...o J^{A_m}_{\lambda_{(n,m)}} x, p) \\ &\leq d(J^{A_2}_{\lambda_{(n,2)}} o J^{A_3}_{\lambda_{(n,3)}} o...o J^{A_m}_{\lambda_{(n,m)}} x, p) \\ &\leq ... \\ &\leq d(J^{A_m}_{\lambda_{(n,m)}} x, p) \\ &< d(x, p). \end{aligned}$$

Now it is clear that $x \notin F(U_{\lambda_n})$ because, otherwise, we would obtain

$$d(x,p) = d(U_{\lambda_n}x, p) < d(x,p),$$

that is a contradiction. Hence, $F(U_{\lambda_n}) \subset \cap_{i=1}^m A_i^{-1}(\mathbf{0})$, which completes the proof. □

Theorem 3.2. *Let X be a Hadamard space with dual X^* and $A_1, A_2, ..., A_m : X \to 2^{X^*}$ be multi-valued monotone operators such that satisfy the range condition and $\cap_{i=1}^m A_i^{-1}(\mathbf{0}) \neq \emptyset$. Suppose $(\lambda_{(n,i)})$ for $i = 1, 2, ..., m$, are some sequences of nonnegative real numbers and (α_n) is a sequence in $[0,1]$ such that satisfy the conditions:*

$$\begin{cases} C1 : \lim_{n\to\infty} \alpha_n = 0, \\ C2 : \sum_{n=1}^{\infty} \alpha_n = \infty, \\ C3 : \lambda_{(n,i)} \geq \lambda > 0, \quad \textit{for all} \;\; n \in \mathbb{N} \;\; \textit{and} \;\; i = 1, 2, ..., m. \end{cases}$$

If $f : X \longrightarrow X$ is a contractive mapping with contractive coefficient $\kappa \in (0, \frac{1}{2}]$, then the sequence generated by (3.1) is convergent strongly to the element p of $\cap_{i=1}^m A_i^{-1}(\mathbf{0})$ such that $p = P_{\cap_{i=1}^m A_i^{-1}(\mathbf{0})} f(p)$, where $P_{\cap_{i=1}^m A_i^{-1}(\mathbf{0})}$ is the projection mapping from $P : X \longrightarrow \cap_{i=1}^m A_i^{-1}(\mathbf{0})$, and p also is the unique solution to the following variational inequality:

$$\langle \overrightarrow{pf(p)}, \overrightarrow{xp} \rangle \geq 0, \quad \forall \; x \in \cap_{i=1}^m A_i^{-1}(\mathbf{0}).$$

Proof. By Remark 2.9, $\cap_{i=1}^m A_i^{-1}(\mathbf{0})$ is convex and closed. Set $U_\lambda = J_\lambda^{A_1} o J_\lambda^{A_2} o J_\lambda^{A_3} o...o J_\lambda^{A_m}$. Then, by Lemma 3.1, U_λ is nonexpansive and $F(U_\lambda) = \cap_{i=1}^m A_i^{-1}(\mathbf{0}) = F(U_{\lambda_n})$. Therefore, by Lemma 2.11, there exists $p \in \cap_{i=1}^m A_i^{-1}(\mathbf{0})$ such that $p = P_{\cap_{i=1}^m A_i^{-1}(\mathbf{0})} f(p)$. We show (x_n) is convergent strongly to p. First, we prove that (x_n) is bounded. From $p = P_{\cap_{i=1}^m A_i^{-1}(\mathbf{0})} f(p)$, we get $0 \in A_i p$, for $i \in \{1, 2, ..., m\}$. Moreover, by the definition of the resolvent operator, we obtain

$$[\frac{1}{\lambda_{(n,i)}} \overrightarrow{z_n^i z_n^{i+1}}] \in A_i(z_n^i), \text{ for } \; i \in \{1, 2, ..., m\}.$$

Hence by the monotonicity of A_i, for $i \in \{1, 2, ..., m\}$ one has

$$\langle [\frac{1}{\lambda_{(n,i)}} \overrightarrow{z_n^i z_n^{i+1}}] - 0, \overrightarrow{p z_n^i} \rangle \geq 0,$$

or equivalently,

$$d^2(z_n^{i+1}, p) - d^2(z_n^i, p) \geq d^2(z_n^{i+1}, z_n^i), \quad \text{for } i \in \{1, 2, ..., m\}. \tag{3.2}$$

By summing the inequality (3.2) from $i = 1$ to $i = m$, we obtain

$$d^2(p, x_n) - d^2(p, z_n^1) \geq \sum_{i=1}^{k} d^2(z_n^{i+1}, z_n^i) \geq 0, \tag{3.3}$$

which implies $d(z_n^1, p) \leq d(x_n, p)$. Therefore, by Lemma 2.10, we have

$$\begin{aligned}
d(x_{n+1}, p) &\leq \alpha_n d(f(x_n), p) + (1-\alpha_n) d(z_n^1, p) \\
&\leq \alpha_n d(f(x_n), f(p)) + \alpha_n d(f(p), p) + (1-\alpha_n) d(x_n, p) \\
&\leq \kappa \alpha_n d(x_n, p) + \alpha_n d(f(p), p) + (1-\alpha_n) d(x_n, p) \\
&\leq (1 - \alpha_n(1-\kappa) d(x_n, p) + \alpha_n d(f(p), p) \\
&\leq \max\{d(x_n, p), \frac{d(f(p), p)}{1-\kappa}\} \\
&\leq ... \leq \max\{d(x_1, p), \frac{d(f(p), p)}{1-\kappa}\}.
\end{aligned}$$

Hence, (x_n) is bounded and so are $(f(x_n)$ and (z_n^1).
Note that

$$\begin{aligned}
d^2(f(x_n), p) - d^2(f(x_n), z_n^1) &= 2\langle f(x_n)p, z_n^1 p\rangle - d^2(z_n^1, p) \\
&= 2\langle f(x_n)f(p), z_n^1 p\rangle + 2\langle f(p)p, z_n^1 p\rangle - d^2(z_n^1, p) \\
&\leq 2d(f(x_n), f(p)) d(z_n^1, p) + d^2(f(p), p) - d^2(z_n^1, f(p)) \\
&\leq 2\kappa d^2(x_n, p) + d^2(f(p), p) - d^2(z_n^1, f(p)).
\end{aligned}$$

Therefore, by Lemma 2.10, we conclude

$$\begin{aligned}
&d^2(x_{n+1}, p) = d^2(\alpha_n f(x_n) \oplus (1-\alpha_n) z_n^1, p) \\
&\quad \leq \alpha_n d^2(f(x_n), p) + (1-\alpha_n) d^2(z_n^1, p) - \alpha_n(1-\alpha_n) d^2(f(x_n), z_n^1) \\
&\quad = (1-\alpha_n) d^2(z_n^1, p) + \alpha_n(d^2(f(x_n), p) - d^2(f(x_n), z_n^1)) + \alpha_n^2 d^2(f(x_n), z_n^1) \\
&\quad \leq (1-\alpha_n) d^2(x_n, p) + 2\kappa\alpha_n d^2(x_n, p) + \alpha_n d^2(f(p), p) - \alpha_n d^2(f(p), z_n^1) \\
&\qquad + \alpha_n^2 d^2(f(x_n), z_n^1) \\
&\quad \leq (1 - \alpha_n(1-2\kappa)) d^2(x_n, p) + \alpha_n(d^2(f(p), p) - d^2(z_n^1, f(p))) + \alpha_n^2 d^2(f(x_n), z_n^1),
\end{aligned}$$

which is,

$$d^2(x_{n+1}, p) \leq (1-\alpha_n(1-2\kappa)) d^2(x_n, p) + \alpha_n(d^2(f(p), p) - d^2(z_n^1, f(p)) + \alpha_n d^2(f(x_n), z_n^1)).$$

By this and Lemma 4.2, for getting to $d(x_{n+1}, p) \longrightarrow 0$, it suffices to show that

$$\limsup_{k\to\infty}(d^2(f(p), p) - d^2(z_{n_k}^1, f(p)) + \alpha_{n_k} d^2(f(x_{n_k}), z_{n_k}^1)) \leq 0,$$

for every subsequence $d^2(x_{n_k}, p)$ of $d^2(x_n, p)$ that satisfies,

$$\liminf_{k\to\infty}(d^2(x_{n_k+1}, p) - d^2(x_{n_k}, p)) \geq 0.$$

By boundedness of $(z^1_{n_k})$, there exists subsequence $(z^1_{n_{k_t}})$ of $(z^1_{n_k})$ such that $(z^1_{n_{k_t}})$ is Δ-convergent to $q \in X$ and

$$\begin{aligned}&\limsup_{k\to\infty}(d^2(f(p), p) - d^2(z^1_{n_k}, f(p)) + \alpha_{n_k} d^2(f(x_{n_k}), z^1_{n_k}))\\ &\qquad = \lim_{t\to\infty}(d^2(f(p), p) - d^2(z^1_{n_{k_t}}, f(p)) + \alpha_{n_{k_t}} d^2(f(x_{n_{k_t}}), z^1_{n_{k_t}})).\end{aligned}$$

Since $d^2(f(p), .)$ is convex and continuous and therefore Δ-lower semicontinuous, by the assumption $C1$, we obtain

$$\begin{aligned}&\limsup_{k\to\infty}(d^2(f(p), p) - d^2(z^1_{n_k}, f(p)) + \alpha_{n_k} d^2(f(x_{n_k}), z^1_{n_k}))\\ &\qquad = \lim_{t\to\infty}(d^2(f(p), p) - d^2(z^1_{n_{k_t}}, f(p)) + \alpha_{n_{k_t}} d^2(f(x_{n_{k_t}}), z^1_{n_{k_t}}))\\ &\qquad \leq d^2(f(p), p) - d^2(q, f(p)),\end{aligned}$$

Hence, by $p = P_{\cap_{i=1}^m A_i^{-1}(\mathbf{0})} f(p)$ and Remark 2.7, for proving $d(x_{n+1}, p) \longrightarrow 0$, it is enough to show $q \in \cap_{i=1}^m A_i^{-1}(\mathbf{0})$ for every subsequence $d^2(x_{n_k}, p)$ of $d^2(x_n, p)$ that satisfies,

$$\liminf_{k\to\infty}(d^2(x_{n_k+1}, p) - d^2(x_{n_k}, p)) \geq 0.$$

For this, suppose that $d^2(x_{n_k}, p)$ is a subsequence of $d^2(x_n, p)$ that satisfies,

$$\liminf_{k\to\infty}(d^2(x_{n_k+1}, p) - d^2(x_{n_k}, p)) \geq 0.$$

Then

$$
\begin{aligned}
0 &\leq \liminf_{k\to\infty}(d^2(x_{n_k+1},p) - d^2(x_{n_k},p)) \\
&\leq \liminf_{k\to\infty}(\alpha_{n_k} d^2(f(x_{n_k}),p) + (1-\alpha_{n_k})d^2(z^1_{n_k},p) \\
&\qquad - \alpha_{n_k}(1-\alpha_{n_k})d^2(f(x_{n_k}),z^1_{n_k}) - d^2(x_{n_k},p)) \\
&\leq \liminf_{k\to\infty}(\alpha_{n_k} d^2(f(x_{n_k}),p) + (1-\alpha_{n_k})d^2(z^1_{n_k},p) - d^2(x_{n_k},p)) \\
&= \liminf_{k\to\infty}(\alpha_{n_k}(d^2(f(x_{n_k}),p) - d^2(z^1_{n_k},p)) + d^2(z^1_{n_k},p) - d^2(x_{n_k},p)) \\
&\leq \limsup_{k\to\infty}(\alpha_{n_k}(d^2(f(x_{n_k}),p) - d^2(z^1_{n_k},p)) + \liminf_{k\to\infty}(d^2(z^1_{n_k},p) - d^2(x_{n_k},p)) \\
&= \liminf_{k\to\infty}(d^2(z^1_{n_k},p) - d^2(x_{n_k},p)) \\
&\leq \limsup_{k\to\infty}(d^2(z^1_{n_k},p) - d^2(x_{n_k},p)) \\
&\leq \limsup_{k\to\infty}(d^2(x_{n_k},p) - d^2(x_{n_k},p)) = 0,
\end{aligned}
$$

follows

$$\lim_{k\to\infty}(d^2(z^1_{n_k},p) - d^2(x_{n_k},p)) = 0, \tag{3.4}$$

which by (3.3) implies

$$\lim_{k\to\infty} d(z^i_{n_k}, z^{i+1}_{n_k}) = 0, \quad \text{for} \quad i = 1,2,...,m. \tag{3.5}$$

This, together with the triangle inequality of the metric d, follows

$$\lim_{k\to\infty} d(x_{n_k}, z^i_{n_k}) = 0, \quad \text{for } i = 1,2,...,m. \tag{3.6}$$

On the other hand, by $C3$ and part (iii) of Theorem 2.8, we obtain

$$d(J^{A_i}_\lambda z^{i+1}_{n_k}, z^{i+1}_{n_k}) \leq d(J^{A_i}_{\lambda_{(n_k,i)}} z^{i+1}_{n_k}, z^{i+1}_{n_k}) = d(z^i_{n_k}, z^{i+1}_{n_k}),$$

which by (3.5) implies

$$d(J^{A_i}_\lambda z^{i+1}_{n_k}, z^{i+1}_{n_k}) \longrightarrow 0, \quad \text{for } i = 1,2,...,m. \tag{3.7}$$

As well as, for all $i \in \{1,2,...,m\}$, we have

$$
\begin{aligned}
d(J^{A_i}_\lambda x_{n_k}, x_{n_k}) &\leq d(J^{A_i}_\lambda x_{n_k}, J^{A_i}_\lambda z^{i+1}_{n_k}) + d(J^{A_i}_\lambda z^{i+1}_{n_k}, z^{i+1}_{n_k}) + d(z^{i+1}_{n_k}, x_{n_k}) \\
&\leq 2d(z^{i+1}_{n_k}, x_{n_k}) + d(J^{A_i}_\lambda z^{i+1}_{n_k}, z^{i+1}_{n_k})
\end{aligned}
$$

that by (3.6) and (3.7) follows

$$d(x_{n_k}, J^{A_i}_\lambda x_{n_k}) \longrightarrow 0, \quad \text{for } i = 1,2,...,m. \tag{3.8}$$

Moreover, $(x_{n_{k_t}})$ is a subsequence of (x_{n_k}) which Δ-converge to $q \in X$, beacuse of (3.6) and Δ-convergence of $(z^1_{n_{k_t}})$ to $q \in X$. Therefore by demicloseness of $J^{A_i}_\lambda$ and (3.8), we conclude $q \in \cap_{i=1}^m A_i^{-1}(\mathbf{0})$, as desired . Hence the sequence (x_n) generated by (3.1) is convergent strongly to $p = P_{\cap_{i=1}^m A_i^{-1}(\mathbf{0})} f(p)$, which by Remark 2.7 and Lemma 2.11 is the unique solution to the following variational inequality

$$\langle \overrightarrow{pf(p)}, \overrightarrow{xp} \rangle \geq 0, \quad \forall\, x \in \cap_{i=1}^m A_i^{-1}(\mathbf{0}).$$

□

4 Approximation to common minimizer of convex functions

One of the most widely used examples of monotone operators that satisfies the range condition, is subdifferential of a convex, proper and lower semicontinuous function. In the following, we approximate a common minimizer of a finite family of proper, convex and lower semicontinuous functions in Hadamard spaces. Let (X, d) be a Hadamard space. In [2], the subdifferential of a proper function on a Hadamard space X was defined, as follows.

Definition 4.1. [2] Let X be a Hadamard space with dual X^* and $f : X \to (-\infty, +\infty]$ be a proper function with efficient domain $D(f) := \{x : f(x) < +\infty\}$, then the subdifferential of f is the multi-valued function $\partial f : X \to 2^{X^*}$ defined by

$$\partial f(x) = \{x^* \in X^* : f(z) - f(x) \geq \langle x^*, \overrightarrow{xz} \rangle \quad (z \in X)\},$$

when $x \in D(f)$ and $\partial f(x) = \emptyset$, otherwise.

Part (iii) of the following theorem shows that subdifferential of a convex, proper and lower semicontinuous function satisfies the range condition in Hadamard spaces.

Theorem 4.2. *[2, Theorem 4.2] [20, Proposition 5.2] Let $f : X \to (-\infty, +\infty]$ be a proper, lower semicontinuous and convex function on a Hadamard space X with dual X^*, then*
(i) f attains its minimum at $x \in X$ if and only if $\mathbf{0} \in \partial f(x)$.
(ii) $\partial f : X \to 2^{X^*}$ is a monotone operator.
(iii) for any $y \in X$ and $\alpha > 0$, there exists a unique point $x \in X$ such that $[\alpha\overrightarrow{xy}] \in \partial f(x)$. (i.e. $D(J^{\partial f}_\lambda) = X \quad \forall \lambda > 0$).

Khatibzadeh and the author in [20, Proposition 5.3.] proved if $f : X \to (-\infty, +\infty]$ is a proper, lower semicontinuous and convex function on a Hadamard space X with dual X^*, then

$$J_\lambda^{\partial f} x = Argmin_{z\in X}\{f(z) + \frac{1}{2\lambda} d^2(z, x)\}, \quad \forall \ \lambda > 0, \ x \in X. \tag{4.1}$$

Therefore, using Theorem 3.2, we can approximate a common minimizer of a finite family of proper, convex and lower semicontinuous functions which is a unique solution of a variational inequality in Hadamard spaces.

Theorem 4.3. *Let X be a Hadamard space with dual X^* and $g_1, g_2, ..., g_m : X \to (-\infty, +\infty]$ be be proper, convex and lower semi-continuous functions such that $\cap_{i=1}^m argmin\ g_i \neq \emptyset$. Suppose $(\lambda_{(n,i)})$ for $i = 1, 2, ..., m$, are some sequences of nonnegative real numbers and (α_n) is a sequence in $[0, 1]$ such that satisfy the conditions:*

$$\begin{cases} C1 : \lim_{n\to\infty} \alpha_n = 0, \\ C2 : \sum_{n=1}^{\infty} \alpha_n = \infty, \\ C3 : \lambda_{(n,i)} \geq \lambda > 0, \quad \text{for all} \ \ n \in \mathbb{N} \ \ \text{and} \ \ i = 1, 2, ..., m. \end{cases}$$

If $f : X \longrightarrow X$ is a contractive mapping with contractive coefficient $\kappa \in (0, \frac{1}{2}]$, then the sequence generated by:

$$\begin{cases} x_1 \in X, \\ z_n^i = argmin_{u\in X}\{g_i(u) + \frac{1}{\lambda_{(n,i)}} d^2(u, z_n^{i+1})\}, & \text{for } i \in \{1, 2, ..., m\}, \\ z_n^{m+1} = x_n, & \text{for all } n \in \mathbb{N}, \\ x_{n+1} = \alpha_n f(x_n) \oplus (1 - \alpha_n) z_n^1, & \text{for all } n \in \mathbb{N}, \end{cases}$$

is convergent strongly to the element p of $\cap_{i=1}^m argmin\ g_i$ such that $p = P_{\cap_{i=1}^m argmin\ g_i} f(p)$ and p also is the unique solution to the following variational inequality:

$$\langle \overrightarrow{pf(p)}, \overrightarrow{xp} \rangle \geq 0, \quad \forall \ x \in \cap_{i=1}^m argmin\ g_i.$$

Proof. Define $A_i := \partial g_i$, for $i = 1, 2, ..., m$, then each operator $A_i := \partial g_i$ is a monotone operator that satisfies the range condition. Therefore, by (4.1), we can use Theorem 3.2 to get the desired result. □

5 Approximation to common fixed point of nonexpansive mappings

Let (X, d) be a Hadamard space and $T : X \to X$ be a nonexpansive mapping (i.e. $d(Tx, Ty) \leq d(x, y) \quad \forall x, y \in X$). Set $Az = [\overrightarrow{Tzz}]$, then $F(T) =$

$A^{-1}(0)$ and [21, Proposition 4.2] shows the operator A is a monotone operator. We refer the reader to [20, Section 6], to consider the range codition for the operator $Az = [\overrightarrow{Tzz}]$. In the following theorem, as a corollary of the Theorem 3.2, a common fixed point of a finite family of nonexpansive mappings, which is a unique solution of a variational inequality, is approximated.

Theorem 5.1. *Let X be a Hadamard space with dual X^* and $T_1, T_2, ..., T_m : X \to 2^{X^*}$ be nonexpansive mappings such that $A_i z = [\overrightarrow{T_i zz}]$ for $i = 1, 2, 3, ..., m$, satisfy the range condition and $\cap_{i=1}^m F(T_i) \neq \emptyset$. Suppose $(\lambda_{(n,i)})$ for $i = 1, 2, ..., m$, are some sequences of nonnegative real numbers and (α_n) is a sequence in $[0, 1]$ such that satisfy the conditions:*

$$\begin{cases} C1 : \lim_{n\to\infty} \alpha_n = 0, \\ C2 : \sum_{n=1}^{\infty} \alpha_n = \infty, \\ C3 : \lambda_{(n,i)} \geq \lambda > 0, \quad \text{for all } n \in \mathbb{N} \text{ and } i = 1, 2, ..., m. \end{cases}$$

If $f : X \longrightarrow X$ is a contractive mapping with contractive coefficient $\kappa \in (0, \frac{1}{2}]$, then the sequence generated by:

$$\begin{cases} x_1 \in X, \\ z_n^i = J_{\lambda_{(n,i)}}^{A_i} z_n^{i+1}, & \text{for } i \in \{1, 2, ..., m\}, \\ z_n^{m+1} = x_n, & \text{for all } n \in \mathbb{N}, \\ x_{n+1} = \alpha_n f(x_n) \oplus (1 - \alpha_n) z_n^1, & \text{for all } n \in \mathbb{N}, \end{cases}$$

is convergent strongly to the element p of $\cap_{i=1}^m F(T_i)$ such that $p = P_{\cap_{i=1}^m F(T_i)} f(p)$ and p also is the unique solution to the following variational inequality:

$$\langle \overrightarrow{pf(p)}, \overrightarrow{xp} \rangle \geq 0, \quad \forall\ x \in \cap_{i=1}^m F(T_i).$$

Proof. Proof is deduced by Theorem 3.2. $\square$

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