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A Generalization of Archimedes' Theorem on the Area of a Parabolic Segment

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Abstract

Archimedes' well known theorem on the area of a parabolic segment says that this area is 4/3 of the area of a certain inscribed triangle. In this paper we generalize this theorem to the n-dimensional euclidean space, $n \geq 3$. It appears that the ratio of the volume of an n-dimensional solid bounded by an $(n-1)$-dimensional hyper-paraboloid and an $(n-1)$-dimensional hyperplane and the volume of a certain inscribed cone (we analogously repeat Archimedes' procedure) depends only on the dimension of the euclidean space and it equals to $2n/(n+1)$.

Introduction

Let us recall Archimedes' (287-212 BC) theorem on the area of a parabolic segment. Let P be a parabola in the euclidean plane $\mathbb{R}^2$ and let AB be a chord of P. Denote by C' the midpoint of AB. Consider the line ℓ which passes through C' and is parallel to the symmetry line of P. Denote by C the intersection of ℓ with the parabola P, see Figure 1.

The point C is often called the center of the arc AB of the parabola P. Denote by S the parabolic segment bounded by the parabola P and the chord AB. Archimedes' famous result says that

$$\text{Area(S)} = \frac{4}{3} \cdot \text{Area}(\triangle\ ABC)\ . \tag{0.1}$$

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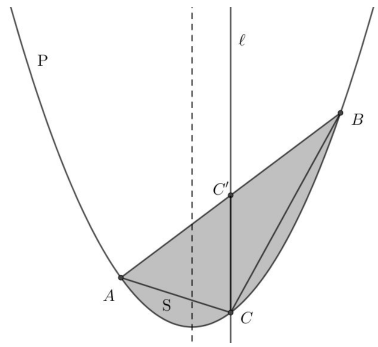

Figure 1: Classic version of Archimedes' theorem.

It is amazing that this relationship between the area of the parabolic segment S and the area of the triangle $\triangle\, ABC$ does not depend on the shape of the parabola P or the chord AB. Of course Archimedes' proof does not involve the integral calculus. In fact he gave two proofs of the above theorem. One of them involves Eudoxus' (408-355 BC) method of exhaustion and another one is based on moving center of masses of certain arcs. For learning Archimedes' original considerations we refer to [3, p. 251], see also [2].

In this paper we generalize Archimedes' theorem to the n-dimensional euclidean space $\mathbb{R}^n$. Let us start with some notations. We denote by $\mathbb{N}$ the set of all positive integers. Let $n \in \mathbb{N}$, $n \geq 3$. For $a_1, \ldots, a_{n-1} \in (0; +\infty)$ we set

$$\mathrm{P}_{n-1} := \left\{ (x_1 \ldots, x_n) \in \mathbb{R}^n \colon x_n = \sum_{k=1}^{n-1} a_k x_k^2 \right\} . \tag{0.2}$$

Notice that P_{n-1} is an $(n-1)$-dimensional elliptic hyper-paraboloid. For $b_1, \ldots, b_n \in \mathbb{R}$ we consider an $(n-1)$-dimensional hyperplane H_{n-1} given by

$$\mathrm{H}_{n-1} := \left\{ (x_1 \ldots, x_n) \in \mathbb{R}^n \colon x_n = \sum_{k=1}^{n-1} b_k x_k + b_n \right\} . \tag{0.3}$$

Obviously, H_{n-1} is not perpendicular to the hyperplane $\{(x_1, \ldots, x_{n-1}, 0) \colon x_i \in \mathbb{R},\ i = 1, \ldots, n-1\}$. Notice that $(x_1, \ldots, x_n) \in \mathrm{P}_{n-1} \cap \mathrm{H}_{n-1}$ if and only if

$$x_n = \sum_{k=1}^{n-1} a_k x_k^2 = \sum_{k=1}^{n-1} b_k x_k + b_n,$$

which is equivalent to

$$(x_1,\dots,x_n)\in \mathrm{H}_{n-1} \quad \text{and} \quad \sum_{k=1}^{n-1}\left(\sqrt{a_k}x_k-\frac{b_k}{2\sqrt{a_k}}\right)^2=b_n+\sum_{k=1}^{n-1}\frac{b_k^2}{4a_k}.$$

This means that the intersection $\mathrm{P}_{n-1}\cap \mathrm{H}_{n-1}$ contains more than one point if and only if

$$b_n+\sum_{k=1}^{n-1}\frac{b_k^2}{4a_k}>0. \tag{0.4}$$

Moreover, if the condition (0.4) is satisfied, then the number

$$R:=\sqrt{b_n+\sum_{k=1}^{n-1}\frac{b_k^2}{4a_k}} \tag{0.5}$$

is well defined and the set

$$\mathrm{E}_{n-2}:=\mathrm{P}_{n-1}\cap \mathrm{H}_{n-1}=\left\{(x_1,\dots,x_n)\in \mathrm{H}_{n-1}\colon \sum_{k=1}^{n-1}\frac{\left(x_k-b_k/(2a_k)\right)^2}{(R/\sqrt{a_k})^2}=1\right\} \tag{0.6}$$

is an $(n-2)$-dimensional ellipsoid. In what follows we assume that the condition (0.4) is satisfied. We denote by c the center of the ellipsoid E_{n-2}, i.e.

$$c=\left(\frac{b_1}{2a_1},\frac{b_2}{2a_2},\dots,\frac{b_{n-1}}{2a_{n-1}},\sum_{k=1}^{n-1}\frac{b_k^2}{2a_k}+b_n\right),$$

and we denote by $p(c)$ the intersection of the line

$$\ell_1(c):=\left\{\left(\frac{b_1}{2a_1},\frac{b_2}{2a_2},\dots,\frac{b_{n-1}}{2a_{n-1}},t\right):t\in\mathbb{R}\right\}$$

with the hyper-paraboloid P_{n-1}. We get

$$p(c)=\mathrm{P}_{n-1}\cap\ell_1(c)=\left(\frac{b_1}{2a_1},\frac{b_2}{2a_2},\dots,\frac{b_{n-1}}{2a_{n-1}},\sum_{k=1}^{n-1}\frac{b_k^2}{4a_k}\right).$$

Finally we denote by C_n the convex hull of the union of the point $p(c)$ and the ellipsoid E_{n-2}, i.e.

$$\mathrm{C}_n=\operatorname{conv}\left(\mathrm{E}_{n-2}\cup p(c)\right):=\{\lambda x+(1-\lambda)y\colon \lambda\in[0;1],\ x,y\in \mathrm{E}_{n-2}\cup p(c)\}. \tag{0.7}$$

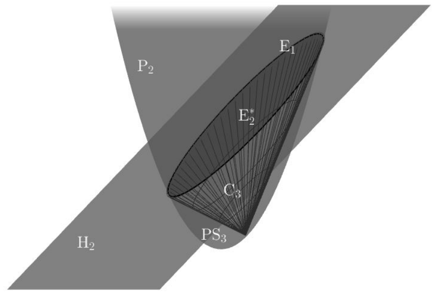

Figure 2: Three dimensional version of Archimedes' theorem.

Notice that C_n is an n-dimensional cone with the vertex at $p(c)$, see Figure 2 for the 3-dimensional case.

In what follows, we refer to C_n as a cone, for short. Comparing to the 2-dimensional case, the role of the parabolic segment plays the set

$$\mathrm{PS}_n := \left\{ (x_1 \ldots, x_n) \in \mathbb{R}^n : \sum_{k=1}^{n-1} a_k x_k^2 \leq x_n \leq \sum_{k=1}^{n-1} b_k x_k + b_n \right\} \tag{0.8}$$

which is an n-dimensional solid body bounded by the hyper-paraboloid P_{n-1} and the hyperplane H_{n-1}. The role of the triangle $\triangle\, ABC$ plays the cone C_n. It appears that the ratio of the volume of the hyper-parabolic segment PS_n and the volume of the cone C_n depends only on the dimension of the euclidean space, i.e. depends only on n. More precisely we prove the following result:

$$\mathrm{Vol}_n(\mathrm{PS}_n) = \frac{2n}{n+1} \cdot \mathrm{Vol}_n(\mathrm{C}_n).$$

In particular,

$$\mathrm{Vol}_3(\mathrm{PS}_3) = \frac{3}{2} \cdot \mathrm{Vol}_3(\mathrm{C}_3).$$

Throughout this paper we write $\mathrm{Vol}_n(E)$ for the n-dimensional Lebesgue measure of a Lebesgue measurable set E, although it would be sufficient to operate on Jordan measurable sets and the Jordan measure.

1 Main results

We will compute multiple integrals by changing variables into spherical n-dimensional coordinates, see [1]. Additionally we will scale along the axes and translate appropriate transformation, in order to integrate comfortably over convex hulls of ellipsoids. The Jacobian (see [4, 234]) of such a transformation is basically well-known, so a technical Lemma 1.2 is not new.

Let u and v be non-negative integers. In what follows we formally replace the product $\prod_{i=u}^{v}$ by 1 if $v < u$, regardless what is the expression under the product symbol. We denote by $\mathbb{N}_{u,v}$ the set of all integers k such that $u \le k \le v$, if $u \le v$. We set $\mathbb{N}_{u,v} := \emptyset$, if $u > v$.

Let $m \in \mathbb{N}$, $m \ge 2$, $\theta_1, \dots, \theta_{m-1} \in \mathbb{R}$ and $\theta_m := 0$ (so $\cos(\theta_m) = 1$). Define the matrix $\mathbf{M}_m \colon \mathbb{N}_{1,m} \times \mathbb{N}_{1,m} \to \mathbb{R}$ by

$$\mathbf{M}_m(p,q) := \begin{cases} \cos(\theta_p) \displaystyle\prod_{j=1}^{p-1} \sin(\theta_j)\ , & \text{if } p \in \mathbb{N}_{1,m} \text{ and } q = 1, \\ \cos(\theta_p)\cos(\theta_{q-1}) \displaystyle\prod_{\substack{j=1 \\ j \ne q-1}}^{p-1} \sin(\theta_j), & \text{if } p \in \mathbb{N}_{2,m} \text{ and } q \in \mathbb{N}_{2,p}, \\ -\displaystyle\prod_{j=1}^{p} \sin(\theta_j)\ , & \text{if } p \in \mathbb{N}_{1,m-1} \text{ and } q = p+1, \\ 0\ , & \text{if } p \in \mathbb{N}_{1,m-2} \text{ and } q \in \mathbb{N}_{p+2,m}. \end{cases}$$

For the convenience of the Reader we give an array form of the matrix $\mathbf{M}_m$:

$$\begin{pmatrix} \cos(\theta_1) & -\sin(\theta_1) & 0 & 0 & \cdots & 0 \\ \cos(\theta_2)\sin(\theta_1) & \cos(\theta_2)\cos(\theta_1) & -\prod_{j=1}^{2}\sin(\theta_j) & 0 & \cdots & 0 \\ \vdots & \vdots & & & \ddots & \vdots \\ \cos(\theta_{m-2})\prod_{j=1}^{m-3}\sin(\theta_j) & \cos(\theta_{m-2})\cos(\theta_1)\prod_{\substack{j=1\\ j\ne 1}}^{m-3}\sin(\theta_j) & \ddots & & \ddots & 0 \\ \cos(\theta_{m-1})\prod_{j=1}^{m-2}\sin(\theta_j) & \cos(\theta_{m-1})\cos(\theta_1)\prod_{\substack{j=1\\ j\ne 1}}^{m-2}\sin(\theta_j) & & \ddots & & -\prod_{j=1}^{m-1}\sin(\theta_j) \\ \prod_{j=1}^{m-1}\sin(\theta_j) & \cos(\theta_1)\prod_{\substack{j=1\\ j\ne 1}}^{m-1}\sin(\theta_j) & \cdots & \cdots & & \cos(\theta_{m-1})\prod_{\substack{j=1\\ j\ne m-1}}^{m-1}\sin(\theta_j) \end{pmatrix}$$

Lemma 1.1. *The determinant of* $\mathbf{M}_m$ *satisfies the following equality*

$$\det(\mathbf{M}_m) = \prod_{k=1}^{m-2} \sin^{m-k-1}(\theta_k)\ .$$

Proof. Obviously, $\det(\mathbf{M}_2) = 1$. Assume $m \in \mathbb{N}$, $m \geq 3$. Using Laplace's expansion along the mth column of the matrix $\mathbf{M}_n$, we get

$$\begin{aligned}\det(\mathbf{M}_m) &= (-1)^{2m-1}\left(-\prod_{j=1}^{m-1}\sin(\theta_j)\right)\sin(\theta_{m-1})\det(\mathbf{M}_{m-1})\\ &\quad+(-1)^{2m}\left(\cos(\theta_{m-1})\prod_{\substack{j=1\\ j\neq m-1}}^{m-1}\sin(\theta_j)\right)\cos(\theta_{m-1})\det(\mathbf{M}_{m-1})\\ &= \prod_{j=1}^{m-2}\sin(\theta_j)\det(\mathbf{M}_{m-1})\ .\end{aligned}$$

By the mathematical induction, we get the desired formula. □

Let $\alpha := (\alpha_1, \ldots, \alpha_m) \in \mathbb{R}^m$, $\beta := (\beta_1, \ldots, \beta_m) \in \mathbb{R}^m$. For each $k \in \mathbb{N}_{1,m}$ we set

$$\varphi_k^{(\alpha_k,\beta_k)}(r, \theta_1, \theta_2, \ldots, \theta_{m-1}) := \alpha_k r\cos(\theta_k)\prod_{j=1}^{k-1}\sin(\theta_j) + \beta_k\ ,$$

where $(r, \theta_1, \theta_2, \ldots, \theta_{m-1}) \in \mathbb{R}^m$. Recall that $\theta_m = 0$, so $\cos(\theta_m) = 1$. We define the mapping $\Phi_m^{(\alpha,\beta)} \colon \mathbb{R}^m \to \mathbb{R}^m$ by

$$\Phi_m^{(\alpha,\beta)} := \left(\varphi_1^{(\alpha_1,\beta_1)}, \ldots, \varphi_m^{(\alpha_m,\beta_m)}\right)\ . \tag{1.1}$$

So $\varphi_k^{(\alpha_k,\beta_k)}$ is the kth coordinate function of $\Phi_m^{(\alpha,\beta)}$. The vector α controls a scaling along the axes while the vector β controls a translation. Of course, the mapping $\Phi_m^{(\alpha,\beta)}$ has continuous partial derivatives of all orders.

Lemma 1.2. *The Jacobian* $\mathrm{J}_{\Phi_m^{(\alpha,\beta)}}$ *of the mapping* $\Phi_m^{(\alpha,\beta)} \colon \mathbb{R}^m \to \mathbb{R}^m$ *satisfies the equality*

$$\mathrm{J}_{\Phi_m^{(\alpha,\beta)}}(r, \theta_1, \theta_2, \ldots, \theta_{m-1}) = r^{m-1}\left(\prod_{k=1}^{m}\alpha_k\right)\prod_{k=1}^{m-2}\sin^{m-k-1}(\theta_k),$$

for all $(r, \theta_1, \theta_2, \ldots, \theta_{m-1}) \in \mathbb{R}^m$.

Proof. For $k \in \mathbb{N}_{1,m}$ we have

$$\frac{\partial \varphi_k^{(\alpha,\beta)}}{\partial r}(r, \theta_1, \theta_2, \ldots, \theta_{m-1}) = \alpha_k \cos(\theta_k) \prod_{j=1}^{k-1} \sin(\theta_j) \ ,$$

$$\frac{\partial \varphi_k^{(\alpha,\beta)}}{\partial \theta_i}(r, \theta_1, \theta_2, \ldots, \theta_{m-1}) = \begin{cases} r\alpha_k \cos(\theta_k)\cos(\theta_i) \prod\limits_{\substack{j=1 \\ j \neq i}}^{k-1} \sin(\theta_j), & \text{if } i \in \mathbb{N}_{1,k-1}, \\ -r\alpha_k \prod\limits_{j=1}^{k} \sin(\theta_j), & \text{if } i = k, \\ 0, & \text{if } i \in \mathbb{N}_{k+1,m-1}. \end{cases}$$

Thus, by Lemma 1.1, we get

$$\begin{aligned} \mathrm{J}_{\Phi_m^{(\alpha,\beta)}}(r, \theta_1, \theta_2, \ldots, \theta_{m-1}) &= r^{m-1} \left(\prod_{k=1}^{m} \alpha_k \right) \det(\mathbf{M}_m) \\ &= r^{m-1} \left(\prod_{k=1}^{m} \alpha_k \right) \prod_{k=1}^{m-2} \sin^{m-k-1}(\theta_k). \end{aligned}$$

□

Remark 1.3. Set $\mathbf{1} := (1, \ldots, 1) \in \mathbb{R}^m$ and $\mathbf{0} := (0, \ldots, 0) \in \mathbb{R}^m$. Then the mapping $\Phi_m^{(\mathbf{1},\mathbf{0})} \colon \mathbb{R}^m \to \mathbb{R}^m$ restricted to the set $[0, +\infty) \times \mathbb{R}^{m-1}$ is a transformation of the classical spherical m-dimensional coordinates onto the Cartesian coordinates. In this case

$$\mathrm{J}_{\Phi_m^{(\mathbf{1},\mathbf{0})}}(r, \theta_1, \theta_2, \ldots, \theta_{m-1}) = r^{m-1} \prod_{k=1}^{m-2} \sin^{m-k-1}(\theta_k),$$

for all $(r, \theta_1, \theta_2, \ldots, \theta_{m-1}) \in \mathbb{R}^m$ (compare this with the formula given in [1]).

Let $r > 0$. Notice, that the mapping $\Phi_m^{(\mathbf{1},\mathbf{0})}$ transforms in a non-injective manner the closed m-orthotope $[0; r] \times [0; \pi]^{m-2} \times [0; 2\pi]$ onto the closed ball $\mathbb{B}_m(r) := \{x \in \mathbb{R}^m \colon \|x\| \leq r\}$ of the radius r (here $\| \, \|$ means the euclidean norm). However, the restriction of the mapping $\Phi_m^{(\mathbf{1},\mathbf{0})}$ to the open m-orthotope

$$\Omega_m(r) := (0, r) \times (0; \pi)^{m-2} \times (0; 2\pi) \tag{1.2}$$

is a diffeomorphism. Moreover

$$\Phi_m^{(\mathbf{1},\mathbf{0})}(\Omega_m(r)) \subset \mathbb{B}_m(r) \quad \text{and} \quad \mathrm{Vol}_m \left(\Phi_m^{(\mathbf{1},\mathbf{0})}(\Omega_m(r)) \right) = \mathrm{Vol}_m \left(\mathbb{B}_m(r) \right).$$

Theorem 1.4. *Let* $a_1, \ldots, a_{n-1} \in (0; +\infty)$, $b_1, \ldots, b_n \in \mathbb{R}$, *and let the condition* (0.4) *be satisfied. Then*

$$\mathrm{Vol}_n(\mathrm{PS}_n) = \frac{2n}{n+1} \cdot \mathrm{Vol}_n(\mathrm{C}_n) \ ,$$

where C_n *and* PS_n *are defined in* (0.7) *and* (0.8)*, respectively.*

Proof. Let us define $\mathrm{E}^*_{n-1} := \mathrm{conv}(\mathrm{E}_{n-2})$. Denote by D_{n-1} the projection of E^*_{n-1} onto the hyperplane $\{(x_1, \ldots, x_{n-1}, 0) \colon x_i \in \mathbb{R},\ i = 1, \ldots, n-1\}$. By (0.6), we have

$$\mathrm{D}_{n-1} = \left\{ (x_1, \ldots, x_{n-1}, 0) \in \mathbb{R}^n \colon \sum_{k=1}^{n-1} \left[\sqrt{a_k} \cdot \left(x_k - \frac{b_k}{2a_k} \right) \right]^2 \leq R^2 \right\},$$

where R is defined as in (0.5). Put $m := n-1$,

$$\alpha := \left(\frac{1}{\sqrt{a_1}}, \ldots, \frac{1}{\sqrt{a_{n-1}}} \right) \quad \text{and} \quad \beta := \left(\frac{b_1}{2a_1}, \ldots, \frac{b_{n-1}}{2a_{n-1}} \right) \ ,$$

and consider the mapping $\Phi^{(\alpha,\beta)}_{n-1} \colon \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ defined by (1.1). The restriction of $\Phi^{(\alpha,\beta)}_{n-1}$ to the open $(n-1)$-orthotope $\Omega_{n-1}(R)$, defined by (1.2), is a diffeomorphism. Moreover,

$$\Phi^{(\alpha,\beta)}_{n-1}(\Omega_{n-1}(R)) \subset D_{n-1} \quad \text{and} \quad \mathrm{Vol}_{n-1}\left(\Phi^{(\alpha,\beta)}_{n-1}(\Omega_{n-1}(R)) \right) = \mathrm{Vol}_{n-1}(D_{n-1}).$$

Hence, by the transformation formula (see [4, 252]) and by Lemma 1.2, we get

$$\mathrm{Vol}_n(\mathrm{PS}_n) = \overbrace{\iint \cdots \int}^{n-1}_{\mathrm{D}_{n-1}} \left(R^2 - \sum_{k=1}^{n-1} a_k \left(x_k - \frac{b_k}{2a_k} \right)^2 \right) \mathrm{d}x_1 \mathrm{d}x_2 \ldots \mathrm{d}x_{n-1}$$

$$= \overbrace{\iint \cdots \int}^{n-1}_{\Omega_{n-1}(R)} \left(R^2 - r^2 \right) \left| \mathrm{J}_{\Phi^{(\alpha,\beta)}_{n-1}}(r, \theta_1, \theta_2, \ldots, \theta_{n-1}) \right| \mathrm{d}\theta_1 \mathrm{d}\theta_2 \ldots \mathrm{d}\theta_{n-2} \mathrm{d}r$$

$$= \int_0^R \int_0^{2\pi} \underbrace{\int_0^\pi \ldots \int_0^\pi}_{n-3} (R^2 - r^2) \left| r^{n-2} \left(\prod_{k=1}^{n-1} \frac{1}{\sqrt{a_k}} \right) \prod_{k=1}^{n-3} \sin^{n-k-2}(\theta_k) \right| \mathrm{d}\theta_1 \mathrm{d}\theta_2 \ldots \mathrm{d}\theta_{n-2} \mathrm{d}r$$

$$= 2^{n-1}\int_0^R \underbrace{\int_0^{\frac{\pi}{2}}\cdots\int_0^{\frac{\pi}{2}}}_{n-2}(R^2r^{n-2}-r^n)\left(\prod_{k=1}^{n-1}\frac{1}{\sqrt{a_k}}\right)\prod_{k=1}^{n-3}\sin^{n-k-2}(\theta_k)\mathrm{d}\theta_1\mathrm{d}\theta_2\ldots\mathrm{d}\theta_{n-2}\mathrm{d}r$$

$$= 2^{n-1}\cdot\left(\frac{R^{n+1}}{n-1}-\frac{R^{n+1}}{n+1}\right)\cdot\left(\prod_{k=1}^{n-1}\frac{1}{\sqrt{a_k}}\right)\cdot\frac{\pi}{2}\cdot\prod_{k=1}^{n-3}\int_0^{\frac{\pi}{2}}\sin^{n-k-2}(\theta_k)\mathrm{d}\theta_k$$

$$= \frac{2^{n-1}\pi R^{n+1}}{n^2-1}\cdot\left(\prod_{k=1}^{n-1}\frac{1}{\sqrt{a_k}}\right)\cdot\prod_{k=1}^{n-3}\frac{\sqrt{\pi}\Gamma\left(\frac{n-k-1}{2}\right)}{2\Gamma\left(\frac{n-k}{2}\right)}$$

$$= \frac{4\pi^{\frac{n-1}{2}}R^{n+1}}{(n^2-1)\Gamma\left(\frac{n-1}{2}\right)}\prod_{k=1}^{n-1}\frac{1}{\sqrt{a_k}}\ .$$

If $n=3$ then $\underbrace{\int_0^{\pi}\cdots\int_0^{\pi}}_{n-3}$ just disappear and the result remains valid.

Now we determine $\mathrm{Vol}_n(\mathrm{C}_n)$. Since C_n is an n-dimensional cone with base E^*_{n-1} and vertex at $p(c)$. Therefore

$$\mathrm{Vol}_n(\mathrm{C}_n) = \frac{1}{n}\,\mathrm{dist}(p(c);\mathrm{H}_{n-1})\cdot\mathrm{Vol}_{n-1}(\mathrm{E}^*_{n-1})\ , \tag{1.3}$$

where $\mathrm{dist}(p(c);\mathrm{H}_{n-1})$ denotes the euclidean distance between $p(c)$ and H_{n-1}. Since

$$(\mathrm{dist}(p(c);\mathrm{H}_{n-1}))^2 = \frac{\left(\sum\limits_{k=1}^{n-1}\frac{b_k^2}{2a_k}-\sum\limits_{k=1}^{n-1}\frac{b_k^2}{4a_k}+b_n\right)^2}{\sum\limits_{k=1}^{n-1}b_k^2+1},$$

so

$$\mathrm{dist}(p(c);\mathrm{H}_{n-1}) = R^2\left(\sum_{k=1}^{n-1}b_k^2+1\right)^{-\frac{1}{2}}\ . \tag{1.4}$$

Furthermore,

$$\mathrm{Vol}_{n-1}(\mathrm{E}^*_{n-1}) = \overbrace{\iint\cdots\int}^{n-1}_{\mathrm{D}_{n-1}}\sqrt{\sum_{k=1}^{n-1}b_k^2+1}\ \mathrm{d}x_1\mathrm{d}x_2\ldots\mathrm{d}x_{n-1}\ .$$

Proceeding as above, we get

$$\begin{aligned}
&\mathrm{Vol}_{n-1}(\mathrm{E}^*_{n-1}) = \\
&= 2^{n-1}\sqrt{\sum_{k=1}^{n-1} b_k^2 + 1}\int_0^R \underbrace{\int_0^{\frac{\pi}{2}}\cdots\int_0^{\frac{\pi}{2}}}_{n-2} \left|\mathrm{J}_{\Phi_{n-1}^{(\alpha,\beta)}}(r,\theta_1,\theta_2,\ldots,\theta_{n-1})\right| \mathrm{d}\theta_1\mathrm{d}\theta_2\ldots\mathrm{d}\theta_{n-2}\mathrm{d}r \\
&= 2^{n-1}\sqrt{\sum_{k=1}^{n-1} b_k^2 + 1}\cdot\frac{R^{n-1}}{n-1}\cdot\left(\prod_{k=1}^{n-1}\frac{1}{\sqrt{a_k}}\right)\cdot\frac{\pi}{2}\cdot\prod_{k=1}^{n-3}\int_0^{\frac{\pi}{2}}\sin^{n-k-2}(\theta_k)\mathrm{d}\theta_k \\
&= \frac{2\pi^{\frac{n-1}{2}}R^{n-1}}{(n-1)\Gamma\left(\frac{n-1}{2}\right)}\sqrt{\sum_{k=1}^{n-1} b_k^2 + 1}\;\prod_{k=1}^{n-1}\frac{1}{\sqrt{a_k}}\ .
\end{aligned}$$

Hence, by (1.3) and (1.4), we have

$$\mathrm{Vol}_n(\mathrm{C}_n) = \frac{2\pi^{\frac{n-1}{2}}R^{n+1}}{n(n-1)\Gamma\left(\frac{n-1}{2}\right)}\prod_{k=1}^{n-1}\frac{1}{\sqrt{a_k}}\ .$$

Finally, we get

$$\frac{\mathrm{Vol}_n(\mathrm{PS}_n)}{\mathrm{Vol}_n(\mathrm{C}_n)} = \frac{2n}{n+1}\ .$$

□

Corollary 1.5. *In the* 3*-dimensional case (see Figure 2) the ratio of the volume of the parabolic segment* PS_3 *and the volume of the cone* C_3 *is equal to* $3/2$.

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