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On some new results for the generalised Lucas sequences

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Abstract

In this paper we introduce the functions which count the number of generalized Lucas and Pell-Lucas sequence terms not exceeding a given value x and, under certain conditions, we derive exact formulae (Theorems 3 and 4) and establish asymptotic limits for them (Theorem 6). We formulate necessary and sufficient arithmetic conditions which can identify the terms of a-Fibonacci and a-Lucas sequences. Finally, using a deep theorem of Siegel, we show that the aforementioned sequences contain only finitely many perfect powers. During the process we also discover some novel integer sequences.

1 Introduction

The classical Fibonacci, Lucas, Pell, Pell-Lucas sequences (and their numerous extensions) have been researched for more than two centuries. Still, numerous interesting properties and applications are being discovered. Many of them have significant theoretical and practical importance (see, e.g., [3], [9]).

In this paper we focus on the *generalized Lucas* sequence $\{U_n(a,b)\}_{n\geq 0}$ and its companion, the *generalized Pell-Lucas* sequence $\{V_n(a,b)\}_{n\geq 0}$, defined for the arbitrary integers a and b by

$$U_{n+2} = aU_{n+1} - bU_n, \quad U_0 = 0,\, U_1 = 1, \quad n = 0, 1, \dots \tag{1}$$

$$V_{n+2} = aV_{n+1} - bV_n, \quad V_0 = 2,\, V_1 = a, \quad n = 0, 1, \dots. \tag{2}$$

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The standard method for analysing these sequences involves the roots of the quadratic $z^2 - az + b = 0$ and its discriminant $D = a^2 - 4b$. We will consider the case $D \neq 0$, when this equation has the distinct roots

$$\alpha = \frac{a+\sqrt{D}}{2}, \quad \beta = \frac{a-\sqrt{D}}{2}.$$

From Viéte's relations, one obtains $\alpha + \beta = a$, $\alpha\beta = b$, while $\alpha - \beta = \sqrt{D}$.

In these notations, the following Binet-type formulae hold

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \frac{1}{\sqrt{D}}\left(\alpha^n - \beta^n\right), \quad n = 0, 1, \ldots, \tag{3}$$

$$V_n = \alpha^n + \beta^n, \quad n = 0, 1, \ldots. \tag{4}$$

These expressions extend naturally to negative indices. For example, we have

$$U_{-1} = \frac{1}{\sqrt{D}}\left(\alpha^{-1} - \beta^{-1}\right) = -\frac{1}{b}, \quad V_{-1} = \alpha^{-1} + \beta^{-1} = \frac{a}{b},$$

and in general, the relations below hold for any integer $n \geq 0$

$$U_{-n} = \frac{1}{\sqrt{D}}\left(\alpha^{-n} - \beta^{-n}\right) = -\frac{1}{b^n}U_n, \quad V_{-n} = \alpha^{-n} + \beta^{-n} = \frac{1}{b^n}V_n.$$

When $b = -1$ and a is a positive integer one obtains the a-Fibonacci and a-Lucas numbers from $F_{a,n} = U_n(a, -1)$ and $L_{a,n} = V_n(a, -1)$, where $D = a^2 + 4$. The Fibonacci and Lucas numbers are obtained for $a = 1$ as $F_n = U_n(1, -1)$ and $L_n = V_n(1, -1)$ with $D = 5$, while the Pell and Pell-Lucas numbers are given by $P_n = U_n(2, -1)$ and $Q_n = V_n(2, -1)$, for $a = 2$ where $D = 8$.

Arithmetic properties of the sequences $\{U_n(a, b)\}_{n\in\mathbb{Z}}$ and $\{V_n(a, b)\}_{n\in\mathbb{Z}}$ were investigated in [4], generalising earlier work on Fibonacci and Lucas sequences in [5]. In [10] Lehmer studied some arithmetic properties of further generalisations of these sequences, where he allows a to be a quadratic algebraic integer.

The structure of the present paper is as follows. In Section 2 we give conditions on the parameters a and b for which the sequences $\{U_n(a, b)\}_{n\geq 0}$ and $\{V_n(a, b)\}_{n\geq 0}$ are increasing. This information is used to derive exact formulae for the functions defined by the number of generalised Lucas and Pell-Lucas terms smaller than a given value. Asymptotic formulae for these functions are given in Section 3. In the paper [1], the authors explore various quadratic Diophantine equations whose solutions are expressed in terms of Fibonacci and Lucas sequences. This is the starting point of Section 4, which deals with the identification of the a-Fibonacci and a-Lucas numbers. As a consequence of a deep theorem which asserts the finiteness of integral points on curves due to Siegel, we show that the sequences of a-Fibonacci and a-Lucas numbers contain only finitely many perfect powers. Some novel associated integer sequences are investigated in Section 5.

2 The functions $u(a,b;x)$ and $v(a,b;x)$

We aim to find expressions for the number of terms of generalised Lucas sequences which are less or equal than a positive real number x. Define

$$u(a,b;x) = \mathrm{card}\{n \in \mathbb{N} \quad : \quad U_n \leq x\} \tag{5}$$
$$v(a,b;x) = \mathrm{card}\{n \in \mathbb{N} \quad : \quad V_n \leq x\}. \tag{6}$$

Clearly, whenever the sequences $\{U_n(a,b)\}_{n\geq 0}$ and $\{V_n(a,b)\}_{n\geq 0}$ are increasing, one has $u(a,b;x) = n$ whenever $U_{n-1} \leq x < U_n$, and $v(a,b;x) = n$ when $V_{n-1} \leq x < V_n$. Throughout this paper we shall assume that $a > 0$ and $D > 0$, in which case $\alpha >\mid \beta \mid$, hence $\lim_{n\to\infty} U_n = \lim_{n\to\infty} V_n = \infty$.

Let us denote by $n_0(a,b)$ and $n_0'(a,b)$ the first indices where consecutive terms are ordered increasingly

$$n_0(a,b) = \min\{n \in \mathbb{N} : U_{n+1} > U_n\},$$
$$n_0'(a,b) = \min\{n \in \mathbb{N} : V_{n+1} > V_n\}.$$

Clearly, these sets are not empty. We prove that for certain choices of the parameters a and b, the sequences $\{U_n(a,b)\}_{n\geq 0}$ and $\{V_n(a,b)\}_{n\geq 0}$ are increasing when $n \geq n_0(a,b)$, or $n \geq n_0'(a,b)$, respectively.

More precisely, we have the following result.

Lemma 1. *Let $a > 0$ and b be integers, such that $a \geq b+1$. Then*

1° *The sequence $\{U_n(a,b)\}_{n\geq 0}$ is increasing for $n \geq n_0(a,b)$, where*

$$n_0(a,b) = \begin{cases} 0 & \text{if } a \geq 2 \text{ and } a \geq b+1 \\ 2 & \text{if } a = 1 \text{ and } b < 0. \end{cases}$$

2° *The sequence $\{V_n(a,b)\}_{n\geq 0}$ is increasing for $n \geq n_0'(a,b)$, where*

$$n_0'(a,b) = \begin{cases} 0 & \text{if } a \geq 3 \text{ and } a \geq b+1 \\ 1 & \text{if } a = 1,2 \text{ and } b \leq 0. \end{cases}$$

Proof. 1° One may write the recurrence relation (1) as

$$U_{n+2} - U_{n+1} = (a-1)U_{n+1} - bU_n, \quad n = 0, 1, \ldots.$$

The sequence $\{U_n\}_{n\geq 0}$ starts with $U_0 = 0$, $U_1 = 1$, $U_2 = a$ and $U_3 = a^2 - b$.

Case 1. $b < 0$. Here $(a-1)U_{n+1} \geq 0$ whenever $U_{n+1} \geq 0$, and $-bU_n > 0$ if $U_n > 0$. Clearly, $U_{n+2} - U_{n+1} > 0$ for $n \geq 1$, and $\{U_n\}_{n\geq 2}$ is increasing for $a = 1$, while $\{U_n\}_{n\geq 0}$ is increasing for $a \geq 2$.

Case 2. $b = 0$. Here $U_n = a^{n-1}$ for $n \geq 1$, and the sequence $\{U_n\}_{n\geq 0}$ is increasing for $a \geq 2$.

Case 3. $b > 0$. In this case, $(a-1)U_{n+1} \geq bU_{n+1}$ for $U_{n+1} \geq 0$, and for $a - 1 \geq b$ one has $U_{n+2} - U_{n+1} \geq b(U_{n+1} - U_n) > 0$, whenever $U_{n+1} > U_n$. Since $U_1 > U_0$, this condition ensures that the sequence $\{U_n\}_{n\geq 0}$ is increasing.

2° The recurrence relation (2) can be written as

$$V_{n+2} - V_{n+1} = (a-1)V_{n+1} - bV_n, \quad n = 0, 1, \ldots.$$

The sequence $\{V_n\}_{n\geq 0}$ begins with the terms $V_0 = 2$, $V_1 = a$, $V_2 = a^2 - 2b$. We distinguish a few cases.

Case $1'$**.** $b < 0$. Here $(a-1)V_{n+1} \geq 0$ whenever $V_{n+1} \geq 0$, and $-bV_n > 0$ whenever $V_n > 0$. The condition $V_n > 0$ is valid even for $n = 0$, hence $\{V_n\}_{n\geq 1}$ is increasing. Moreover, when $a \geq 3$, the sequence $\{V_n\}_{n\geq 0}$ is increasing.

Case $2'$**.** $b = 0$. $V_n = a^n$, $n \geq 1$, and $\{V_n\}_{n\geq 1}$ is increasing for $a = 2$, and $\{V_n\}_{n\geq 0}$ is increasing for $a \geq 3$.

Case $3'$**.** $b > 0$. In this case, $(a-1)V_{n+1} \geq bV_{n+1}$ whenever $V_{n+1} \geq 0$, and for $a - 1 \geq b \geq 1$ one has $V_{n+2} - V_{n+1} \geq b(V_{n+1} - V_n) > 0$, whenever $V_{n+1} > V_n$. If $a \geq 3$, we have $V_1 - V_0 = a - 2 > 0$, so $\{V_n\}_{n\geq 0}$ is increasing. When $a = 2$ and $b = 1$ the sequence is constant, as whenever $a - 1 = b$, we have $V_n = b^n + 1$ for $n \geq 0$, which is only increasing for $b \geq 2$. □

Various bounds for β can be established in terms of a and b.

Lemma 2. *Let $a \geq 1$ and b be integers such that $D = a^2 - 4b > 0$. The following assertions hold*

1° $\mid \beta \mid \leq 1$ *if and only if* $\mid b + 1 \mid \leq a$.

2° $\mid \beta \mid < 1$ *if and only if* $\mid b + 1 \mid < a$.

3° $\mid \beta \mid < \frac{\sqrt{2}}{2}$ *if and only if* $a = 1$ *and* $b > -\frac{1+\sqrt{2}}{2}$, *or* $a \geq 2$ *and* $(2b+1)^2 < 2a^2$.

4° $\mid \beta \mid < \frac{1}{2}$ *if and only if* $a = 1$ *and* $b > -\frac{3}{4}$, *or* $a \geq 2$ *and* $\mid 4b + 1 \mid < 2a$.

Proof. 1° Since $\beta = \frac{a-\sqrt{D}}{2}$, the condition $\mid \beta \mid \leq 1$ is equivalent to

$$a - 2 \leq \sqrt{D} \leq a + 2.$$

When $a = 1$, the left-hand side condition is fulfilled, while the other gives $b \geq -2$. Same happens for $a = 2$, when the condition satisfied by b is $b \geq -3$.

When $a \geq 3$, by squaring we obtain $a^2 - 4a + 4 \leq a^2 - 4b \leq a^2 + 4a + 4$, which is written equivalently as

$$-a - 1 \leq b \leq a - 1,$$

that is $\mid b + 1 \mid \leq a$. In fact, this condition unifies all the cases $a = 1, 2, \dots$. Clearly, when $b = -1$ (such as for example for the a-Fibonacci numbers), the condition is satisfied for all positive integers a.

2° One can just replace $\leq$ by $<$ in the proof for 1°.

3° The condition $\mid \beta \mid < \frac{\sqrt{2}}{2}$ is equivalent to $a - \sqrt{2} \leq \sqrt{D} \leq a + \sqrt{2}$.

When $a = 1$, the left-hand side condition is fulfilled automatically, while the other side gives $b > -\frac{1+\sqrt{2}}{2}$. Clearly, the case $a = 1$, $b = -1$ required for Fibonacci and Lucas sequences falls in this category.

For $a \geq 2$, we get $-2\sqrt{2}a < -4b - 2 < 2\sqrt{2}a$, which is equivalent to $\mid 2b + 1 \mid < \sqrt{2}a$, or $(2b + 1)^2 < 2a^2$.

4° The condition $\mid \beta \mid < \frac{1}{2}$ is equivalent to $a - 1 < \sqrt{D} < a + 1$.

When $a = 1$, the left inequality is automatically fulfilled, while the right one gives $b > -\frac{3}{4}$. For $a \geq 2$, we get $-2a < -4b-1 < 2a$, i.e., $\mid 4b+1 \mid < 2a$. $\square$

Theorem 3. *Let a and b be integers satisfying $\mid b + 1 \mid \leq a$ and $\sqrt{D} > 2$. When $x > 1$, the following identity holds*

$$u(a, b; x) = \left\lfloor \frac{\ln\left(\sqrt{D}x + 1\right)}{\ln \alpha} \right\rfloor + 1. \tag{7}$$

Proof. By Lemma 1, as $a \geq b + 1$ the sequence $(U_n)_{n \geq 0}$ is increasing. For $x > 1$ we can find n for which $U_{n-1} \leq x < U_n$, that is $u(a, b; x) = n$. To prove the desired identity it suffices to show that

$$n \leq \frac{\ln\left(\sqrt{D}U_n + 1\right)}{\ln \alpha} \leq \frac{\ln\left(\sqrt{D}\left(U_{n+1} - 1\right) + 1\right)}{\ln \alpha} < n + 1.$$

The left-hand inequality reduces to $\alpha^n \leq \alpha^n - \beta^n + 1$. By Lemma 2 1° we have $\mid \beta \mid \leq 1$, hence $\mid \beta^n \mid \leq 1$.

The second inequality holds since $U_n \leq U_{n+1} - 1$.

The right-hand inequality is equivalent to

$$\alpha^{n+1} - \beta^{n+1} - \sqrt{D} + 1 < \alpha^{n+1},$$

which is true since $1 - \beta^{n+1} < \sqrt{D}$ holds for $\sqrt{D} > 2$ and $\mid \beta \mid \leq 1$. $\square$

Theorem 4. *Let a and b be integers such that $a \geq b+1$ and $\mid \beta \mid< \frac{\sqrt{2}}{2}$. When $x > \max\{2, a\}$, then we have*

$$v(a, b; x) = \left\lfloor \frac{\ln\left(x + \frac{1}{2}\right)}{\ln \alpha} \right\rfloor + 1. \tag{8}$$

Proof. By Lemma 1, the sequence $(V_n)_{n\geq 2}$ is increasing, hence for x large enough one can find n such that $V_{n-1} \leq x < V_n$, in which case $v(a, b; x) = n$. To prove this it suffices to show that

$$n \leq \frac{\ln\left(V_n + \frac{1}{2}\right)}{\ln \alpha} \leq \frac{\ln\left((V_{n+1} - 1) + \frac{1}{2}\right)}{\ln \alpha} < n + 1.$$

The left inequality reduces to

$$\alpha^n \leq \alpha^n - \beta^n + \frac{1}{2},$$

or equivalently, $\beta^n \leq \frac{1}{2}$. This holds since $\mid \beta^n \mid=\mid \beta \mid^n\leq\mid \beta \mid^2< \frac{1}{2}$, for $n \geq 2$.

The middle inequality holds since the sequence $V_n \leq V_{n+1} - 1$.

The right inequality is equivalent to

$$\alpha^{n+1} + \beta^{n+1} - 1 + \frac{1}{2} < \alpha^{n+1},$$

or $\beta^{n+1} < \frac{1}{2}$, which holds for $n \geq 1$ by the previous argument. □

Remark 5. *In the particular case when $b = -1$, we recover results which hold for the sequences of a-Fibonacci and a-Lucas numbers.*

3 Asymptotic results for $u(a, b; x)$ and $v(a, b; x)$

Theorem 6. *If a and b are integers satisfying $a > 0$, $a \geq b + 1$ and $D = a^2 - 4b > 0$, then the following asymptotic formulae hold*

$$\lim_{x\to\infty} \frac{u(a, b; x)}{\ln x} = \lim_{x\to\infty} \frac{v(a, b; x)}{\ln x} = \frac{1}{\ln \alpha}. \tag{9}$$

Proof. From the Binet-type formulae (3) and (4) we have

$$\alpha^n = \frac{1}{2}\left(V_n + U_n\sqrt{D}\right), \quad n = 0, 1, \ldots \tag{10}$$

Since $a \geq b+1$, by Lemma 1 the sequences $\{U_n\}_{n\geq 2}$ and $\{V_n\}_{n\geq 2}$ are increasing. For $x \geq \max\{U_2, V_2\}$, one can find m such that $U_m \leq x < U_{m+1}$, that is $u(a, b; x) = m + 1$.

Since $a > 0$ and $D > 0$ we have $\alpha >\mid \beta \mid$, hence $\mid \frac{\beta}{\alpha} \mid< 1$, we can write

$$\lim_{n\to\infty} \frac{U_n}{V_n} = \frac{1}{\sqrt{D}} \lim_{n\to\infty} \frac{\alpha^n - \beta^n}{\alpha^n + \beta^n} = \frac{1}{\sqrt{D}} \lim_{n\to\infty} \frac{1 - \left(\frac{\alpha}{\beta}\right)^n}{1 + \left(\frac{\alpha}{\beta}\right)^n} = \frac{1}{\sqrt{D}},$$

so the sequence $\left(\frac{U_n}{V_n}\right)_{n\geq 0}$ is bounded, hence we can find k_1 and k_2 such that

$$0 < k_1 \leq \frac{U_n}{V_n} \leq k_2, \quad n = 1, 2, \dots. \tag{11}$$

Using (10) and (11), it follows that

$$\alpha^m = \frac{1}{2}\left(V_m + U_m\sqrt{D}\right) \leq \frac{1}{2}\left(\frac{1}{k_1}U_m + U_m\sqrt{D}\right) \leq \frac{1}{2}\left(\frac{1}{k_1} + \sqrt{D}\right)x, \tag{12}$$

and

$$\begin{aligned}\alpha^{m+1} &= \frac{1}{2}\left(V_{m+1} + U_{m+1}\sqrt{D}\right) \geq \frac{1}{2}\left(\frac{1}{k_2}U_{m+1} + U_{m+1}\sqrt{D}\right) \\ &= \frac{1}{2}\left(\frac{1}{k_2} + \sqrt{D}\right)U_{m+1} > \frac{1}{2}\left(\frac{1}{k_2} + \sqrt{D}\right)x.\end{aligned} \tag{13}$$

By (12) and (13) we obtain

$$\frac{\ln\frac{1}{2}\left(\frac{1}{k_2} + \sqrt{D}\right) + \ln x}{\ln\alpha} < u(a, b; x) \leq \frac{\ln\frac{1}{2}\left(\frac{1}{k_1} + \sqrt{D}\right) + \ln x}{\ln\alpha} + 1,$$

hence $\lim_{x\to\infty} \frac{u(a,b;x)}{\ln x} = \frac{1}{\ln\alpha}$.

For the second limit, we assume that $V_l \leq x < V_{l+1}$, i.e., $v(a, b; x) = l + 1$. Similarly, by (11) we obtain

$$\begin{aligned}\alpha^l &= \frac{1}{2}\left(V_l + U_l\sqrt{D}\right) \leq \frac{1}{2}\left(1 + k_2\sqrt{D}\right)V_l \leq \frac{1}{2}\left(1 + k_2\sqrt{D}\right)x, \\ \alpha^{l+1} &= \frac{1}{2}\left(V_{l+1} + U_{l+1}\sqrt{D}\right) \geq \frac{1}{2}\left(1 + k_1\sqrt{D}\right)V_{l+1} > \frac{1}{2}\left(1 + k_1\sqrt{D}\right)x.\end{aligned}$$

Therefore,

$$\frac{\ln\frac{1}{2}\left(1 + k_1\sqrt{D}\right) + \ln x}{\ln\alpha} < v(x) \leq \frac{\ln\frac{1}{2}\left(1 + k_2\sqrt{D}\right) + \ln x}{\ln\alpha} + 1,$$

hence $\lim_{x\to\infty} \frac{v(a,b;x)}{\ln x} = \frac{1}{\ln\alpha}$, which ends the proof. □

Remark 7. *The limits* (9) *can also be obtained from the exact formulae* (7) *and* (8)*, using the inequality* $t-1 \leq \lfloor t \rfloor \leq t$*, provided that the hypotheses of Theorems 3 and 4 are satisfied.*

The limits obtained for some classical sequences are given in Table 1. The limit is the same for $(a,b)=(1,-2)$, $(a,b)=(3,2)$, this phenomena being explained by the fact that the pairs give the same value for $\alpha = \frac{a+\sqrt{a^2-4b}}{2}$.

(a,b)	D	(α,β)	$U_n(a,b)$	$V_n(a,b)$	$\frac{1}{\ln\alpha}$
$(1,-2)$	9	$(2,-1)$	Jacobsthal A001045	Jacobsthal-Lucas A014551	$\simeq 1.4427$
$(1,-1)$	5	$\left(\frac{1+\sqrt{5}}{2},\frac{1-\sqrt{5}}{2}\right)$	Fibonacci A000045	Lucas A000032	$\simeq 2.0781$
$(2,-1)$	8	$(1+\sqrt{2},1-\sqrt{2})$	Pell A000129	Pell-Lucas A002203	$\simeq 1.1346$
$(3,-1)$	13	$\left(\frac{3+\sqrt{13}}{2},\frac{3-\sqrt{13}}{2}\right)$	bronze Fibonacci A006190	bronze Lucas A006497	$\simeq 0.8370$
$(3,1)$	5	$\left(\frac{3+\sqrt{5}}{2},\frac{3-\sqrt{5}}{2}\right)$	bisection of Fibonacci A001906	bisection of Lucas A005248	$\simeq 1.0390$
$(3,2)$	1	$(2,1)$	Mersenne A000225	Pisot A000051	$\simeq 1.4427$

Table 1: Sequences $\{U_n(a,b)\}_{n\geq 0}$ and $\{V_n(a,b)\}_{n\geq 0}$ and asymptotic limits of the functions $u(a,b;x)$ and $v(a,b;x)$.

4 Identifying generalized Lucas and Pell-Lucas numbers

Here we discuss the next identification question: *given a positive integer N, can one decide whether N is a term of $\{U_n(a,b)\}_{n\geq 0}$ or $\{V_n(a,b)\}_{n\geq 0}$?*

4.1 Motivation

Recall that $\{U_n(2,-1)\}_{n\geq 0}$ and $\{V_n(2,-1)\}_{n\geq 0}$ are the sequences of Pell numbers, and Pell-Lucas numbers, respectively. The following proposition will serve as a motivating example for the aforementioned question.

Proposition 8. 1° *A positive integer N is a Pell number if and only if $2N^2 \pm 1$ is a perfect square.*

2° *A positive integer N is a Pell-Lucas number if and only if $2N^2 \pm 2$ is a perfect square.*

Proof. 1° We have that $2N^2 \pm 1 = M^2$, where $M \in \mathbb{Z}$. Rearranging, one obtains the Pell (or negative Pell) equation

$$M^2 - 2N^2 = \pm 1,$$

so N is a Pell number by definition.

2° Note that if $2N^2 \pm 2 = M^2$ with $M \in \mathbb{Z}$, then $M = 2M_1$ where $M_1 \in \mathbb{Z}$. Dividing by 2 and rearranging we get the Pell (or negative Pell) equation

$$N^2 - 2M_1^2 = \mp 1,$$

so N is a Pell-Lucas number. The converse trivially holds.

The numbers N obtained from the positive (negative) equation correspond to even (odd) terms in the respective sequences $\{U_n(2,-1)\}_{n\geq 0}$ and $\{V_n(2,-1)\}_{n\geq 0}$. □

Let us now explore a more general setting.

Lemma 9. *The following relation holds for every integer $n \geq 0$*

$$V_n^2 - DU_n^2 = 4b^n. \tag{14}$$

Proof. From (3) and (4) we get $V_n + \sqrt{D}U_n = 2\alpha^n$ and $V_n - \sqrt{D}U_n = 2\beta^n$, therefore $V_n^2 - DU_n^2 = 4(\alpha\beta)^n = 4b^n$, for every $n \geq 0$. □

Corollary 10. *For every $n \geq 0$, the general Pell equations*

$$x^2 - Dy^2 = 4b^n \tag{15}$$

are solvable.

Proof. We just note that (V_n, U_n) is a solution to (15). □

Corollary 11. *The following statements hold.*

1° *If N is a term of $\{U_n(a,b)\}_{n\geq 0}$, then $DN^2 + 4b^n$ is a perfect square for some $m \in \mathbb{N}$.*

2° *If N is a term of $\{V_n(a,b)\}_{n\geq 0}$ then $DN^2 - 4Db^k$ is a perfect square for some $k \in \mathbb{N}$.*

Proof. For 1°, notice that if $N = U_m$, then by using (14) we conclude that $DN^2 + 4b^m = DU_m^2 + 4b^m = V_m^2$.

Similarly for 2°, if $N = V_k$, using again formula (14), one can deduce that $DN^2 - 4Db^k = DV_k^2 - 4Db^k = (DU_k)^2$, and the conclusion follows. □

Such relations in the particular case $b = -1$ are found in [12, Chapter 8], where the study of converse implications is also mentioned.

We will restrict (without loosing generality) on square free integers $D > 1$.

4.2 The quadratic ring $\mathbb{Z}[\sqrt{D}]$ and Pell-type equations

Let us denote by $\mathbb{Z}[\sqrt{D}] = \{a + b\sqrt{D} : a, b \in \mathbb{Z}\}$. This is a subring of the algebraic integers in the quadratic field $\mathbb{Q}(\sqrt{D})$. The conjugate of an element $\alpha = a+b\sqrt{D} \in \mathbb{Z}[\sqrt{D}]$ is denoted by $\overline{\alpha} = a-b\sqrt{D}$, while the notation $\mathbb{Z}[\sqrt{D}]^\times$ represents the group of multiplicative units in the ring $\mathbb{Z}[\sqrt{D}]$.

Definition 12. *The norm of* $\alpha \in \mathbb{Z}[\sqrt{D}]$ *is defined as* $\mathrm{Norm}(\alpha) = \alpha \cdot \overline{\alpha} \in \mathbb{Z}$.

One can easily check that Norm is multiplicative on $\mathbb{Z}[\sqrt{D}]$. Moreover, $N(\alpha) = \pm 1$ if and only if α belongs to $\mathbb{Z}[\sqrt{D}]^\times$. The equations

$$X^2 - DY^2 = 1, \tag{16}$$

$$X^2 - DY^2 = -1 \tag{17}$$

are called the (positive), and the negative Pell equations, respectively.

The elements $\alpha \in \mathbb{Z}[\sqrt{D}]^\times$ that have positive norm are in bijection with the integer solutions to (16), whereas $\alpha \in \mathbb{Z}[\sqrt{D}]^\times$ of negative norm are in a one-to-one correspondence with the integer solutions to (17).

An old algebraic result (see, e.g., [8]) asserts that $\mathbb{Z}[\sqrt{D}]^\times \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}$, where the finite part consists of the roots of unity $\{-1, 1\}$ in $\mathbb{Z}[\sqrt{D}]$, whereas the infinite part is composed of all the integer powers of ϵ_D, an element which is called the fundamental unit.

For every square-free $D > 1$, there are elements $\alpha \in \mathbb{Z}[\sqrt{D}]^\times$ which have norm 1, as one can see by computing the norm of ϵ_D^2. This explains why Pell's equation (16) is solvable for any such D. On the other hand, the question of whether there are elements $\alpha \in Z[\sqrt{D}]^\times$ such that $\mathrm{Norm}(\alpha) = -1$, i.e. the solvability of the negative Pell equation (17) is much more complicated. For example, one can see that such an α does not exist if D contains a prime factor $p \equiv 3 \pmod 4$, since -1 is not a quadratic residue modulo such primes. Note that $X^2 - DY^2 = -1$ is solvable if and only if $\mathrm{Norm}(\epsilon_D) = -1$.

We now prove the following result which emphasises the connection between the units $\mathbb{Z}[\sqrt{D}]^\times$, the set of solutions to Pell equation (16) and the set of solutions to the negative Pell equation (17).

Proposition 13. 1° *If* $X^2 - DY^2 = -1$ *is solvable, then the set of solutions are in bijection with* $\{\pm\epsilon_D^k : k \in \mathbb{Z}$ *is odd*$\}$. *In this case, the set of solutions of* $X^2 - DY^2 = 1$ *are in bijection with* $\{\pm\epsilon_D^k : k$ *is even*$\}$.

2° *If the equation* $X^2 - DY^2 = -1$ *is not solvable, then the set of solutions of* $X^2 - DY^2 = 1$ *are in bijection with* $\mathbb{Z}[\sqrt{D}]^\times = \{\pm\epsilon_D^k : k \in \mathbb{Z}\}$.

Proof. First, notice that the restriction $\mathrm{Norm} : \mathbb{Z}[\sqrt{D}]^\times \to \{-1, 1\}$ is a group homomorphism.

1° The equation $X^2 - DY^2 = -1$ is solvable if and only if the homomorphism above is surjective, i.e., if and only if $\mathrm{Norm}(\epsilon_D) = -1$. Every solution $(x, y) \in \mathbb{Z}^2$ to the negative Pell equation $X^2 - DY^2 = -1$ corresponds to an element $x + y\sqrt{D} \in \mathbb{Z}[\sqrt{D}]^\times$ of norm -1. By the structure of the group $\mathbb{Z}[\sqrt{D}]^\times$, we have that $x + y\sqrt{D} = \pm\epsilon_D^k$, where k is odd.

Here, the solutions $(x, y) \in \mathbb{Z}^2$ to the Pell equation $X^2 - DY^2 = 1$ correspond to elements $x + y\sqrt{D} \in \mathbb{Z}[\sqrt{D}]^\times$ having norm 1.

The set of all such elements make up for the kernel of the restricted Norm homomorphism above, an index 2 subgroup of $\mathbb{Z}[\sqrt{D}]^\times$. By the structure of the unit group $\mathbb{Z}[\sqrt{D}]^\times$, every such element $x + y\sqrt{D}$ is equal to $\pm\epsilon_D^k$, where k is an even integer.

2° The equation $X^2 - DY^2 = -1$ is not solvable if and only if the kernel of the restricted Norm homomorphism is the whole group of units $\mathbb{Z}[\sqrt{D}]^\times$. To be precise, any solution $(x, y) \in \mathbb{Z}^2$ to the Pell equation $X^2 - DY^2 = 1$ corresponds bijectively to an element $x + y\sqrt{D} \in \mathbb{Z}[\sqrt{D}]^\times$. □

Remark. If $X^2 - DY^2 = -1$ is solvable, then it is easy to see that if we consider the minimal solution (x_0, y_0), the element $x_0 + y_0\sqrt{D}$ must be a generator of the infinite part of $\mathbb{Z}[\sqrt{D}]^\times$. In other words, $x_0 + y_0\sqrt{D} \in \{\epsilon_D^{\pm 1}\}$. The minimal solution (x_1, y_1) to the Pell equation $X^2 - DY^2 = 1$ corresponds to the element $x_1 + y_1\sqrt{D} \in \{\epsilon_D^{\pm 2}\}$. We have just showed that one can recover all the integer solutions to the Pell equation (16) by taking even powers of the unit corresponding to the minimal solution $x_0 + y_0\sqrt{D}$ to the negative Pell equation (17) and multiplying the result by ± 1.

The equations

$$X^2 - DY^2 = 4, \tag{18}$$

$$X^2 - DY^2 = -4 \tag{19}$$

are called the special Pell equation, and the negative special Pell equation, respectively. They will play an important role in some theorems in this section.

We have the following easy corollary of Proposition 13.

Corollary 14. *Let $D = a^2 + 4$ be a square-free positive integer. There are unique minimal solutions $(a, 1)$ of (19) and (x_1, y_1) of (18). Moreover, they satisfy the relation*

$$2(x_1 + y_1\sqrt{D}) = \left(a + \sqrt{D}\right)^2.$$

Proof. The fact that (18) and (19) have unique minimal solutions follows from Theorem 4.4.1 in [2]. In the ring $\mathbb{Z}[\sqrt{D}]$ this translates into the fact that, up to units, the only element of norm 4 is 2 and that $\mathrm{Norm}(\epsilon_D) = -1$.

Any solution $(x, y) \in \mathbb{Z}^2$ to (19) gives an element $x + y\sqrt{D}$ which must be equal to $2 \cdot \epsilon_D^{2k+1}$ for some integer $k \in \mathbb{Z}$. Similarly, any solution $(x, y) \in \mathbb{Z}^2$ to (18) is given by $2 \cdot \epsilon_D^{2k}$, for some integer k. The conclusion follows. □

4.3 Identifying a-Fibonacci and a-Lucas numbers

As seen above, the converse of the properties 1° and 2° from Corollary 10 are related to the structure of the unit group in the ring $\mathbb{Z}[\sqrt{D}]$. Moreover, the multiple number of ways in which one can write $4b^n$ as a product of elements in $\mathbb{Z}[\sqrt{D}]$ contributes to the difficulty. This is intimately connected with the class group of $\mathbb{Q}(\sqrt{d})$.

Here we prove the converse properties mentioned above in the special case where $b = -1$. In this situation, one obtains the a-Fibonacci numbers $U_n(a, -1)$ and the a-Lucas numbers $V_n(a, -1)$. The main results are stated in the following two theorems.

Theorem 15. *Let $a > 0$ be such that $D = a^2 + 4$ is square-free. The positive integer N is an a-Fibonacci number if and only if $(a^2 + 4)N^2 \pm 4$ is a perfect square.*

Proof. The left to right implication is contained in Corollary 11. For the converse, suppose that $(a^2 + 4)N^2 + 4 = M^2$ for some integer M. The special Pell equation

$$u^2 - (a^2 + 4)v^2 = 4 \tag{20}$$

is solvable and one solution is $(u_1, v_1) = (a^2 + 2, a)$. However, we do not know whether this is a minimal solution.

By Theorem 4.4.1 in [2], we obtain that (20) has just one infinite class of solutions and, moreover, all solutions $(u, v) \in \mathbb{Z}^2$ to (20) can be recovered from

$$\frac{1}{2}(u + v\sqrt{D}) = \pm\left(\frac{u_0 + v_0\sqrt{D}}{2}\right)^n, \tag{21}$$

where (u_0, v_0) is the minimal solution to (20) and $D = a^2 + 4$.

On the other hand, the aforementioned theorem asserts that the negative special Pell equation

$$u^2 - (a^2 + 4)v^2 = -4 \tag{22}$$

has the minimal solution $(a, 1)$. We have explained in Corollary 14 that the minimal solution to (20) is given by

$$u_0 + v_0\sqrt{D} = \left(\frac{a + \sqrt{D}}{2}\right)^2 = a^2 + 2 + a\sqrt{D}.$$

It follows that all the solutions to (20) in positive integers are

$$\frac{1}{2}(u+v\sqrt{D}) = \left(\frac{a+\sqrt{D}}{2}\right)^{2n}, \quad n = 1, 2, \ldots$$

We now have

$$\frac{1}{2}(M+N\sqrt{D}) = \left(\frac{a+\sqrt{D}}{2}\right)^{2n},$$

for some positive integer n. At the same time,

$$\frac{1}{2}(M-N\sqrt{D}) = \left(\frac{a-\sqrt{D}}{2}\right)^{2n},$$

hence

$$N = \frac{1}{\sqrt{D}}\left(\left(\frac{a+\sqrt{D}}{2}\right)^{2n} - \left(\frac{a-\sqrt{D}}{2}\right)^{2n}\right) = \frac{1}{\sqrt{D}}(\alpha^{2n}-\beta^{2n}) = U_{2n}(a,-1).$$

If $(a^2+4)N^2-4 = M^2$ for some positive integer M, then the negative special Pell equation (22) has all solutions given by

$$\frac{1}{2}(u+v\sqrt{D}) = \left(\frac{a+\sqrt{D}}{2}\right)^{2n+1}, \quad n = 0, 1, 2, \ldots$$

Then, it follows that $N = U_{2n+1}(a,-1)$, for some positive integer n. □

Theorem 16. *Let $a > 0$ be an integer such that $D = a^2+4$ is square-free. A positive integer N is an a-Lucas number if and only if the number $(a^2+4)N^2 \pm 4(a^2+4)$ is a perfect square.*

Proof. Suppose that $(a^2+4)N^2 \pm 4(a^2+4) = M^2$ for some positive integer M. Since a^2+4 is square-free, we have that $a^2+4 \mid M$, hence $N^2 \pm 4 = (a^2+4)M_1^2$, where M_1 is a positive integer. Now we use the same arguments as in the proof of Theorem 15, for the special Pell equations (20) and (22). We obtain that $N = V_{2n+1}(a,-1)$ for the positive signed equation, and $N = V_{2n}(a,-1)$ for the negative signed equation, respectively. □

We present the following two corollaries. The first of these recovers known results for the sequences of Fibonacci and Lucas numbers, whereas the second is a new result concerning the bronze Fibonacci and bronze Lucas numbers (see subsection 5.3 for the definitions).

Corollary 17. 1° *(Gessel, Theorem 5.4 in [9]) A positive integer N is a Fibonacci number if and only if $5N^2 \pm 4$ is a perfect square.*

2° *(Wulczyn, Theorem 5.10 in [9]) A positive integer N is a Lucas number if and only if $5N^2 \pm 20$ is a perfect square.*

Regarding 2°, we remark that in Theorem 5.10 of [9] the authors just presented the proof for the easy left to right implication, mentioning that they omitted the proof of the converse since it was too complicated.

Corollary 18. 1° *A positive integer N is a bronze Fibonacci number if and only if $13N^2 \pm 4$ is a perfect square.*

2° *A positive integer N is a bronze Lucas number if and only if $13N^2 \pm 52$ is a perfect square.*

4.4 Perfect powers among $\{U_n(a, \pm 1)\}_{n\geq 0}$ and $\{V_n(a, \pm 1)\}_{n\geq 0}$

In the previous subsections we have discussed methods for identifying whether a positive integer N was a term of $\{U_n(a,b)\}_{n\geq 0}$ or $\{V_n(a,b)\}_{n\geq 0}$. Here we prove a result in a slightly different direction. Numerical evidence suggests that perfect powers appear very rarely as terms of these sequences. Moreover, the same numerical experience suggests that perfect powers appear only among the first terms of such a sequence. We will now show that this is a consequence of a deep theorem of Siegel. Namely, we prove that for any given k, the sequences $\{U_n(a,b)\}_{n\geq 0}$ and $\{V_n(a,b)\}_{n\geq 0}$ contain only finitely many kth powers.

Let us make a few remarks about the finiteness of the set of integral points on various curves. For any bivariate polynomial $f \in \mathbb{Z}[X,Y]$, let $C_f := \{(x,y) \in \overline{\mathbb{Q}}^2 : f(x,y) = 0\}$ be an affine algebraic curve. For any $S \subseteq \overline{\mathbb{Q}}$, we denote by $C_f(S) = C_f \cap S^2$. Curves can be classified by their genus, a non-negative integer associated to their projectivization. The genus is a geometric invariant. A classical result in number theory is the the following.

Theorem 19 (Siegel, 1929)**.** *If $f \in \mathbb{Z}[X,Y]$ defines an irreducible curve C_f of genus $g(C_f) > 0$, then $C_f(\mathbb{Z})$ is finite.*

We now fix $b \in \{-1, 1\}$ and D any positive integer, not necessarily square free. Recall from Lemma 9 that the terms U_n and V_n of such a sequence satisfy the relation $V_n(a,\pm 1)^2 - DU_n(a, \pm 1)^2 = 4(\pm 1)^n$. We now present our result.

Theorem 20. *Fix $b \in \{-1, 1\}$ and let $k \geq 2$ and $D = a^2 - 4b$ be positive integers. The sequences $\{U_n(a,b)\}_{n\geq 0}$ and $\{V_n(a,b)\}_{n\geq 0}$ contain only finitely many perfect kth powers.*

Proof. Denote by $f_+ = X^2 - DY^{2k} - 4$ and by $f_- = X^2 - DY^{2k} + 4$ two bivariate polynomials with integer coefficients.

If $U_n(a, \pm 1)$ is a perfect kth power, then the pair $(V_n(a, \pm 1), U_n(a, \pm 1))$ gives an integral point on one of the hyperelliptic curves C_{f_+} or C_{f_-}. Both of the latter have genus $g = \frac{2k-2}{2} = k - 1 \geq 1$, therefore by Siegel's theorem contain finitely many points.

We just showed that the infinite sequence $U_n(a, \pm 1)$ contains finitely many kth perfect powers. The proof of the analogous result for the sequence $V_n(a, \pm 1)$ is similar. □

Recall that for $a = 1$ and $b = -1$, the sequence $\{U_n(1, -1)\}_{n \geq 0}$ is the Fibonacci sequence. On page 64 of [12], the author shows that the only perfect squares in the Fibonacci sequence are $F_0 = 1$, $F_1 = 1$, $F_2 = 1$ and $F_{12} = 144$. A very deep result in number theory whose proof makes use of Wiles' modularity theorem is Theorem 1 in [7], which asserts that the only perfect powers in the Fibonacci sequence are the perfect squares above and the cube $F_6 = 8$.

Remark. We expect that a result analogous to Theorem 20 holds for fixed general $b \in \mathbb{Z}$, but its proof does not follow from Siegel's theorem (or any result that guarantees finiteness of integral points on a given curve). To see this, we observe that a perfect power among the terms of the sequence $\{U_n(a, b)\}_{n \geq 0}$ gives rise to an integral point on the infinite family of curves C_{f_n}, defined by $f_n = X^2 - DY^{2k} - 4b^n$, where D, k and b are fixed.

Although Siegel's theorem implies that there are finitely many integral points on each one of this curves, this does not prove that there are finitely many integral points on the union (over $n \in \mathbb{N}$) of the curves C_{f_n}. In fact, one can easily see that for even n and $b \geq 2$, the point $(2b^{n/2}, 0)$ is always an integral point on C_{f_n}, hence the union of all such curves contains infinitely many points. We would be surprised if results analogous to the aforementioned theorem could be proved for general b without making use of a major future breakthrough in the fields of Diophantine Equations or Arithmetic Geometry.

Let us take, for instance, $a = 1$ and $b = -2$, in which case $\{U_n(1, -2)\}_{n \geq 0}$ is the sequence of Jacobsthal numbers (see subsection 5.6). For every $n \geq 0$, the term $U_n(1, -2)$ is equal to $\frac{2^n - (-1)^n}{3}$. We propose following conjecture.

Conjecture. *Let $k \geq 2$ be a fixed integer. Then the sequence $\left\{ \frac{2^n - (-1)^n}{3} \right\}_{n \geq 0}$ of Jacobsthal numbers contains only finitely many kth powers.*

Theorem 1 in [6] implies that for even values of n, $\frac{2^n - 1}{3}$ is never a perfect power of some integer $y > 1$. On the other hand, for odd values of $n > 1$, the results of [11] assert that $\frac{2^n + 1}{3}$ is never a perfect square. To complete our conjecture and show that given $k \geq 2$, the number $\frac{2^n + 1}{3}$ is a perfect kth power just for finitely many odd values of n seems to be a difficult task.

5 Some associated integer sequences

In this section we are going to analyse the sequences $u(a,b;x)$ and $v(a,b;x)$ for the parameter values given in Table 1, and we find some new integer sequences, not currently indexed in the Online Encyclopedia [13]. We will compare the exact results obtained by direct counting of terms from the definitions (5), against the formulae proposed in Theorems 3 and 4.

5.1 $(a,b) = (1,-1)$ (Fibonacci and Lucas sequences)

If $n \geq 0$ is an integer, then $U_n(1,-1)$ and $V_n(1,-1)$ are the nth Fibonacci and Lucas numbers. The sequence $u(1,-1;n)$ counting the Fibonacci numbers $F_k \leq n$ is indexed as $A108852$ in [13]. The number of Lucas terms L_k less or equal to n given by $v(1,-1;n)$ is indexed as $A130245$ in [13]. The encyclopedia gives little information about these sequences, but this includes some formulae which are particular cases of (7) and (8). The Fibonacci and Lucas sequences satisfy the conditions in Theorems 3 and 4, so the formulae (7) and (8) are in perfect agreement with the exact count. The starting values for these sequences are given in Table 2.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	$\cdots$
$U_n(1,-1)$	0	1	1	2	3	5	8	13	21	34	55	89	144	233	377	610	$\cdots$
$u(1,-1;n)$	1	3	4	5	5	6	6	6	7	7	7	7	7	8	8	8	$\cdots$
Formula (7)	1	3	4	5	5	6	6	6	7	7	7	7	7	8	8	8	$\cdots$
$V_n(1,-1)$	2	1	3	4	7	11	18	29	47	76	123	199	322	521	843	1364	$\cdots$
$v(1,-1;n)$	0	1	2	3	4	4	4	5	5	5	5	6	6	6	6	6	$\cdots$
Formula (8)	0	1	2	3	4	4	4	5	5	5	5	6	6	6	6	6	$\cdots$

Table 2: Values for Fibonacci and Lucas sequences.

5.2 $(a,b) = (2,-1)$ (Pell and Pell-Lucas sequences)

If $n \geq 0$ is an integer, then $U_n(2,-1)$ and $V_n(2,-1)$ are the nth Pell and Pell-Lucas numbers. The sequences $u(2,-1;n)$ counting the terms for which $P_k \leq n$, and $v(2,-1;n)$ counting the terms with $Q_k \leq n$ are not in [13].

The Pell and Pell-Lucas sequences satisfy the conditions in Theorems 3 and 4, hence the formulae (7) and (8) match the count for $\{U_n\}_{n\geq 0}$ and $\{V_n\}_{n\geq 1}$. The first terms of these sequences are shown in Table 3.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	$\cdots$
$U_n(2,-1)$	0	1	2	5	12	29	70	169	408	985	2378	5741	13860	33461	$\cdots$
$u(2,-1;n)$	1	2	3	3	3	4	4	4	4	4	4	4	5	5	$\cdots$
Formula (7)	1	2	3	3	3	4	4	4	4	4	4	4	5	5	$\cdots$
$V_n(2,-1)$	2	2	6	14	34	82	198	478	1154	2786	6726	16238	39202	94642	$\cdots$
$v(2,-1;n)$	0	0	2	2	2	2	3	3	3	3	3	3	3	3	$\cdots$
Formula (8)	0	1	2	2	2	2	3	3	3	3	3	3	3	3	$\cdots$

Table 3: Values for Pell and Pell-Lucas sequences.

5.3 $(a,b) = (3,-1)$ (bronze Fibonacci and bronze Lucas sequences)

When $n \geq 0$ is an integer, $U_n(3,-1)$ is the nth bronze Fibonacci number (linked fatty acids [14]), while $V_n(3,-1)$ is the nth bronze Lucas number. The sequences $u(3,-1;n)$ counting the terms $U_k(3,-1) \leq n$, and $v(3,-1;n)$ counting the terms which $V_k(3,-1) \leq n$, are not currently indexed in [13].

The Pell and Pell-Lucas sequences satisfy the conditions in Theorems 3 and 4, so the formulae (7) and (8) match the exact count for $\{U_n\}_{n\geq 0}$ and $\{V_n\}_{n\geq 2}$. The first terms of these sequences are shown in Table 4.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	$\cdots$
$U_n(3,-1)$	0	1	3	10	33	109	360	1189	3927	12970	42837	141481	467280	$\cdots$
$u(3,-1;n)$	1	2	2	3	3	3	3	3	3	3	4	4	4	$\cdots$
Formula (7)	1	2	2	3	3	3	3	3	3	3	4	4	4	$\cdots$
$V_n(3,-1)$	2	3	11	36	119	393	1298	4287	14159	46764	154451	510117	1684802	$\cdots$
$v(3,-1;n)$	0	0	1	2	2	2	2	2	2	2	2	3	3	$\cdots$
Formula (8)	0	1	1	2	2	2	2	2	2	2	2	3	3	$\cdots$

Table 4: Values for bronze Fibonacci and bronze Lucas sequences.

5.4 $(a,b) = (3,1)$ (bisection of Fibonacci and of Lucas sequences)

If $n \geq 0$ is an integer, then $U_n(3,1)$ and $V_n(3,1)$ are the nth term of the bisection of Fibonacci, and the bisection of Lucas numbers, respectively. The sequence $u(3,1;n)$ counting the terms for which $U_k(3,1) \leq n$, is indexed as $A130260$, while the sequence $v(3,1;n)$ counting $V_k(3,1) \leq n$ is new.

The bisection of Fibonacci and Lucas sequences satisfy the conditions in Theorems 3 and 4, so the formulae (7) and (8) match the exact count once $\{U_n\}_{n\geq 0}$ and $\{V_n\}_{n\geq 1}$ are increasing. The first terms are shown in Table 5.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	$\cdots$
$U_n(3,1)$	0	1	3	8	21	55	144	377	987	2584	6765	17711	46368	$\cdots$
$u(3,1;n)$	1	2	2	3	3	3	3	3	4	4	4	4	4	$\cdots$
Formula (7)	1	2	2	3	3	3	3	3	4	4	4	4	4	$\cdots$
$V_n(3,1)$	2	3	7	18	47	123	322	843	2207	5778	15127	39603	103682	$\cdots$
$v(3,1;n)$	0	0	1	2	2	2	2	3	3	3	3	3	3	$\cdots$
Formula (8)	0	1	1	2	2	2	2	3	3	3	3	3	3	$\cdots$

Table 5: The bisection of Fibonacci and bisection of Lucas sequences.

5.5 $(a, b) = (3, 2)$ (Mersenne and Pisot numbers)

If $n \geq 0$ is an integer, then $U_n(3,2)$ and $V_n(3,2)$ are the nth Mersenne and Pisot numbers. The sequences $u(3,2;n)$ counting how many terms satisfy $U_k(3,2) \leq n$, and $v(3,2;n)$ counting terms which satisfy $V_k(3,2) \leq n$ are new.

The Mersenne sequence satisfies (barely) the conditions in Theorem 3, hence the formula (7) is in perfect agreement with the exact count for $\{U_n\}_{n\geq 0}$. The Pisot sequence does not meet the conditions in Theorem 4, and the results from formula (8) for $\{V_n\}_{n\geq 1}$ deviate from the exact count for $v(3,2;n)$.

The first terms of these sequences are shown in Table 6.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	$\cdots$
$U_n(3,2)$	0	1	3	7	15	31	63	127	255	511	1023	2047	4095	$\cdots$
$u(3,2;n)$	1	2	2	3	3	3	3	4	4	4	4	4	4	$\cdots$
Formula (7)	1	2	2	3	3	3	3	4	4	4	4	4	4	$\cdots$
$V_n(3,2)$	2	3	5	9	17	33	65	129	257	513	1025	2049	4097	$\cdots$
$v(3,2;n)$	0	0	1	2	2	3	3	3	3	4	4	4	4	$\cdots$
Formula (8)	0	1	2	2	3	3	3	3	4	4	4	4	4	$\cdots$

Table 6: Values for the Mersenne and Pisot sequences.

5.6 $(a,b) = (1,-2)$ (Jacobsthal and Jacobsthal-Lucas numbers)

When $n \geq 0$ is an integer, $U_n(1,-2)$ and $V_n(1,-2)$ are the nth Jacobsthal and Lucas-Jacobsthal numbers. The sequence $u(1,-2;n)$ counting the terms for which $U_k(1,-2) \leq n$, is indexed as $A130253$ in [13], where the following formulae are also given $u(1,-2,n) = \lfloor \log_2(3n+1) \rfloor + 1 = \lceil \log_2(3n+2) \rceil$.

The sequence $v(1,-2;n)$ counting terms with $V_k(1,-2) \leq n$ is new. The Jacobsthal sequence satisfies (barely, $\beta = -1$) the conditions in Theorem 3, hence the formula (7) is in perfect agreement with the exact count for $\{U_n\}_{n\geq 0}$. The Jacobsthal-Lucas sequence does not meet the conditions in Theorem 4, hence the results from (8) for $\{V_n\}_{n\geq 1}$ deviate from the exact count.

The first terms of these sequences are shown in Table 7.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	$\cdots$
$U_n(1,-2)$	0	1	1	3	5	11	21	43	85	171	341	683	1365	$\cdots$
$u(1,-2;n)$	1	3	3	4	4	5	5	5	5	5	5	6	6	$\cdots$
Formula (7)	1	3	3	4	4	5	5	5	5	5	5	6	6	$\cdots$
$V_n(1,-2)$	2	1	5	7	17	31	65	127	257	511	1025	2047	4097	$\cdots$
$v(1,-2;n)$	0	1	2	2	2	3	3	4	4	4	4	4	4	$\cdots$
Formula (8)	0	1	2	2	3	3	3	3	4	4	4	4	4	$\cdots$

Table 7: Values for the Jacobsthal and Jacobsthal-Lucas sequences.

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