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Sums and products of intervals in ordered semigroups

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Abstract

We show a simple example for ordered semigroup $\mathbb{S} = \mathbb{S}(+, \leqslant)$ that $\mathbb{S} \subseteq \mathbb{R}$ ($\mathbb{R}$ denotes the real line) and $]a, b[+]c, d[=]a + c, b + d[$ for all a, b, c, $d \in \mathbb{S}$ such that $a < b$ and $c < d$, but the intervals are no translation invariant, that is, the equation $c+]a, b[=]c + a, c + b[$ is not always fulfilled for all elements a, b, $c \in \mathbb{S}$ such that $a < b$.

The multiplicative version of the above example is shown too.

The product of open intervals in the ordered ring of all integers (denoted by $\mathbb{Z}$) is also investigated. Let $I_x := \{1, 2, \ldots, x\}$ for all $x \in \mathbb{Z}_+$ and defined the function $g : \mathbb{Z}_+ \to \mathbb{Z}_+$ by

$$g(x) := \max\{y \in \mathbb{Z}_+ \mid I_y \subseteq I_x \cdot I_x\}$$

for all $x \in \mathbb{Z}_+$. We give the function g implicitly using the famous Theorem of Chebishev.

Finally, we formulate some questions concerning the above topics.

1 Introduction

It is well known from elementary real analysis [12], [2] that if a, b, c, d are real numbers such that $a < b$ and $c < d$, then

$$]a, b[+]c, d[=]a + c, b + d[\,, \tag{1}$$

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moreover, if $0 \leqslant a < b$ and $0 \leqslant c < d$ then

$$]a, b[\cdot]c, d[=]ac, bd[\,. \tag{2}$$

We investigate the case when (1) or (2) remains valid in more general settings. Our references for ordered algebraic structure are [4], [9], [10], [11], [13].

Now we give a short list of necessary concepts:

We say that $X = X(\leqslant)$ is a partially ordered set or poset, if X is a set and $\leqslant$ is a relation on X such that it is reflexive, symmetric and transitive.

A poset $X = X(\leqslant)$ is said to be linearly ordered or loset, if $x \leqslant y$ or $y \leqslant x$ including the case $x = y$ for all $x, y \in X$.

We say, that the poset $X = X(\leqslant)$ is lattice ordered if every two elements of X have a unique supremum (also called a least upper bound or join) and a unique infimum (also called a greatest lower bound or meet).

Let $X = X(*)$ be a groupoid in the sense that X is a nonempty set and $*$ is a binary operation on X. Let A, B be sets and $a \in X$. Define the sets and $A * B$ and $a * B$ by

$$\begin{aligned} A * B &:= \{a * b \in X | a \in A, b \in B\}\,, \\ a * B &:= \{a\} * B. \end{aligned}$$

Let $X = X(\leqslant)$ be a poset and $a, b \in X$ such that $a < b$, that is, $a \leqslant b$, but $a \neq b$, then the *(open) interval with endpoints a and b* is defined by

$$]a, b[\, := \{x \in X | a < x \text{ and } x < b\}.$$

An ordered semigroup $\mathbb{S} = \mathbb{S}(+, \leqslant)$ is a semigroup together with a partial ordering of its elements that is compatible with the group operations in the sense that if $x \leqslant y$, then $x + z \leqslant y + z$ for all $x, y, z \in \mathbb{S}$.

An ordered group $\mathbb{G} = \mathbb{G}(+, \leqslant)$ is a group together with a partial ordering of its elements that is compatible with the group operation.

Let $X = X(+, \leqslant)$ be an ordered semigroup (group). We say that the open intervals have translation invariant property if

$$c +]a, b[=]c + a, c + b[\tag{3}$$

for all a, b, $c \in \mathbb{S}$ such that $a < b$.

Let $\mathbb{S} = \mathbb{S}(\cdot, \leqslant)$ be an ordered semigroup. We say that the open intervals have homothety invariant property if

$$c \cdot]a, b[=]ca, cb[\tag{4}$$

for all a, b, $c \in X$ such that $a < b$.

A poset X is said to be dense if $]x,y[\neq\emptyset$ for all $x,y\in X$ with $x<y$.

Let Y be a poset and $A\subseteq B\subseteq Y$. The set A is said to be dense in the set B if $]b_1,b_2[\cap A\neq\emptyset$ for all b_1, $b_2\in B$ such that $b_1<b_2$.

Our paper is structured as follows:

In section 2 we give example for ordered semigroup $\mathbb{S}(+,\leqslant)$ where $\leqslant$ is a lattice order and the intervals of this semigroup are translation invariant and thus the equation (1) is fulfilled.

In section 3 we investigate the product of open intervals of ordered ring of integers denoted by $\mathbb{Z}$ using the famous number-theoretical theorem of Chebishev.

In section 4 among the others we give examples for semigroups $\mathbb{S}(+,\leqslant)$ that equation (1) is fulfilled without equation (3).

In section 5 we formulate conjectures and problems concerning structures which are similar to the structures presented in the previous chapter.

2 Sums of intervals in ordered semigroup

Definition 2.1. Let $\mathbb{S}=\mathbb{S}(+,\leqslant)$ be an ordered Abelian semigroup ($\leqslant$ is a partial order). Consider the following properties:

1. $\mathbb{S}=\mathbb{S}(+)$ is cancellative in the sense that $x+z=y+z$ implies $x=y$ for all $x,y,z\in\mathbb{S}$.

2. If $x<y$ then there exists an element $z\in\mathbb{S}$ such that $y=x+z$ for all $x,y,z\in\mathbb{S}$.

3. $x\neq x+y$ for all $x,y\in\mathbb{S}$.

4. The strickly order $<$ is co-directed in the sense that for all $x,y\in\mathbb{S}$ there exists an element $z\in\mathbb{S}$ such that $z<x$ and $z<y$.

Now we show that the intervals of the ordered semigroup $\mathbb{S}$ with properties of Definitions 2.1 are translation invariant.

Theorem 2.2. *Let $\mathbb{S}=\mathbb{S}(+,\leqslant)$ be an ordered semigroup with properties of Definitions 2.1. Let $\alpha,\beta,\gamma\in\mathbb{S}$ such that $\alpha<\beta$. Then*

$$\gamma+]\alpha,\beta[=]\gamma+\alpha,\gamma+\beta[\ \text{ and } \gamma+]\alpha,\beta]=]\gamma+\alpha,\gamma+\beta]$$

where $]x,y]:=\{z\in\mathbb{S}|x<z$ and $z\leqslant y\}$ for all $x,y\in\mathbb{S}$.

Proof. The proof is trivial. □

Theorem 2.3. *Let $\mathbb{S} = \mathbb{S}(+, \leqslant)$ be an ordered semigroup with properties of Definition 2.1. Let a, b, c, $d \in \mathbb{S}$ such that $a < b$ and $c < d$. Then*

$$]a, b[+]c, d[=]a + c, b + d[.$$

Proof. It is easy to see that $]a, b[+]c, d[\subseteq]a + c, b + d[$. One can easily obtain the converse inclusion by the Theorem 2.2. □

Example 2.4. Let $X := \mathbb{Q}^2$. Then $X(+, \leqslant)$ is an ordered semigroup with the partial order $\leqslant$ and the operation $+$ defined by

$$(x_1, x_2) \leqslant (y_1, y_2) :\Longleftrightarrow x_1 \leqslant y_1 \text{ and } x_2 \leqslant y_2,$$
$$(x_1, x_2) + (y_1, y_2) := (x_1 + y_1, x_2 + y_2)$$

for all (x_1, x_2), $(y_1, y_2) \in X$. Then the relation $\leqslant$ is lattice order and X has properties of Definition 2.1.

3 The products of open intervals in ordered group of integers

It is well-known that the sum of nonempty open interval of the ring of all integers (denoted by $\mathbb{Z}$) is also an interval, but the endpoints of the sum are not equal to the sum of endpoints. This fact is used in our paper [7] when we investigate restricted Pexider additive functional equations in the cases when the additivity is satisfied on a rectangular or on the union of two rectangles of $\mathbb{Z}^2$.

It is also well-known that the product of nonempty open interval of the ordered ring $\mathbb{Z}$ is not always an open interval. For example let

$$I_x := \{1, 2, \ldots, x\} \qquad (x \in \mathbb{Z}_+).$$

Then I_3 is an open interval of $\mathbb{Z}$, but $I_3 \cdot I_3 = \{1, 2, 3, 4, *, 6, *, *, 9\}$ is not.

Define the function $g : \mathbb{Z}_+ \to \mathbb{Z}_+$ by

$$g(x) := \max\{y \in \mathbb{Z}_+ \mid I_y \subseteq I_x \cdot I_x\} \qquad (x \in \mathbb{Z}_+).$$

It is easy to see that for example $g(3) = 4$. Now we give a table which contains the value of x and $g(x)$ for some small integer x.

x	1	$\boxed{2}$	$\boxed{3}$	4	$\boxed{5}$	6	$\boxed{7}$	8	9	10	$\boxed{11}$	12	$\boxed{13}$
$g(x)$	1	$\boxed{2}$	$\boxed{4}$	4	$\boxed{6}$	6	$\boxed{10}$	10	10	10	$\boxed{12}$	12	$\boxed{16}$

The above table suggests the following Theorem. The proof of this Theorem is based on the the Bertrand's postulate which states that there exists a prime number in the interval $[n, 2n]$ for all $n \in \mathbb{Z}_+$. This postulate was proved for the first time by P. L. Chebishev in 1850 and simplified later by P. Erdős in 1932 [3] due to M. El Bachraoui [1].

Theorem 3.1. *The function g has the following properties:*

1. *the function g is increasing;*
2. *$g(x-1) < g(x)$ if and only if x is prime;*
3. *$g(p_n) = p_{n+1} - 1$ where p_1, p_2, $\ldots$ is the increasing sequence of all prime numbers.*

Proof. Let us assume that there exist u, $v \in \mathbb{Z}_+$ such that $p_n \leqslant v < uv < p_{n+1}$. Since $v < uv$, then $u < 2$. By the Theorem of Chebyshev $v \leqslant p_{n+1} \leqslant 2v < uv$ which is a contradiction. Thus we have that if $x \in \mathbb{Z}_+$ such that $p_n < x < p_{n+1}$, then there exist $u, v \in \mathbb{Z}_+$ such that $u, v \leqslant p_n$ and $x = uv$. $\square$

4 Additional examples

In this section we only investigate linearly ordered semigroups. For this we give some notations.

Define the set K_1 by

$$K_1 := \left\{ a + b\sqrt{2} \mid a \in \mathbb{Q}_+, b \in \mathbb{Q}_+ \right\}.$$

Then $K_1 = K_1(+, \leqslant)$ and $K_1 = K_1(\cdot, \leqslant)$ are ordered semigroups where $+$, $\cdot$ and $\leqslant$ are the usual addition, multiplication and order in the real line.

Define the set $\mathbb{Q}(\sqrt{2})$ by

$$\mathbb{Q}(\sqrt{2}) := \left\{ a + b\sqrt{2} \mid a \in \mathbb{Q}, b \in \mathbb{Q} \right\}.$$

Then $\mathbb{Q}(\sqrt{2}) = \mathbb{Q}(\sqrt{2})(+, \cdot, \leqslant)$ is an ordered field where the operation $+$, $\cdot$ and the order $\leqslant$ is the usual field operation and order in the real line $\mathbb{R}$.

Let p_1, p_2, $\ldots$, p_n be pairwise different prim numbers. Define the set X_n by:

$$X_n := \{x_1, x_2, \ldots x_{2^n}\}$$

where $x_1 = 1$, $x_2 = \sqrt{p_1}$, $\ldots$, $x_{n+1} = \sqrt{p_n}$, $x_{n+2} = \sqrt{p_1 p_2}$, $\ldots$, $x_{2^n} = \sqrt{p_1 \ldots p_n}$. Then we can construct two sets from the set X_n by:

$$K_n := \left\{ \sum_{i=1}^{2^n} a_i x_i \mid a_i \in \mathbb{Q}_+, x_i \in X_n \quad (i = 1, \ldots, 2^n) \right\},$$

$$\mathbb{Q}_n := \left\{ \sum_{i=1}^{2^n} a_i x_i \mid a_i \in \mathbb{Q}, x_i \in X_n \quad (i = 1, \ldots, 2^n) \right\}.$$

In both of the above cases $+$, $\cdot$ and $\leqslant$ are the usual addition, multiplication and order in the real line respectively. It is easy to see that $K_n = K_n(+, \cdot, \leqslant)$ is an ordered dense semiring and $\mathbb{Q}_n = \mathbb{Q}_n(+, \cdot, \leqslant)$ is an ordered field, consequently all of equations (3), (1), (4) and (2) are fulfilled in $\mathbb{Q}_n$ [8]. It is also easy to see that $\mathbb{Q}_n = \mathbb{Q}(\sqrt{p_1}, \dots, \sqrt{p_n})$ is an ordered field ($\mathbb{Q}(\sqrt{p_1}, \dots, \sqrt{p_n})$ denotes the smallest subfield of the real line that contains both elements $\sqrt{p_1}, \dots, \sqrt{p_n}$ and all elements of the field $\mathbb{Q}$) [5].

For numerical examples we use constants which are defined by:

$$\begin{aligned} &\alpha := 0.5 + 0.5\sqrt{2} = 1.2071\dots, &&\beta := 0.5 + \sqrt{2} = 1.9142\dots, \\ &\gamma := 2 + \sqrt{2} = 3.4142\dots, &&\delta := 1 + 2\sqrt{2} = 3.8284\dots. \end{aligned} \tag{5}$$

Thus we obtain that $\alpha + \gamma = 4.6213\dots$, $\beta + \delta = 5.7426\dots$, $\alpha \cdot \gamma = 4.1213\dots$, $\beta \cdot \delta = 7.3284\dots$.

Proposition 4.1. Let a, b, c, $d \in K_1$ such that $a < b$ and $c < d$. Then

$$]a, b[+]c, d[=]a + c, b + d[\,.$$

Proof. The inclusion $]a, b[+]c, d[\subseteq]a + c, b + d[$ is clear. For the converse inclusion let

$$x_0 := u + v\sqrt{2} \in]a + c, b + d[\,.$$

We find elements x^*, $y^* \in K_1$ such that

$$x^* = x + y\sqrt{2} \in]a, b[\,, \qquad y^* = x_0 - x^* \in]c, d[\,.$$

Now we list the necessary and sufficient conditions for $x^* \in]a, b[$ and $y^* \in]c, d[$:

1. $a < x^*$ which is equivalent to the inequality $e_1(x) < y$ where

$$e_1(x) = -\frac{1}{\sqrt{2}}x + A_1 \qquad \text{and} \qquad A_1 = \frac{a}{\sqrt{2}};$$

2. $x^* < b$ which is equivalent to the inequality $y < e_2(x)$ where

$$e_2(x) = -\frac{1}{\sqrt{2}}x + A_2 \qquad \text{and} \qquad A_2 = \frac{b}{\sqrt{2}};$$

3. $c < y^*$ which is equivalent to the inequality $y < e_3(x)$ where

$$e_3(x) = -\frac{1}{\sqrt{2}}x + A_3 \qquad \text{and} \qquad A_3 = \frac{x_0 - c}{\sqrt{2}};$$

4. $y^* < d$ which is equivalent to the inequality $e_4(x) < y$ where

$$e_4(x) = -\frac{1}{\sqrt{2}}x + A_4 \qquad \text{and} \qquad A_4 = \frac{x_0 - d}{\sqrt{2}}.$$

Now we list the necessary and sufficient conditions of $x^* \in K_1$ and $y^* \in K_1$:

(5) $u - x > 0$ which is equivalent to the inequality $x < u$;

(6) $v - y > 0$ which is equivalent to the inequality $y < v$.

Define the domains D_1 and D_2 by

$$D_1 := \left\{(x, y) \in \mathbb{Q}_+^2 \mid e_1(x) < y, y < e_2(x), y < e_3(x), e_4(x) < y\right\},$$
$$D_2 := \left\{(x, y) \in \mathbb{Q}_+^2 \mid x < u \text{ and } y < v\right\}.$$

D_1 is the set of all rational points of the first quadrant which points are contained by two strips. These strips are determined by the pairs of parallel lines e_1, e_2 and e_4, e_3 respectively.

The intersection of these two strips is not empty if and only if $]A_1, A_2[\cap]A_4, A_3[\neq \emptyset$ (as open intervals of the real line) which is equal to $A_1 < A_3$ and $A_4 < A_2$. These last two inequalities are fulfilled.

Since $e_1(u) < v$, $e_4(u) = v$ and $\mathbb{Q}^2$ is dense in $\mathbb{R}^2$ thus $D_1 \cap D_2 \neq \emptyset$. Chose a point $P = P(x, y) \in D_1 \cap D_2$ arbitrarily, and define the numbers x^*, y^* by

$$x^* := x + y\sqrt{2} \qquad \text{and} \qquad y^* := (u - x) + (v - y)\sqrt{2}.$$

Then $x^* \in]a, b[$ and $y^* \in]c, d[$ whence we obtain that $]a + c, b + d[\subseteq]a, b[+]c, d[$ indeed. $\square$

Example 4.2. Define the numbers α, β, γ, δ by (5). Let

$$x_0 := 4.8 + 0.1\sqrt{2} = 4.9414\cdots \in]\alpha + \gamma, \beta + \delta[\,.$$

We show that there exist constants $x^* \in]\alpha, \beta[$ and $y^* \in]\gamma, \delta[$ such that $x_0 = x^* + y^*$. Preserving the notations of Proporition 4.1. we obtain that

$$A_1 = 0.8535\ldots, \quad A_2 = 1.3535\ldots, \quad A_3 = 1.0798\ldots, \quad A_4 = 0.7870\ldots.$$

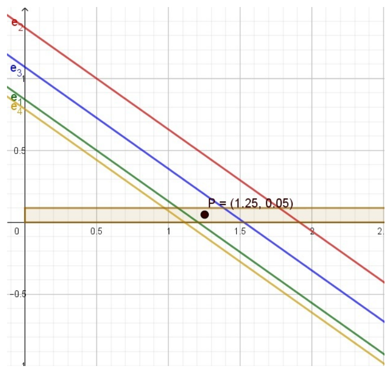

Figure 1: Sums of intervals

Consider the Figure 1 *:
It is easy to see that the choice of $P := (1.25, 0.05)$ is convenient. Thus we obtain that

$$x^* = 1.25 + 0.05\sqrt{2} \in]\alpha, \beta[\,,$$
$$y^* = 3.55 + 0.05\sqrt{2} \in]\gamma, \delta[$$

with $x^* + y^* = x_0$.

Proposition 4.3. Let a, b, c, $d \in K_1$ such that $a < b$ and $c < d$. Then

$$]a, b[\cdot]c, d[=]ac, bd[\,.$$

Proof. The inclusion $]a, b[\cdot]c, d[\subseteq]ac, bd[$ is clear. For the converse inclusion let

$$x_0 := u + v\sqrt{2} \in]ac, bd[\,.$$

We find elements x^*, $y^* \in K_1$ such that

$$x^* = x + y\sqrt{2} \in]a, b[\,, \qquad y^* = \frac{x_0}{x^*} \in]c, d[\,.$$

Now we list the necessary and sufficient conditions for $x^* \in]a, b[$ and $y^* \in]c, d[$.

1. $a < x^*$ which is equivalent to the inequality $e_1(x) < y$ where

$$e_1(x) = -\frac{1}{\sqrt{2}}x + A_1 \qquad \text{and} \qquad A_1 = \frac{a}{\sqrt{2}};$$

*The Figure 1 was made with the program GeoGebra

2. $x^* < b$ which is equivalent to the inequality $y < e_2(x)$ where

$$e_2(x) = -\frac{1}{\sqrt{2}}x + A_2 \qquad \text{and} \qquad A_2 = \frac{b}{\sqrt{2}};$$

3. $c < y^*$ which is equivalent to the inequality $y < e_3(x)$ where

$$e_3(x) = -\frac{1}{\sqrt{2}}x + A_3 \qquad \text{and} \qquad A_3 = \frac{x_0}{\sqrt{2}c};$$

4. $y^* < d$ which is equivalent to the inequality $e_4(x) < y$ where

$$e_4(x) = -\frac{1}{\sqrt{2}}x + A_4 \qquad \text{and} \qquad A_4 = \frac{x_0}{\sqrt{2}d}.$$

Now we list the necessary and sufficient conditions for $x^* \in K_1$ and $y^* \in K_1$:

Since

$$\frac{4x - 2vy}{x^2 - 2y^2} > 0, \quad \text{and} \quad \frac{-uy + vx}{x^2 - 2y^2} > 0, \tag{6}$$

thus (6) is fulfilled if and only if, $y < \mu x$ or $\nu x < y$ where the constants μ, $\nu \in \mathbb{R}_+$ are defined by

$$\mu := \min\left\{\frac{1}{\sqrt{2}}, \frac{2}{v}, \frac{v}{u}\right\}, \qquad \nu := \max\left\{\frac{1}{\sqrt{2}}, \frac{2}{v}, \frac{v}{u}\right\}. \tag{7}$$

Define the domains D_1, D_2 by

$$D_1 := \left\{(x,y) \in \mathbb{Q}_+^2 \mid e_1(x) < y, y < e_2(x), y < e_3(x), e_4(x) < y\right\},$$
$$D_2 := \left\{(x,y) \in \mathbb{Q}_+^2 \mid y < \mu x \text{ or } \nu x < y\right\}.$$

Since $\mathbb{Q}^2$ is dense in $\mathbb{R}^2$ thus we obtain that $D_1 \cap D_2 \neq \emptyset$. Choose a point $P = P(x,y) \in (D_1 \cap D_2)$ arbitrarily and define the numbers x^* and y^* by

$$x^* := x + \sqrt{2}y \qquad \text{and} \qquad y^* := \frac{x_0}{x^*}.$$

Thus $x^* \in\,]a,b[$ and $y^* \in]c,d[$ whence we obtain that $]ac, bd[\, \subseteq\,]a,c[\, \cdot\,]b,c[$ indeed. $\square$

Example 4.4. Define the numbers α, β, γ, δ by (5). Let

$$y_0 := 4 + 0.1\sqrt{2} = 4.1414\cdots \in\,]\alpha\gamma, \beta\delta[\,.$$

We show that there exist constants $x^* \in\,]\alpha,\beta[$ and $y^* \in\,]\gamma,\delta[$ such that $y_0 = x^* y^*$. Preserving the notations of Proporition 4.3. we obtain that

$$A_1 = 0.8535\ldots, \quad A_2 = 1.3535\ldots, \quad A_3 = 0.8577\ldots, \quad A_4 = 0.7649\ldots,$$

moreover, $\mu = 0.025$, $\nu = 20$.

Consider the following figures †:

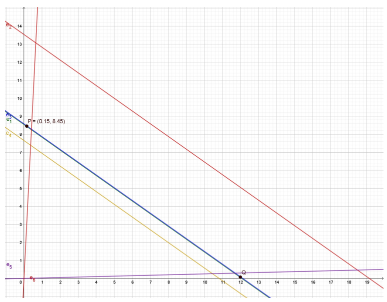

Figure 2: Products of intervals

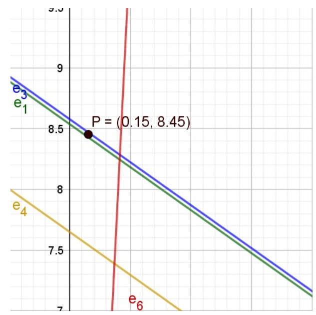

Figure 3: Magnification of Figure 2

The Figure 2 and Figure 3 were taken at ten times magnification for better illustration. The lines e_1, e_2, e_3, e_4 are defined by

$$e_1 = -0.71x + 8.54, \qquad e_3 = -0.71x + 8.58,$$
$$e_2 = -0.71x + 13.54, \qquad e_4 = -0.71x + 7.65.$$

The lines e_5, e_6 are determined by (7) and defined by

$$e_5 = 0.03x, \qquad e_6 = 20x.$$

It is easy to see that the point $P := (0.015, 0.845)$ is convenient. (The point Q is also convenient.) Thus we obtain that

$$x^* = 0.015 + 0.845\sqrt{2} = 1.21\cdots \in \,]\alpha, \beta[\,,$$
$$y^* = \frac{0.109 + 3.3785\sqrt{2}}{1.427825} = 3.4226\cdots \in \,]\gamma, \delta[$$

with $x^* y^* = y_0$.

Theorem 4.5. *Let a, b, c $d \in K_n$ such that $a < b$ and $c < d$. Then*

$$]a, b[+]c, d[=]a + c, b + d[\,,$$
$$]a, b[\cdot]c, d[=]ac, bd[\,.$$

Proof. The proof is easy by induction using Proposition 4.1. and Proposition 4.3. □

†The Figure 2 and Figure 3 were made with the program GeoGebra

Summarising the above Propositions and Theorem concerning the structure K_1 (or K_n) we can infer, that:

K_1 has no property (2) of Definition 2.1. For example $\alpha < \beta$ but $\beta - \alpha = 0.5\sqrt{2} \notin K_1$.

Equality (1) is fulfilled but equality (3) is not. For example let $x_0 = 4.8 + 0.1\sqrt{2} = 4.9414\ldots$. Then $x_0 \in]\alpha+\gamma, \alpha+\delta[$ but $x_0 - \alpha = 4.3 - 0.4\sqrt{2} \notin K_1$

Equality (2) is fulfilled but equality (4) is not. For example let $y_0 = 4 + 0.1\sqrt{2} = 4.1412$ then $y_0 \in]\alpha\gamma, \alpha\delta[$ but $\frac{y_0}{\alpha} = -7.6 + 7.8\sqrt{2} \notin K_1$.

5 Results and Problems

The example of K_n and $\mathbb{Q}_n$ motivates the problems bellow.

Problem 5.1. How to characterise the semirings that satisfy equalities (1) and (2) but do not satisfy equalities (3) and (4)?

The first author of this article has proven the following Theorem in [6]:

Theorem *Let $\mathbb{G}(+, \leqslant)$ be an Archimedean ordered dense Abelian group, $Y(+)$ be a group, x_0, $y_0 \in \mathbb{G}$, $\varepsilon \in \mathbb{G}_+$, moreover, let*

$$\begin{aligned} &f :]x_0 + y_0 - 2\varepsilon, x_0 + y_0 + 2\varepsilon[\to Y, \\ &g :]x_0 - \varepsilon, x_0 + \varepsilon[\to Y, \\ &h :]y_0 - \varepsilon, y_0 + \varepsilon[\to Y \end{aligned}$$

be functions such that

$$f(x+y) = g(x) + h(y) \qquad (x \in]x_0 - \varepsilon, x_0 + \varepsilon[\,, y \in]y_0 - \varepsilon, y_0 + \varepsilon[),$$

then there exists an additive function $a : \mathbb{G} \to Y$ and exist constants c, $d \in Y$ such that

$$\begin{aligned} f(w) &= a(w) + c + d \qquad (w \in]x_0 + y_0 - 2\varepsilon, x_0 + y_0 + 2\varepsilon[), \\ g(u) &= a(u) + c \qquad (u \in]x_0 - \varepsilon, x_0 + \varepsilon[), \\ h(v) &= a(v) + d \qquad (w \in]y_0 - \varepsilon, y_0 + \varepsilon[). \end{aligned}$$

The proof is based on the equalities (1) and (3).

Problem 5.2. Can a theorem analogous to the above one be proved in that case when the role of group $\mathbb{G}$ is taken over by a semigroup $\mathbb{S}$ in which equality (1) is satisfied but equality (3) is not?

Conjecture 5.3. If $\mathbb{S} = \mathbb{S}(+, \leqslant)$ is a dense ordered semigroup, then (1) is satified, that is

$$]a, b[+]a, d[=]a + c, b + d[$$

for all $a, b, c, d \in \mathbb{S}$ such that $a < b$, $c < d$.

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