DOI: 10.2478/auom-2020-0041
An. Şt. Univ. Ovidius Constanţa **Vol. 28(3),2020, 193–216**

δ_{ss}-supplemented modules and rings

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Abstract

In this paper, we introduce the concept of δ_{ss}-supplemented modules and provide the various properties of these modules. In particular, we prove that a ring R is δ_{ss}-supplemented as a left module if and only if $\frac{R}{Soc(_RR)}$ is semisimple and idempotents lift to $Soc(_RR)$ if and only if every left R-module is δ_{ss}-supplemented. We define projective δ_{ss}-covers and prove the rings with the property that every (simple) module has a projective δ_{ss}-cover are δ_{ss}-supplemented. We also study on δ_{ss}-supplement submodules.

1 Introduction

Throughout this paper, all rings are associative with identity and all modules are unitary left modules. Let R be such a ring and M be an R-module. The notation $N \subseteq M$ means that N is a submodule of M. $Soc(M)$ and $Rad(M)$ will stand for the socle of M and the radical of M. Let M be a module. A submodule $L \subseteq M$ is said to be *essential* in M, denoted as $L \trianglelefteq M$, if $L \cap N \neq 0$ for every non-zero submodule $N \subseteq M$. A module M is called *singular* if $M \cong \frac{N}{L}$ for some module N and an essential submodule $L \trianglelefteq N$. As a dual to the notion of an essential submodule, a submodule N of M is said to be *small* in M, denoted by $N \ll M$, if $M \neq N + K$ for every proper submodule K of M ([13, 19.1]). A non-zero module M is called *hollow* if every proper submodule of M is small in M, and it is called *local* if it is hollow and finitely generated.

Key Words: semisimple module, strongly δ-local module, δ_{ss}-supplemented module, left δ_{ss}-perfect ring, projective δ_{ss}-cover
2010 Mathematics Subject Classification: Primary 16D10; 16D60 Secondary 16D99.
Received: 08.10.2019.
Accepted: 09.01.2020.

Let M be a module and U, V be submodules of M. The submodule V is said to be *supplement* of U in M or U is said to have a *supplement* V in M if V is minimal with respect to $M = U + V$. It is well known that a submodule V of M is a supplement of U in M if and only if $M = U + V$ and $U \cap V \ll V$. M is called *supplemented* if every submodule U of M has a supplement in M. A submodule U of M has *ample supplements* in M if every submodule L of M such that $M = U + L$ contains a supplement of U in M. The module M is called *amply supplemented* if every submodule of M has ample supplements in M. Semisimple modules and hollow modules are (amply) supplemented ([13, 41]).

Zhou [15] generalizes small submodules to δ-small submodules of a module M as follows. A submodule $N \subseteq M$ is said to be *δ-small* in M and indicated by $N \ll_\delta M$ if $M \neq N + K$ for every proper submodule K of M with $\frac{M}{K}$ singular. It is clear that every small submodule or projective semisimple submodule of M is δ-small in M. By $\delta(M)$ we will denote the sum of all δ-small submodules of M as in [15, Lemma 1.5 (2)]. Since $\mathrm{Rad}(M)$ is the sum of all small submodules of M, it follows that $\mathrm{Rad}(M) \subseteq \delta(M)$ for a module M. For an arbitrary ring R, let $\delta(R) = \delta({}_RR)$.

Let M be a module. In [7], M is said to be *δ-supplemented* if every submodule U of M has a *δ-supplement* V in M, that is, $M = U + V$ and $U \cap V \ll_\delta V$. The module M is called *amply δ-supplemented* if, whenever $M = U + V$, U has a δ-supplement $V^{'} \subseteq V$. Clearly, every (amply) supplemented module is (amply) δ-supplemented. For characterizations of supplemented and δ-supplemented modules we refer to [1], [7] and [13].

In [6], the authors define ss-supplemented modules as a proper generalization of semisimple modules. A module M is said to be *ss-supplemented* if every submodule U of M has a supplement V in M such that $U \cap V$ is semisimple. They give in the same paper the structure of ss-supplemented modules. In particular, it is shown in [6, Theorem 41] that a ring R is semiperfect and $\mathrm{Rad}(R) \subseteq Soc({}_RR)$ if and only if every left R-module is ss-supplemented if and only if ${}_RR$ is the finite sum of strongly local submodules. Here a module M is called *strongly local* if it is local and the radical is semisimple ([6]).

Motivated by these results, we introduce the concept of δ_{ss}-supplemented modules. In this paper, we study on δ_{ss}-supplemented modules and we obtain the various properties of these modules. We show that strongly δ-local (see below) modules are δ_{ss}-supplemented. Every direct sum of strongly δ-local modules and projective semisimple modules is coatomic. The class of δ_{ss}-supplemented modules is closed under finite sums and factor modules. We prove that a module M with δ-small $\delta(M)$ is δ_{ss}-supplemented if and only if it δ-supplemented and $\delta(M) \subseteq Soc(M)$. We study on the rings with the property that every left module is δ_{ss}-supplemented and call these rings left

δ_{ss}-perfect. We also show that a ring R is left δ_{ss}-perfect if and only if ${}_RR$ is δ_{ss}-supplemented if and only if $\frac{R}{Soc({}_RR)}$ is semisimple and idempotents lift to $Soc({}_RR)$ if and only if for any module every maximal submodule has a δ_{ss}-supplement in the module. We define projective δ_{ss}-covers and prove that a ring is left δ_{ss}-perfect if and only if every left module has a projective δ_{ss}-cover if and only if every semisimple left module has a projective δ_{ss}-cover if and only if every simple left R-module has a projective δ_{ss}-cover. We also study on δ_{ss}-supplement submodules.

The following lemma follows from [15, Lemma 1.2] and we will use it throughout the paper.

Lemma 1.1. *Let M be a module. A submodule $N \subseteq M$ is δ-small in M if and only if whenever $X + N = M$ there exists a projective semisimple submodule $N^{'}$ of N such that $X \oplus N^{'} = M$.*

It is obvious that a module M is projective semisimple if and only if $M \ll_\delta M$. A ring R is called *local* if ${}_RR$ (or R_R) is a local module.

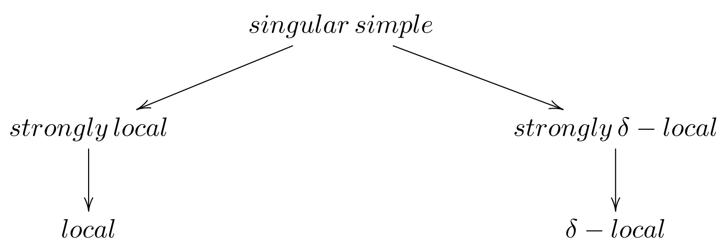
Remark 1.2. Let R be a commutative domain (which is not field) or a local ring and M be a non-zero R-module. Suppose that a submodule N of M is δ-small in M. Let $M = N + K$ for some submodule K of M. Then there exists a projective semisimple submodule $N^{'}$ of N such that $M = N^{'} \oplus K$. By [12, Proposition 2.5], we get that $N^{'} = 0$ and so $K = M$. It means that N is a small submodule of M.

2 Strongly δ-Local Modules

It is well known that M is local if and only if $\mathrm{Rad}(M) \ll M$ and $\mathrm{Rad}(M)$ is maximal. Using this characterization, δ-local modules are defined in [4]. A module M is called *δ-local* if $\delta(M) \ll_\delta M$ and $\delta(M)$ is maximal. Maybe, it is expected that local modules are also δ-local. But unfortunately, it is not the case. Let S be a simple module. Since S is projective or singular, it is $\delta(S) = S$ or δ-local. It follows that a projective simple module is local but not δ-local.

As we have mentioned in the introduction, a module M is *strongly local* if it is local and $\mathrm{Rad}(M)$ is semisimple ([6]). Note that every simple module is strongly local.

We say that a module M *strongly δ-local* if it is δ-local and $\delta(M) \subseteq Soc(M)$. It is clear that every strongly δ-local module is δ-local but the converse is not true in general. For example, let M be the left $\mathbb{Z}$-module $\mathbb{Z}_8$. Then M is δ-local but not strongly δ-local. Then we have the following implications on modules:



We start the next lemma which are taken from [15, Lemma 1.3 and Lemma 1.5]. Recall that a module M *coatomic* if every proper submodule of M is contained in a maximal submodule of M. Note that a coatomic module has small radical.

Lemma 2.1. *Let M be a module.*

(1) For any submodules N and L of M, $N+L \ll_\delta M$ if and only if $N \ll_\delta M$ and $L \ll_\delta M$.

(2) If $K \ll_\delta M$ and $f : M \longrightarrow N$ is a homomorphism, then $f(K) \ll_\delta N$. In particular, if $M \subseteq N$, then $K \ll_\delta N$.

(3) If $f : M \longrightarrow N$ is a homomorphism, then $f(\delta(M)) \subseteq \delta(N)$.

(4) If $M = \bigoplus_{i\in I} M_i$, then $\delta(M) = \bigoplus_{i\in I} \delta(M_i)$.

(5) If M is coatomic, then $\delta(M)$ is the unique largest δ-small submodule of M.

It is well known that every (strongly) local module is indecomposable. On the other hand, the following theorem gives a characterization of a semisimple module which is strongly δ-local. Firstly we need the following facts.

Lemma 2.2. *Let M be a module and let N be a semisimple submodule of M such that $N \subseteq \delta(M)$. Then $N \ll_\delta M$.*

Proof. Let K be a submodule such that $M = N + K$. Since N is semisimple, then there exists a semisimple submodule X of N such that $N = (N\cap K)\oplus X$. Therefore $M = [(N \cap K) \oplus X] + K = X \oplus K$.

Next we prove that X is projective. Let $X = \oplus_{i\in I} S_i$, where I is some index set and each S_i is simple. Since $X \subseteq N \subseteq \delta(M)$, by the modular law, we have $\delta(M) = \delta(M)\cap M = \delta(M)\cap(X\oplus K) = X\oplus(K\cap\delta(M)) = X\oplus\delta(K)$. Note that, by Lemma 2.1 (4), $\delta(M) = \delta(X) \oplus \delta(K)$. Therefore $X = \delta(X)$. Let $\pi_i : X \longrightarrow S_i$ be the canonical projection. It follows from Lemma 2.1 (3) that $S_i = \pi_i(X) = \pi_i(\delta(X)) \subseteq \delta(S_i)$ and so $\delta(S_i) = S_i$, for all $i \in I$. This implies that each S_i is projective for all $i \in I$. Then $X = \oplus_{i \in I} S_i$ is projective as the direct sum of projective submodules. Hence $N \ll_\delta M$. □

Observe from Lemma 2.2 that a module M is strongly δ-local if and only if $\delta(M)$ is maximal and semisimple. It follows that a semisimple module is strongly δ-local if and only if $\delta(M)$ is maximal. The following result is a direct consequence of Lemma 2.2.

Corollary 2.3. *Let M be a module. Then M is semisimple and $\delta(M) = M$ if and only if it is projective semisimple.*

Theorem 2.4. *Let M be a semisimple module. Then M is strongly δ-local if and only if M has the decomposition $M = M_1 \oplus M_2$, where M_1 is a projective semisimple submodule and M_2 is a singular simple submodule.*

Proof. ($\Longrightarrow$) Let M be a strongly δ-local module. Since $\delta(M)$ is maximal and M is semisimple, there exists a simple submodule M_2 of M such that $M = \delta(M) \oplus M_2$. Put $M_1 = \delta(M)$. Since $M_1 = \delta(M) \ll_\delta M$, it follows from Lemma 2.1 (2) that $M_1 \ll_\delta M_1$ and so M_1 is semisimple projective by Corollary 2.3. Therefore $\delta(M_2) \subseteq \delta(M) \cap M_2 = 0$ and so $\delta(M_2) = 0$. It means that M_2 is singular. Hence we get the decomposition $M = M_1 \oplus M_2$ as desired.

($\Longleftarrow$) Clearly, $\delta(M_1) = M_1 \ll_\delta M_1$ and $\delta(M_2) = 0 \ll_\delta M$. It follows from Lemma 2.1 (2)-(4) that $\delta(M) = \delta(M_1) \oplus \delta(M_2) = M_1 \oplus 0$ is δ-small in M. Since $\delta(M)$ is maximal, we deduce that M is strongly δ-local. □

Observe from Theorem 2.4 that any factor module (in particular, direct summand) of a strongly δ-local module need not be strongly δ-local in general.

Proposition 2.5. *Let M be an indecomposable module. If M is strongly δ-local, then it is strongly local.*

Proof. If M is simple, then it is singular simple because M is strongly δ-local. Suppose that M is not singular simple. Since M is indecomposable, we get that $Soc(M) \subseteq \text{Rad}(M)$. This implies that $Soc(M) \ll M$. Since M is strongly δ-local, we have $\delta(M) \subseteq Soc(M)$ and so $\delta(M) = Soc(M)$ is maximal. Therefore $Soc(M) = \text{Rad}(M)$. Thus M is strongly local. □

Proposition 2.6. *Let R be a local ring. If M is a strongly δ-local R-module, then it is a strongly local R-module.*

Proof. By Remark 1.2. □

Proposition 2.7. *Let M be a module. Assume that $\frac{M}{\delta(M)}$ is semisimple. Then M is coatomic if and only if $\delta(M)$ is δ-small in M.*

Proof. ($\Longrightarrow$) By Lemma 2.1 (5).

($\Longleftarrow$) If $\delta(M) = M$, then clearly $M \ll_\delta M$ and so M is projective semisimple. Let $\delta(M) \neq M$ and let U be any submodule of M. If $U + \delta(M) = M$, then there exists a (projective) semisimple submodule S of $\delta(M)$ such that $U \oplus S = M$. Let $S = \oplus_{i\in I} S_i$, where $(i \in I)$ S_i is simple and I is some index set. For some $i_0 \in I$, put $U^{'} = U \oplus (\oplus_{i\in I\setminus\{i_0\}} S_i)$. Then clearly $U \subseteq U^{'}$. Therefore $\frac{M}{U'} \cong S_{i_0}$ and hence $U^{'}$ is a maximal submodule of M. Suppose that $U + \delta(M) \neq M$. Then $\frac{U+\delta(M)}{\delta(M)}$ is a proper submodule of $\frac{M}{\delta(M)}$. Since $\frac{M}{\delta(M)}$ is semisimple, there exists a maximal submodule $\frac{K}{\delta(M)}$ of $\frac{M}{\delta(M)}$ such that $\frac{U+\delta(M)}{\delta(M)} \subseteq \frac{K}{\delta(M)}$. So K is a maximal submodule of M which contains U. It means that M is coatomic. □

Recall that a module M is called *radical* if M has no maximal submodules, that is, $M = Rad(M)$. Let $P(M)$ be the sum of all radical submodules of M. It is easy to see that $P(M)$ is the largest radical submodule of M. If $P(M) = 0$, M is called *reduced*.

Corollary 2.8. *Any strongly δ-local module is reduced and coatomic.*

Proof. Let M be a strongly δ-local module. Therefore $\delta(M) \subseteq Soc(M)$. Since $\text{Rad}(M) \subseteq \delta(M)$, it follows that M is reduced. Since $\frac{M}{\delta(M)}$ is simple, we get M is coatomic by Proposition 2.7. □

Theorem 2.9. *Let $M = \oplus_{i\in I} M_i$, where each M_i is either strongly δ-local or projective semisimple. Then M is coatomic.*

Proof. Note that $\frac{M}{\delta(M)} = \pi(M) \cong \bigoplus_{i\in I} \frac{M_i}{\delta(M_i)}$. Let $i_0 \in I$. If M_{i_0} is projective semisimple, then $\delta(M_{i_0}) = M_{i_0}$ and so the factor module $\frac{M_{i_0}}{\delta(M_{i_0})} = 0$. It follows that we can consider the module $\frac{M}{\delta(M)}$ is the direct sum of simple modules $\frac{M_k}{\delta(M_k)}$, where $(k \in \Lambda)$ M_k is strongly δ-local and $\Lambda \subseteq I$. Thus $\frac{M}{\delta(M)}$ is semisimple.

By Proposition 2.7, it is enough to prove that $\delta(M)$ is δ-small in M. By the hypothesis, we have $\delta(M_i) \subseteq Soc(M_i)$. Applying Lemma 2.1 (4) and [13, 21.2 (5)], we obtain that $\delta(M) = \oplus_{i\in I}\delta(M_i) \subseteq \oplus_{i\in I} Soc(M_i) = Soc(M)$. That is, $\delta(M)$ is semisimple. It follows from Lemma 2.2 that $\delta(M)$ is δ-small in M. This completes the proof. □

3 δ_{ss}-Supplement Submodules

Let M be a module. By $Soc_s(M)$ we denote the sum of all simple submodules of M that are small in M as in [14]. Since every small submodule of M

is δ-small in M, the notation motives us to introduction the sum all simple submodules of M that are δ-small in M. For a module M, let

$$Soc_\delta(M) = \sum\{\, S \subseteq M \mid S\, is\, simple\, and\, S \ll_\delta M \,\}.$$

The properties of $Soc_\delta(M)$ for a module M are given in the next proposition.

Proposition 3.1. *Let R be a ring and M be a left R-module. Then:*

(1) $Soc_\delta(M) = Soc(M) \cap \delta(M)$,

(2) $Soc_\delta(M) \ll_\delta M$,

(3) $Rad(Soc_\delta(M)) = 0$,

(4) $Soc_\delta(M) = M$ if and only if M is projective semisimple,

(5) If $M^{'}$ is a left R-module and $f : M \longrightarrow M^{'}$ is a homomorphism, then $f(Soc_\delta(M)) \subseteq Soc_\delta(f(M))$.

Proof. (1) Let $x \in \delta(M) \cap Soc(M)$. Then $Rx \ll_\delta M$ and Rx is semisimple. So there exist $m \in \mathbb{Z}^+$ and simple submodules S_i of M for every $i \in \{1, 2, \ldots, m\}$ such that $Rx = S_1 \oplus S_2 \oplus \cdots \oplus S_m$ by [10, Proposition 3.3]. Since $Rx \ll_\delta M$, it follows from Lemma 2.1 (2) that each $S_i \ll_\delta M$. Thus $x \in Rx \subseteq Soc_\delta(M)$. The converse is clear by the definition of $Soc_\delta(M)$.

(2) Clearly, $Soc_\delta(M)$ is semisimple. Then the proof follows from Lemma 2.2.

(3) Since semisimple modules have zero radical, it is clear.

(4) Let $Soc_\delta(M) = M$. By (1), we get M is semisimple and $\delta(M) = M$. Hence M is projective semisimple by Corollary 2.3. The converse is clear.

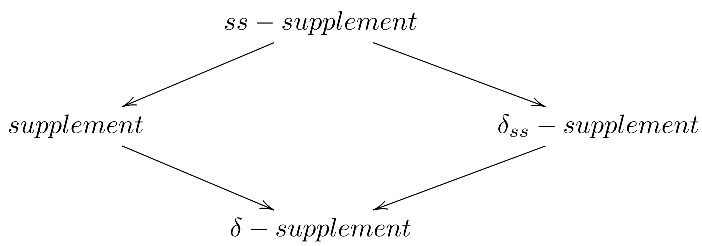
(5) Let $f : M \longrightarrow M^{'}$ be a homomorphism of modules and $x \in f(Soc_\delta(M))$. Then $x = f(m)$ for some element $m \in Soc_\delta(M)$. Applying (1), we obtain that $m \in Soc(M) \cap \delta(M)$. Therefore $x = f(m) \in f(Rm) \subseteq Soc(f(M))$ by [13, 21.2 (1)] and $x = f(m) \in f(Rm) \subseteq \delta(f(M))$ by Lemma 2.1 (3). It means that $x \in Soc(f(M)) \cap \delta(f(M))$. Again applying (1), we have $x \in Soc_\delta(f(M))$. □

Let M be a module and S be a simple submodule of M. Then $S \ll M$ or we have the decomposition $M = S \oplus K$ for some submodule K of M. Using this fact we have:

Corollary 3.2. *Let M be a module and let S be a simple submodule of M. Then $S \ll_\delta M$ if and only if S is projective or small in M.*

Proof. Let $S \ll_\delta M$. Suppose that S is not small in M. Then we get $M = S \oplus K$. By the assumption, S is projective as desired. The converse is clear. □

Let M be a module and U, V be submodules of M. Following [6], V is called *ss-supplement of* U *in* M if $M = U + V$ and $U \cap V \subseteq Soc_s(V)$. For any left module X, we have $Soc_s(X) \subseteq Soc_\delta(X)$ and so it is natural to introduce another notion that we called δ_{ss}-supplement. A submodule V of M is called δ_{ss}*-supplement of* U *in* M if $M = U + V$ and $U \cap V \subseteq Soc_\delta(V)$. Under given definitions we obtain the following diagram:



Modifying of [6, Lemma 3] we characterize δ_{ss}-supplement submodules of a module M. Note that we shall freely use the next lemma without reference in this paper.

Lemma 3.3. *Let M be a module and U, V be submodules of M. Then the following statements are equivalent.*

(1) V is a δ_{ss}-supplement of U in M,

(2) $M = U + V$, $U \cap V \subseteq \delta(V)$ and $U \cap V$ is semisimple,

(3) $M = U + V$, $U \cap V \ll_\delta V$ and $U \cap V$ is semisimple.

Proof. Using Proposition 3.1, we have clearly (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (1).
(2) $\Rightarrow$ (3) It follows from Lemma 2.2. □

Proposition 3.4. *Let M be a module and U be a maximal submodule of M. If U has a δ_{ss}-supplement V in M, then V is strongly δ-local or projective semisimple*

Proof. Let V be a δ_{ss}-supplement of U in M. Then $M = U + V$, $U \cap V \subseteq \delta(V)$ and $U \cap V$ is semisimple. Note that $\frac{M}{U} \cong \frac{V}{U \cap V}$ is simple and thus $U \cap V$ is a maximal submodule of V. Hence $\delta(V) = U \cap V$ or $\delta(V) = V$. If $\delta(V) = U \cap V$, then $\delta(V) \subseteq Soc(V)$. Therefore V is strongly δ-local. Now suppose that $\delta(V) = V$. By [11, Lemma 2.22], we get that V is projective semisimple. □

Proposition 3.5. *Let M be module and let $V \subseteq M$ be a δ_{ss}-supplement in M.*

(1) If L is a submodule of V, then $\frac{V}{L}$ is a δ_{ss}-supplement in $\frac{M}{L}$,

(2) Whenever $V \subseteq K \subseteq M$, V is also a δ_{ss}-supplement in K,

(3) $Soc_\delta(V) = V \cap Soc_\delta(M)$.

Proof. Since V is a δ_{ss}-supplement in M, then there exists a submodule U of M such that $M = U + V$, $U \cap V \ll_\delta V$ and $U \cap V$ is semisimple.

(1) Since $M = U + V$, we have $\frac{M}{L} = (\frac{U+L}{L}) + \frac{V}{L}$. Let $\pi : V \longrightarrow \frac{V}{L}$ be the canonical homomorphism. Then by Lemma 2.1 (2), we obtain that $\pi(U \cap V) = \frac{(U \cap V)+L}{L} = \frac{(U+L)\cap V}{L} = (\frac{U+L}{L}) \cap \frac{V}{L} \ll_\delta \frac{V}{L}$. It follows from [5, 8.1.5 (2)] that $\pi(U \cap V) = (\frac{U+L}{L}) \cap \frac{V}{L}$ is semisimple. It means that $\frac{V}{L}$ is a δ_{ss}-supplement of $\frac{U+L}{L}$ in $\frac{M}{L}$.

(2) By the modular law, we have $K = K \cap M = K \cap (U + V) = U \cap K + V$. Therefore $(U \cap K) \cap V = U \cap V \subseteq Soc_\delta(V)$.

(3) It follows from Proposition 3.1, [4, Corollary 2.5] and [13, 21.2 (2)] that we can write $V \cap Soc_\delta(M) = V \cap [Soc(M) \cap \delta(M)] = [V \cap Soc(M)] \cap [V \cap \delta(M)] = Soc(V) \cap \delta(V) = Soc_\delta(V)$. □

Lemma 3.6. *Let M be a module and let K be a direct summand of M. Then a submodule $V \subseteq K$ is a δ_{ss}-supplement in K if and only if it is a δ_{ss}-supplement in M.*

Proof. ($\Longrightarrow$) By the hypothesis, we have $M = K \oplus L$ where $L \subseteq M$. Since V is a δ_{ss}-supplement in K, then there exists a submodule U of K such that $K = U + V$, $U \cap V << _\delta V$ and $U \cap V$ is semisimple. So $M = (U + V) \oplus L = (U \oplus L) + V$. It can be seen that $(U \oplus L) \cap V = U \cap V$. Hence V is a δ_{ss}-supplement of $U \oplus L$ in M.

($\Longleftarrow$) By Proposition 3.5 (2). □

Theorem 3.7. *Let M be a module. Then M is a δ_{ss}-supplement in every extension if and only if it is a δ_{ss}-supplement in $E(M)$, where $E(M)$ is the injective hull of M.*

Proof. One direction is clear. Conversely, let $M \subseteq N$. Then we have $E(M) \subseteq E(N)$. So by [10, Theorem 2.15], $E(N) = E(M) \oplus L$ for some submodule L of $E(N)$. Since M is a δ_{ss}-supplement in $E(M)$, it follows from Lemma 3.6 that it is a δ_{ss}-supplement in $E(N)$. Hence M is a δ_{ss}-supplement in N by Proposition 3.5 (2). □

Proposition 3.8. *Let M be a module with $Soc_\delta(M) = 0$. Then M is a δ_{ss}-supplement in $E(M)$ if and only if it is injective.*

Proof. Let M be a δ_{ss}-supplement in $E(M)$. Then there exists a submodule N of $E(M)$ such that $E(M) = N + M$ and $N \cap M \subseteq Soc_\delta(M)$. Since $Soc_\delta(M) = 0$, we obtain that $N \cap M = 0$. Thus $E(M) = N \oplus M$. It means that M is injective. The converse is clear. □

Let R be a commutative domain and M be an R-module. We denote by $Tor(M)$ the set of all elements m of M for which there exists a non-zero element r of R such that $rm = 0$, i.e. $Ann(m) \neq 0$. Then $Tor(M)$, which is a submodule of M, called *the torsion submodule* of M. If $M = Tor(M)$, then M is called a *torsion module* and M is called *torsion-free* provided $Tor(M) = 0$.

Let R be a commutative domain which is not field and M be an R-module. Suppose that S is a simple submodule of M. Let m be a non-zero element of S. Then $Rm = S$ and so we can write $S \cong \frac{R}{Ann(m)}$. Since R is not field, $Ann(m) \neq 0$. Therefore, for some non-zero element $r \in R$, we get $rm = 0$. So $m \in Tor(M)$. It means that $Soc(M) \subseteq Tor(M)$. Using this fact and Proposition 3.8, we obtain that the next result. By Remark 1.2, we get that δ_{ss}-supplements are ss-supplements in this case.

Corollary 3.9. *Let R be a commutative domain which is not field and M be a torsion-free R-module. Then M is a ss-supplement in $E(M)$ if and only if it is injective.*

Proof. Since M is torsion-free, we get that $Soc_\delta(M) = 0$. It follows from Proposition 3.8 that the proof is clear. □

Let R be a commutative domain which is not field. R is said to be *one dimensional* if, for every non-zero ideal I of R, $\frac{R}{I}$ is an artinian ring.

Corollary 3.10. *Let R be a one dimensional domain and M be a torsion-free R-module. Then the following statements are equivalent:*

(1) M is a ss-supplement in $E(M)$,

(2) M is injective,

(3) M is radical, i.e. M has no maximal submodules.

Proof. By Corollary 3.9 and [2, Lemma 4.4] □

Proposition 3.11. *Let R be a Dedekind domain and M be an R-module. Then M is a ss-supplement in $E(M)$ if and only if it is injective.*

Proof. Let M be a ss-supplement of some submodule N in $E(M)$. For every non-zero element $r \in R$, we can write $E(M) = rE(M) = rN + rM = N + rM$ and so, by the minimality of M, we obtain that M is divisible. By [2, Lemma 4.4], M is injective. □

A module M is said to be π-*projective* if whenever U and V are submodules of M such that $M = U + V$, there exists an endomorphism f of M such that $f(M) \subseteq U$ and $(1 - f)(M) \subseteq V$. Hollow (local) modules and self-projective modules are π-projective.

Lemma 3.12. *Let M be a π-projective module and U, V be submodules of M. If U and V are mutual δ-supplements in M, then they are mutual δ_{ss}-supplements in M.*

Proof. It follows from [7, Lemma 2.15]. □

Recall from [13, 41.16 (1)] that every supplement submodule of a π-projective supplemented module is a direct summand. Analogous to that we have:

Corollary 3.13. *Let M be a π-projective and δ-supplemented module. Then every δ-supplement in M is δ_{ss}-supplement in M.*

Proof. Let V be a δ-supplement of some submodule U in M. Then $M = U+V$ and $U \cap V \ll_\delta V$. Since M is π-projective and δ-supplemented, it follows from [1, Theorem 4.4] that it is amply δ-supplemented. So V has a δ_{ss}-supplement $U^{'} \subseteq U$ in M. Therefore V and $U^{'}$ are mutual δ-supplements in M. Thus by Lemma 3.12, V is a δ_{ss}-supplement in M. □

Theorem 3.14. *The following conditions are equivalent for a module M with non-zero $\delta(M)$.*

(1) every cyclic submodule of M is a δ_{ss}-supplement in M,

(2) every cyclic submodule of M is a δ-supplement in M,

(3) M is projective semisimple.

Proof. (3) $\Longrightarrow$ (1) and (1) $\Longrightarrow$ (2) are clear.

(2) $\Longrightarrow$ (3) Let $0 \neq m \in \delta(M)$. By (2), there exists a submodule U of M such that $M = U + Rm$ and $U \cap Rm \ll_\delta Rm$. Since $Rm \ll_\delta M$, we can write $M = X \oplus Rm$, where X is a projective semisimple submodule of U. Since Rm is a δ-supplement in M, it follows from [4, Corollary 2.5] that $\delta(Rm) = Rm \cap \delta(M) = Rm$. By Lemma 2.1 (2), we get $\delta(Rm) = Rm \ll_\delta Rm$ and so Rm is projective semisimple. Hence $M = X \oplus Rm$ is projective semisimple. □

Theorem 3.15. *The following conditions are equivalent for a module M with zero $\delta(M)$.*

(1) every (resp., cyclic) submodule of M is a δ_{ss}-supplement in M,

(2) every (resp., cyclic) submodule of M is a δ-supplement in M,

(3) M is (resp., regular) semisimple.

Proof. (3) $\Longrightarrow$ (1) and (1) $\Longrightarrow$ (2) are clear.

(2) $\Longrightarrow$ (3) Since $\delta(M) = 0$, every (cyclic) submodule of M is a direct summand of M and so M is (regular) semisimple. □

It is well known that a ring R is semisimple if and only if, for every left R-module, every submodule is direct summad (see [13, 20.7]). Using Theorem 3.14 and Theorem 3.15, we generalize this fact.

Corollary 3.16. *Let R be a ring. Then R is semisimple if and only if, for every left R-module M, every submodule of M is δ_{ss}-supplement in M.*

4 δ_{ss}-Supplemented Modules

In this section, we define the concept of δ_{ss}-supplemented modules and obtain the basic properties of such modules.

Let M be a module. We say that M *a δ_{ss}-supplemented module* if every submodule U of M has a δ_{ss}-supplement V in M, and M *amply δ_{ss}-supplemented* if in case $M = U + V$ implies that U has a δ_{ss}-supplement $V^{'} \subseteq V$. It is clear that every (amply) ss-supplemented module is (amply) δ_{ss}-supplemented, and (amply) δ_{ss}-supplemented modules are (amply) δ-supplemented.

Now we begin by giving some examples of module to seperate (amply) ss-supplemented, (amply) δ_{ss}-supplemented and (amply) δ-supplemented. Firstly we need the following facts:

Lemma 4.1. *Every strongly δ-local module is δ_{ss}-supplemented.*

Proof. Let M be a strongly δ-local module and U be any submodule of M. If $U \subseteq \delta(M)$, then U is semisimple since $\delta(M)$ is semisimple. By Lemma 2.2, we get $U \ll_{\delta} M$. Thus M is the δ_{ss}-supplement of U in M. Let $U \nsubseteq \delta(M)$. Since $\delta(M)$ is maximal, we can write the equality $M = U + \delta(M)$. Then there exists a projective semisimple submodule V of $\delta(M)$ such that $M = U \oplus V$ because $\delta(M) \ll_{\delta} M$. Hence M is δ_{ss}-supplemented. □

π-projective supplemented modules are amply supplemented. Similarly, we show that π-projective δ_{ss}-supplemented modules are amply δ_{ss}-supplemented. The proof is virtually the same that of [13, 41.15], but we give it for completeness.

Proposition 4.2. *Let M be a π-projective and δ_{ss}-supplemented module. Then M is amply δ_{ss}-supplemented.*

Proof. Let U and V be submodules of M such that $M = U + V$. Since M is π-projective, there exists an endomorphism f of M such that $f(M) \subseteq U$ and $(1-f)(M) \subseteq V$. Note that $(1-f)(U) \subseteq U$. Let $V^{'}$ be a δ_{ss}-supplement of U in M. Then $M = f(M)+(1-f)(M) = f(M)+(1-f)(U+V^{'}) \subseteq U+(1-f)(V^{'}) \subseteq M$, so that $M = U + (1-f)(V^{'})$. Here $(1-f)(V^{'})$ is a submodule of V. Let $y \in U \cap (1-f)(V^{'})$. Then, $y \in U$ and $y = (1-f)(x) = x - f(x)$ for some $x \in V^{'}$. We have $x = y + f(x) \in U$ so that $y \in (1-f)(U \cap V^{'})$. Since $U \cap V^{'} \ll_{\delta} V^{'}$, we get $U \cap (1-f)(V^{'}) = (1-f)(U \cap V^{'}) \ll_{\delta} (1-f)(V^{'})$ by [15, Lemma 1.3 (2)]. Also $U \cap (1-f)(V^{'}) = (1-f)(U \cap V^{'})$ is semisimple. Thus, $(1-f)(V^{'})$ is a δ_{ss}-supplement of U in M. Therefore M is amply δ_{ss}-supplemented. □

Combining Proposition 4.2 and Lemma 4.1, we obtain the next result:

Corollary 4.3. *A projective strongly δ-local module is amply δ_{ss}-supplemented.*

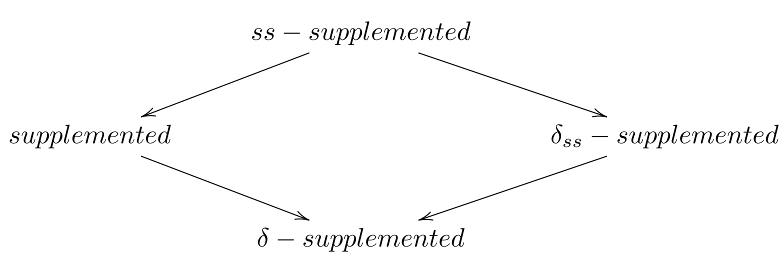
Example 4.4. (1) Consider the non-noetherian commutative ring S which is the direct product $\prod_{i\geq 1}^{\infty} F_i$, where $F_i = \mathbb{Z}_2$. Suppose that R is the subring of S generated by $\bigoplus_{i=1}^{\infty} F_i$ and 1_S. Let $M =_R R$. Then M is a regular module which is not semisimple. Therefore $Soc(M)$ is maximal. By [15, Example 4.1], we have $Soc(M) = \delta(M) \ll_{\delta} M$. This means that M is strongly δ-local. Since M is projective, it follows from Lemma 4.1 and Corollary 4.3 that M is amply δ_{ss}-supplemented. On the other hand, it is not (amply) ss-supplemented because $Rad(M) = 0$.

(2) Let M be the local $\mathbb{Z}$-module $\mathbb{Z}_{p^k}$, for p is any prime integer and $k \geq 3$. It is clearly that M is amply δ-supplemented. Since $Soc_{\delta}(\mathbb{Z}_{p^k}) = Soc(\mathbb{Z}_{p^k}) \cong \mathbb{Z}_p$ and $\delta(M) = Rad(M) = p\mathbb{Z}_{p^k}$, M is not (amply) δ_{ss}-supplemented.

It is well known that artinian modules are (amply) δ-supplemented. Example 4.4 (2) also shows that in general artinian modules need not to be δ_{ss}-supplemented. Now, we have the following implications the classes of modules:

$$\text{artinian} \Longrightarrow \text{supplemented} \Longrightarrow \delta\text{-supplemented}$$

and



Now we study on the various properties of δ_{ss}-supplemented modules.

Proposition 4.5. *Let M be a δ-local module. Then M is δ_{ss}-supplemented if and only if it is strongly δ-local.*

Proof. ($\Longrightarrow$) Since M is δ-local, it suffices to show that $\delta(M) \subseteq Soc(M)$. Let $m \in \delta(M)$. Then $Rm \ll_\delta M$. Since M is δ_{ss}-supplemented, Rm has a δ_{ss}-supplement V in M. Therefore $M = Rm + V$ and $Rm \cap V$ is semisimple. So we can write $M = S \oplus V$, where S is a projective semisimple submodule of Rm. Applying the modular law, we have $Rm = Rm \cap M = Rm \cap (S \oplus V) = S \oplus (Rm \cap V)$. So Rm is semisimple as the sum of two semisimple submodules. Hence $Rm \subseteq Soc(M)$. It means that $\delta(M) \subseteq Soc(M)$.

($\Longleftarrow$) By Lemma 4.1. □

Proposition 4.6. *Let M be a δ-supplemented module with $\delta(M) \subseteq Soc(M)$. Then M is δ_{ss}-supplemented.*

Proof. Let $U \subseteq M$. Since M is δ-supplemented, there exists a submodule V of M such that $M = U + V$ and $U \cap V \ll_\delta V$. Then $U \cap V \subseteq \delta(V) \subseteq \delta(M)$. By the hypothesis, $U \cap V \subseteq Soc(M)$. Therefore V is a δ_{ss}-supplement of U in M. It means that M is δ_{ss}-supplemented. □

Proposition 4.7. *Let M be a δ_{ss}-supplemented module. Then $\frac{M}{Soc_\delta(M)}$ is semisimple.*

Proof. Let $Soc_\delta(M) \subseteq U \subseteq M$. Since M is δ_{ss}-supplemented, there exists a submodule V of M such that $M = U + V$ and $U \cap V \subseteq Soc_\delta(V)$. Then $U \cap V \subseteq Soc_\delta(M)$ and so the sum $\frac{M}{Soc_\delta(M)} = \frac{U}{Soc_\delta(M)} + \frac{V + Soc_\delta(M)}{Soc_\delta(M)}$ is direct sum. Hence $\frac{M}{Soc_\delta(M)}$ is semisimple. □

In order to prove that every finite sum of δ_{ss}-supplemented modules is δ_{ss}-supplemented, we use the following standard lemma (see, [13, 41.2]).

Lemma 4.8. *Let M be a module and U be a submodule of M. Suppose that a submodule M_1 of M is δ_{ss}-supplemented. If $M_1 + U$ has a δ_{ss}-supplement in M, U has also a δ_{ss}-supplement in M.*

Proof. Suppose that X is a δ_{ss}-supplement of $M_1 + U$ in M and Y is a δ_{ss}-supplement of $(X+U) \cap M_1$ in M_1. So $M = M_1 + U + X$, $M_1 = (X+U) \cap M_1 + Y$, $(M_1 + U) \cap Y \ll_\delta Y$, $(X + U) \cap Y \ll_\delta Y$, $(M_1 + U) \cap Y$ and $(X + U) \cap Y$ is semisimple. Then $M = (X + U) \cap M_1 + Y + U + X = U + X + Y$ and by [11, Lemma 2.1 (2)] $U \cap (X+Y) \subseteq X \cap (U+Y) + Y \cap (U+X) \subseteq X \cap (U+M_1) + Y \cap (U + X) \ll_\delta X + Y$. Moreover, $X \cap (Y + U)$ is semisimple as a submodule of semisimple module $X \cap (Y + U)$. Note that $Y \cap [(X + U) \cap M_1] = Y \cap (X + U)$

is semisimple. It follows from [5, 8.1.5] that $(X+Y)\cap U$ is semisimple. Hence $X+Y$ is a δ_{ss}-supplement of U in M. □

Proposition 4.9. *The class of δ_{ss}-supplemented modules is closed under finite sums.*

Proof. Let M_i, $i=1,2,\dots,n$ be any finite collection of δ_{ss}-supplemented modules and let $M=M_1+M_2+\dots+M_n$. To prove that M is δ_{ss}-supplemented by induction on n, it is sufficient to prove this in the case, where $n=2$. Hence, suppose $n=2$. Let M_1, M_2 be any submodules of a module M such that $M=M_1+M_2$. If M_1 and M_2 are δ_{ss}-supplemented, M is δ_{ss}-supplemented. Let U be any submodule of M. The trivial submodule 0 is δ_{ss}-supplement of $M=M_1+M_2+U$ in M. Since M_1 is δ_{ss}-supplemented, M_2+U has a δ_{ss}-supplement in M by Lemma 4.8. Again applying Lemma 4.8, we have that U has a δ_{ss}-supplement in M. This shows that M is δ_{ss}-supplemented. □

A submodule U of a module M is said to be *cofinite* if $\frac{M}{U}$ is finitely generated (see [2]). Note that maximal submodules of M are cofinite.

Proposition 4.10. *Let M be a module. Then the following conditions are equivalent.*

(1) M is the sum of strongly δ-local or projective semisimple submodules,

(2) M is coatomic and every cofinite submodule of M has a δ_{ss}-supplement in M,

(3) M is coatomic and every maximal submodule of M has a δ_{ss}-supplement in M.

Proof. (1) $\Longrightarrow$ (2) Let $M=\sum_{i\in I}M_i$, where I is some index set and each M_i is strongly δ-local submodules or projective semisimple submodules. Put $N=\bigoplus_{i\in I}M_i$. It follows from Theorem 2.9 that N is coatomic. Consider the epimorphism $\psi: N\longrightarrow M$ via $\psi((m_i)_{i\in I})=\sum_{i\in I}m_i$ for all $(m_i)_{i\in I}\in N$. By [16, Lemma 1.5 (a)], we get M is coatomic.

Let U be any cofinite submodule of M. Then $\frac{M}{U}$ is finitely generated and so there exists a finite subset $\Lambda\subseteq I$ such that $M=U+\sum_{i\in\Lambda}M_i$. By Lemma 4.1 and Proposition 4.9, we obtain that $\sum_{i\in\Lambda}M_i$ is δ_{ss}-supplemented as the finite sum of δ_{ss}-supplemented submodules. Hence U has a δ_{ss}-supplement in M according to Lemma 4.8.

(2) $\Longrightarrow$ (3) Clear.

(3) $\Longrightarrow$ (1) Let X be the sum of all strongly δ-local submodules or semisimple projective submodules. Suppose that $X\neq M$. Since M is coatomic, there exists a submodule U of M such that $X\subseteq U\subset M$. By the assumption, U

has a δ_{ss}-supplement, say V, in M. It follows from Proposition 3.4 that V is projective simple or V is strongly δ-local. Then $V \subseteq X \subseteq U$. This is a contradiction. □

It is clear that every submodule of a finitely generated module is cofinite. Using this fact and Proposition 4.10, we obtain the following result:

Corollary 4.11. *Let M be a finitely generated module. Then the following conditions are equivalent:*

(1) $M = \sum_{i=1}^{n} M_i$, where each M_i is strongly δ-local or projective semisimple,

(2) M is δ_{ss}-supplemented,

(3) every maximal submodule of M has a δ_{ss}-supplement in M.

Theorem 4.12. *Let M be a module. Then M is δ_{ss}-supplemented if and only if every submodule U of M containing $Soc(M)$ has a δ_{ss}-supplement in M.*

Proof. One direction is clear. Conversely, let $U \subseteq M$. By the assumption, $Soc(M)+U$ has a δ_{ss}-supplement V in M. Since $Soc(M)$ is δ_{ss}-supplemented, it follows from Lemma 4.8 that U has a δ_{ss}-supplement in M. Hence M is δ_{ss}-supplemented. □

It is trivial to show that:

Corollary 4.13. *Let R be a ring and M be an R-module.*

(1) *$Soc(M)$ has a δ_{ss}-supplement in M if and only if $Soc(M)$ has a δ-supplement in M.*

(2) *If R is a commutative domain, then $Soc(M)$ has a δ_{ss}-supplement in M if and only if $Soc(M)$ has a supplement in M.*

Proposition 4.14. *If M is a (amply) δ_{ss}-supplemented module, then every factor module of M is (amply) δ_{ss}-supplemented.*

Proof. Let M be a δ_{ss}-supplemented module and $\frac{M}{L}$ be a factor module of M. By the assumption, for any submodule U of M which contains L, there exists a submodule V of M such that $M = U + V$, $U \cap V \ll_\delta V$ and $U \cap V$ is semisimple. Let $\pi : M \longrightarrow \frac{M}{L}$ be the canonical projection. Then we have that $\frac{M}{L} = \frac{U}{L} + \frac{V+L}{L}$ and $\frac{U}{L} \cap \frac{V+L}{L} = \frac{(U\cap V)+L}{L} = \pi(U \cap V) \ll_\delta \pi(V) = \frac{V+L}{L}$ by Lemma 2.1 (2). Since $U \cap V$ is semisimple, it follows from [5, 8.1.5 (2)] that $\pi(U \cap V) = \frac{(U\cap V)+L}{L}$ is semisimple. That is, $\frac{V+L}{L}$ is a δ_{ss}-supplement of $\frac{U}{L}$ in $\frac{M}{L}$, as required.

It can be proved similarly that if M is amply δ_{ss}-supplemented, then $\frac{M}{L}$ is amply δ_{ss}-supplemented for every submodule L of M. □

Lemma 4.15. *Let M be a δ_{ss}-supplemented module and $N \ll_\delta M$. Then $N \subseteq Soc_\delta(M)$.*

Proof. Let K be a δ_{ss}-supplement of N in M. Then $M = N + K$, $N \cap K \ll_\delta K$ and $N \cap K$ is semisimple. Since $N \ll_\delta M$, there exists a semisimple projective submodule N' of N such that $M = N' \oplus K$. By the modular law, we obtain that $N = N' \oplus (N \cap K)$. Hence N is semisimple. □

Corollary 4.16. *Let M be a coatomic module and M be a δ_{ss}-supplemented module. Then* $\mathrm{Rad}(M) \subseteq \delta(M) \subseteq Soc(M)$.

The following result is a generalization of Corollary 2.3.

Proposition 4.17. *Let M be a δ_{ss}-supplemented module and $\delta(M) = M$. Then M is projective semisimple.*

Proof. Let m be any element of M. It follows from $\delta(M) = M$ that $Rm \ll_\delta M$. By the assumption and Lemma 4.15, we have $Rm \subseteq Soc_\delta(M) \subseteq Soc(M)$ and so $m \in Soc(M)$. Therefore M is semisimple. Hence it is projective semisimple by Corollary 2.3. □

Note that a hollow module is either radical or local. Observe from Proposition 4.17 that a hollow-radical module is not δ_{ss}-supplemented.

Proposition 4.18. *Let M be a hollow module. If M is δ_{ss}-supplemented, then it is strongly local.*

Proof. Let M be a δ_{ss}-supplemented module. If $\delta(M) = M$, it follows from Proposition 4.17 that M is projective semisimple and so M is projective simple because M is hollow. Assume that $\delta(M) \neq M$. Since $Rad(M) \subseteq \delta(M)$ and M is hollow, M is local. Therefore we have $Rad(M) = \delta(M)$ is maximal and small in M. It follows from Lemma 4.15 that $\delta(M) \subseteq Soc_\delta(M) \subseteq Soc(M)$. It means that M is strongly local. □

In the following next theorem we give the structure of a δ_{ss}-supplemented module M with δ-small $\delta(M)$ in terms of δ-supplemented modules.

Theorem 4.19. *Let M be a module and $\delta(M) \ll_\delta M$. Then the following statements are equivalent:*

(1) M is δ_{ss}-supplemented,

(2) M is δ-supplemented and $\delta(M)$ has a δ_{ss}-supplement in M,

(3) M is δ-supplemented and $\delta(M) \subseteq Soc(M)$.

Proof. Clearly we have (1) $\Longrightarrow$ (2), and (2) $\Longrightarrow$ (3) follows from Lemma 4.15.
(3) $\Longrightarrow$ (1) By Proposition 4.6. □

5 Rings whose modules are δ_{ss}-supplemented

It follows from [15] that a projective module P is called a *projective δ-cover* of a module M if there exists an epimorphism $f : P \longrightarrow M$ with $Ker(f) \ll_\delta P$. A ring R is called *δ-semiperfect* if every simple R-module has a projective δ-cover, and it is called *δ-perfect* if every left R-module has a projective δ-cover. It is proven in [7, Theorem 3.3 and Theorem 3.4] that a ring R is δ-perfect (respectively, δ-semiperfect) if and only if every left (respectively, finitely generated) R-module is δ-supplemented. Now we characterize the rings the property that every left R-module is (amply) δ_{ss}-supplemented.

Lemma 5.1. *Let M be a module. If every submodule of M is δ_{ss}-supplemented, then M is amply δ_{ss}-supplemented.*

Proof. Let U and V be submodules of M such that $M = U+V$. Since V is δ_{ss}-supplemented, there exists a submodule $V^{'}$ of V such that $V = (U \cap V) + V^{'}$, $U \cap V^{'} \ll_\delta V^{'}$ and $U \cap V^{'}$ is semisimple. Note that $M = U + V = U + (U \cap V) + V^{'} = U + V^{'}$. It means that U has ample δ_{ss}-supplements in M. Hence M is amply δ_{ss}-supplemented. □

A module M is called *locally projective* in case whenever $g : N \longrightarrow K$ is an epimorphism and $f : M \longrightarrow K$ is a homomorphism then for every finitely generated submodule M_0 of M there exists a homomorphism $h : M \longrightarrow N$ such that $gh|_{M_0} = f|_{M_0}$. Every projective module is locally projective. Also, a finitely generated locally projective module is projective.

Proposition 5.2. *Let M be a locally projective module and $N \subseteq Soc(M)$. Then $N \ll_\delta M$.*

Proof. Let $M = N + K$ for some submodule K of M. Since N is semisimple, we can write $N = (N \cap K) \oplus X$ where X is a semisimple submodule of N. Therefore the sum $M = X + K$ is direct sum. Since being locally projective is inherited by direct summands, it follows that every direct summand of X is locally projective and so every simple submodule of X is projective. Therefore X is projective as the direct sum projective simple submodules. Hence $N \ll_\delta M$. □

Theorem 5.3. *Let R be a ring. Then the following statements are equivalent.*

(1) *${}_RR$ is δ_{ss}-supplemented,*

(2) *R is a δ-semiperfect ring and $\delta(R) = Soc({}_RR)$,*

(3) *$\frac{R}{Soc({}_RR)}$ is semisimple and idempotents lift to $Soc({}_RR)$,*

(4) every projective left R-module is δ_{ss}-supplemented,

(5) every left R-module is (amply) δ_{ss}-supplemented,

(6) for every left R-module M every maximal submodule has δ_{ss}-supplement in M,

(7) every left maximal ideal of R has a δ_{ss}-supplement in R.

Proof. (1) $\Longrightarrow$ (2) By the hypothesis, ${}_RR$ is δ-supplemented and so it follows from [7, Theorem 3.3] that R is a δ-semiperfect ring. Since ${}_RR$ is coatomic, it follows from Lemma 2.1 (5) that $\delta(R)$ is δ-small in ${}_RR$. Applying Theorem 4.19, we get that $\delta(R) \subseteq Soc({}_RR)$. On the other hand, by Proposition 5.2, $Soc({}_RR) \subseteq \delta(R)$ and so we obtain that the equality $\delta(R) = Soc({}_RR)$.

(2) $\Longrightarrow$ (3) By [15, Theorem 3.6].

(3) $\Longrightarrow$ (4) Let P be a projective left R-module. Since $\frac{R}{Soc({}_RR)}$ is artinian semisimple, it follows from [15, Corollary 1.7] that $\delta(R) = Soc({}_RR)$ and so $\delta(P) = \delta(R)P = Soc({}_RR)P \subseteq Soc(P)$ by [15, Theorem 1.8]. According to Proposition 4.6, it suffices to prove that P is δ-supplemented. Since semisimple rings are perfect, it follows from assumption and [15, Theorem 3.8] that R is a δ-perfect ring. By [7, Theorem 3.4], we obtain that P is δ-supplemented.

(4) $\Longrightarrow$ (5) Let M be a left R-module. Since every left R-module is a homomorphic image of a free left R-module, it follows from Proposition 4.14 that every submodule of M is δ_{ss}-supplemented. By Lemma 5.1, it is amply δ_{ss}-supplemented.

(5) $\Longrightarrow$ (6) and (6) $\Longrightarrow$ (7) Clear.

(7) $\Longrightarrow$ (1) By Corollary 4.11. □

Hence we have the following strict containments of classes of rings:

$$\{\text{rings in [6, Theorem 41]}\} \subset \{\text{rings in Theorem 5.3}\} \subset \{\delta\text{-perfect rings}\}$$

Examples for showing these implications are not invertible can be found [15, Example 4.1 and Example 4.3]. So we say that a ring R is *left δ_{ss}-perfect* if the equal conditions satisfy in the above theorem. Right δ_{ss}-perfect rings are defined similarly. R is said to be *δ_{ss}-perfect* if it is both a right and a left δ_{ss}-perfect.

Proposition 5.4. *Let R be a left δ_{ss}-perfect ring. Then $Rad(R)$ is semisimple. In particular, $(Rad(R))^2 = 0$.*

Proof. Since R is a left δ_{ss}-perfect ring, it follows from Theorem 5.3 that $Rad(R) \subseteq \delta(R) = Soc({}_RR)$. It means that $Rad(R)$ is semisimple. By [13, 21.12 (4)], we obtain that $(Rad(R))^2 = 0$. □

A ring R is called *a left max ring* if every left R-module has a maximal submodule. It is well known that a ring R is left max if and only if every non-zero left R-module is coatomic.

Proposition 5.5. *Let R be a left δ_{ss}-perfect ring. Then it is a left max ring.*

Proof. Let M be a radical module, that is, $\mathrm{Rad}(M) = M$. Then $\delta(M) = M$. Since R is a left δ_{ss}-perfect ring, by Theorem 5.3, $\frac{R}{Soc(_RR)} = \frac{R}{\delta(R)}$ is a semisimple ring. By [15, Theorem 1.8], we obtain that $\delta(M) = \delta(R)M = Soc(_RR)M \subseteq Soc(M)$. Then $M = Soc(M)$. Since semisimple modules are zero radical, we get $M = Rad(M) = 0$. This means that R is a left max ring. □

Now we characterize the left δ_{ss}-perfect rings via a different kind of projective δ-covers. Let M be a module and $f : P \longrightarrow M$ be an epimorphism. We call the module P a *δ_{ss}-cover* of M if $ker(f)$ is semisimple and δ-small in P, and call a δ_{ss}-cover P a *projective δ_{ss}-cover* of M in case P is projective.

Theorem 5.6. *Let M be a projective module. Then the following conditions are equivalent.*

(1) M is δ_{ss}-supplemented,

(2) every submodule of M has a δ_{ss}-supplement that is a direct summand of M,

(3) for any submodule N of M, M has the decomposition $M = N^{'} \oplus K$ such that $N^{'} \subseteq N$ and $N \cap K \subseteq Soc_{\delta}(M)$,

(4) every factor module of M has a projective δ_{ss}-cover.

Proof. (1) $\Longrightarrow$ (4) Let U be a submodule of M. It follows that U has a δ_{ss}-supplement, say V, in M. Since $M = U + V$, the homomorphism $g : V \longrightarrow \frac{M}{U}$ via $g(v) = v + U$ is an epimorphism. Let $\pi : M \longrightarrow \frac{M}{U}$ be the canonical projection. Since M is projective, there exists a homomorphism $f : M \longrightarrow V$ such that $gf = \pi$. Then it can be seen that $M = U + f(M)$. Applying the modular law, we get $V = U \cap V + f(M)$. Therefore we can write $V = S \oplus f(M)$ for some projective semisimple submodule S of V because $U \cap V \ll_{\delta} V$. Since $U \cap f(M) \subseteq U \cap V \ll_{\delta} V$, then $U \cap f(M) \ll_{\delta} V$ by Lemma 2.1 (2). It follows from [15, Lemma 1.3 (3)] that $U \cap f(M) \ll_{\delta} f(M)$ since $f(M)$ is a direct summand of V. This means that $f(M)$ is a δ_{ss}-supplement of U in M. Since M is projective and δ_{ss}-supplemented, by Proposition 4.2, it is amply δ_{ss}-supplemented and so $f(M)$ has a δ_{ss}-supplement $U^{'} \subseteq U$ in M. Therefore $f(M)$ and $U^{'}$ are mutual δ_{ss}-supplements in M. Using [7, Lemma 2.15], we obtain that $f(M)$ is projective.

Now we consider the epimorphism $\varphi : f(M) \longrightarrow \frac{M}{U}$ via $\varphi(x) = x + U$ for all $x \in f(M)$. Since $M = U + f(M)$, we obtain that $ker(\varphi) = U \cap f(M)$ is semisimple and δ-small in $f(M)$. Hence $f(M)$ is a projective δ_{ss}-cover of $\frac{M}{U}$ as desired.

(4) $\Longrightarrow$ (3) It follows from [15, Lemma 2.4].

(3) $\Longrightarrow$ (2) and (2) $\Longrightarrow$ (1) Clear. □

The next result is crucial.

Corollary 5.7. *The following conditions are equivalent for a ring R.*

(1) R is a left δ_{ss}-perfect ring,

(2) every left R-module has a projective δ_{ss}-cover,

(3) every semisimple left R-module has a projective δ_{ss}-cover,

(4) every simple left R-module has a projective δ_{ss}-cover.

Proof. (1) $\Longrightarrow$ (2) Let M be a left R-module. Then there exist a projective module P and an epimorphism $\Psi : P \longrightarrow M$. By the assumption and Theorem 5.3, we get that P is δ_{ss}-supplemented. It follows from Theorem 5.6 that M has a projective δ_{ss}-cover as a factor module of P.

(2) $\Longrightarrow$ (3) and (3) $\Longrightarrow$ (4) are clear.

(4) $\Longrightarrow$ (1) It follows from [15, Lemma 2.4] and Theorem 5.3. □

Proposition 5.8. *A commutative δ_{ss}-perfect domain is field.*

Proof. Let R be a commutative δ_{ss}-perfect domain and $a \in R$. It follows that R is a local ring. If $a \in R \backslash Rad(R)$, we have that $Ra = R$ and so a is an invertible element of R. Suppose that $a \in Rad(R)$. By Proposition 5.4, $a^2 \in (Rad(R))^2 = 0$. Therefore $a = 0$ since R is a domain. Thus R is field. □

Let R be a ring. Next we will give a necessary and sufficient condition for the δ_{ss}-perfect ring R to be ss-supplemented as a left R-module. Recall from Lomp [8] that a module M is said to be *semilocal* if $\frac{M}{Rad(M)}$ is semisimple, and a ring R is said to be *semilocal* if it is semilocal as a left (right) module over itself. It is shown in [8, Teorem 3.5] that a ring R is semilocal if and only if every left R-module is semilocal.

It is shown in [4, Proposition 4.2] that a projective semilocal, δ-supplemented module M with small radical is supplemented. From this fact we see that the condition "small radical" is necessary for M to be a supplemented. However, we show by the following proposition that a projective semilocal, δ_{ss}-supplemented module is ss-supplemented without necessity of this condition.

Proposition 5.9. *Let M be a projective module. If M is semilocal and δ_{ss}-supplemented, then it is ss-supplemented.*

Proof. Let M be a semilocal and δ_{ss}-supplemented module. Then $Soc(M) = X \oplus Soc_s(M)$, where $X \subseteq Soc(M)$. Since M is semilocal, we can write $M = X + Y$ and $X \cap Y \subseteq Rad(M)$ for some submodule Y of M. Now $X \cap Y \subseteq X \cap Rad(M) = [X \cap Soc(M)] \cap Rad(M) = X \cap [Soc(M) \cap Rad(M)] = X \cap Soc_s(M) = 0$. Therefore $M = X \oplus Y$ and $Soc(Y) \subseteq Rad(Y) = Rad(M)$. Then Y is projective as a direct summand of the projective module M. By the proof of [4, Proposition 4.2], we have $Rad(Y) = \delta(Y)$. Since M is δ_{ss}-supplemented, it follows from Proposition 4.14 that Y is δ_{ss}-supplemented.

Let U be a submodule of Y. By Theorem 5.6, there exists a direct summand V of Y such that $Y = U + V$ and $U \cap V \subseteq Soc_\delta(V)$. Then $U \cap V \subseteq \delta(V) \subseteq \delta(Y) = Rad(Y)$ and hence $U \cap V$ is small in Y. It follows from [13, 19.3 (5)] that $U \cap V \ll V$. This means that Y is ss-supplemented. Hence $M = X \oplus Y$ is ss-supplemented by [6, Corollary 3.13]. □

For a ring R, let $\mathfrak{X}(R) = \frac{Soc(_RR)}{Soc_s(_RR)}$ as in [4].

Corollary 5.10. *Let R be a ring. Then the following statements are equivalent:*

(1) $_RR$ is ss-supplemented,

(2) R is left δ_{ss}-perfect and semilocal,

(3) R is left δ_{ss}-perfect and $\mathfrak{X}(R)$ is finitely generated.

Proof. (1) $\iff$ (2) By Proposition 5.9.
(2) $\iff$ (3) It follows from [4, Lemma 4.1]. □

Observe from Corollary 5.10 that if a left δ_{ss}-perfect ring is left noetherian, then it is a left artinian ring.

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