DOI: 10.2478/auom-2019-0036
An. Şt. Univ. Ovidius Constanţa

sciendo
Vol. 27(3),2019, 97–112

f-biharmonic and bi-f-harmonic submanifolds of generalized (k, μ)-space-forms

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Abstract

Here we have studied f-biharmonic and bi-f-harmonic submanifolds of generalized (k, μ)-space-forms and obtained a necessary and sufficient condition on a submanifold of generalized (k, μ)-space-form to be f-biharmonic and bi-f-harmonic submanifold. We have also studied f-biharmonic hypersurfaces of said ambient space forms.

1 Introduction

An almost contact metric manifold $\bar{M}(\phi, \xi, \eta, g)$ is called generalized (k, μ)-space-form if there exist $f_i \in C^\infty(\bar{M}),\ i = 1, 2, 3, 4, 5, 6$, such that its curvature tensor $\bar{R}$ satisfies [5]

$$\bar{R}(X,Y)Z = (\sum_{i=1}^{6} f_i \bar{R}_i)(X,Y)Z, \tag{1.1}$$

where $\bar{R}_i(X,Y)Z,\ i = 1, 2, 3, 4, 5, 6$ are defined as

$$\bar{R}_1(X,Y)Z = g(Y,Z)X - g(X,Z)Y,$$

Key Words: generalized (k, μ)-space-forms, f-biharmonic submanifold, bi-f-biharmonic submanifold, f-biharmonic hypersurface.
2010 Mathematics Subject Classification: Primary 53C42, 53C43.

Received: 26.09.2018
Revised: 03.10.2018
Accepted: 08.01.2019
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$$\bar{R}_2(X,Y)Z = g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z,$$
$$\bar{R}_3(X,Y)Z = \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi,$$
$$\bar{R}_4(X,Y)Z = g(Y,Z)hX - g(X,Z)hY + g(hY,Z)X - g(hX,Z)Y,$$
$$\bar{R}_5(X,Y)Z = g(hY,Z)hX - g(hX,Z)hY + g(\phi hX,Z)\phi hY - g(\phi hY,Z)\phi hX,$$
$$\bar{R}_6(X,Y)Z = \eta(X)\eta(Z)hY - \eta(Y)\eta(Z)hX + g(hX,Z)\eta(Y)\xi - g(hY,Z)\eta(X)\xi$$

for all $X, Y, Z \in \chi(\bar{M})$, where $h = \frac{1}{2}\pounds_\xi \phi$ and $\pounds$ is the usual Lie derivative. Such a manifold of dimension $(2n+1)$, $n > 1$ is denoted by $\bar{M}^{2n+1}(f_1, f_2, \cdots, f_6)$.

In particular, if $f_1 = \frac{c+3}{4}, f_2 = \frac{c-1}{4}, f_3 = \frac{c+3}{4} - k, f_4 = 1, f_5 = \frac{1}{2}$ and $f_6 = 1 - \mu$ then $\bar{M}^{2n+1}(f_1, f_2, \cdots, f_6)$ turns out to the notion of (k, μ)-space-forms.

For any two Riemannian manifolds (M, g) and $(N, g^{'})$, the map $\psi : M \to N$ is called harmonic map if it is a critical point of the energy functional

$$E(\psi) = \frac{1}{2}\int_M |d\psi|^2 dv_g. \tag{1.2}$$

Several authors studied biharmonic submanifolds such as [1], [6], [7], [8], [9], [15], [16] etc.

As a natural generalization of biharmonic map, Lu [10] defined f-biharmonic map. A map $\psi : M \to N$ is said to be f-biharmonic if it is a critical point of the f-bienergy functional

$$E_{2,f}(\psi) = \frac{1}{2}\int_M f|\tau(\psi)|^2 dv_g, \tag{1.3}$$

where $\tau(\psi)$ is the tension field of ψ. The Euler-Lagrange equation for f-biharmonic map is

$$\tau_{2,f}(\psi) = f\tau_2(\psi) + (\Delta f)\tau(\psi) + 2\nabla^{\psi}_{grad\ f}\tau(\psi), \tag{1.4}$$

where $\tau_2(\psi)$ is bitension field of ψ.

In [13], Ouakkas et al. introduced bi-f-harmonic map. For $f \in C^\infty(M)$, the map $\psi : M \to N$ is said to be bi-f-harmonic if it is a critical point of the bi-f-energy functional

$$E^2_f(\psi) = \frac{1}{2}\int_M f|\tau_f(\psi)|^2 dv_g, \tag{1.5}$$

where $\tau_f(\psi) = f\tau(\psi) + d\psi(grad\ f)$. The bi-$f$-harmonic map are characterized by the following Eular-Lagrange equation

$$\tau^2_f(\psi) = fJ^{\psi}(\tau_f(\psi)) - \nabla^{\psi}_{grad\ f}\tau_f(\psi) = 0, \tag{1.6}$$

where $J^{\psi}(X) = -[Tr._g \nabla^{\psi}\nabla^{\psi}X - \nabla^{\psi}_{\nabla^M}X - R^N(d\psi, X)d\psi]$. The f-biharmonic submanifolds are also studied in [11], [12], [13].

In this paper we have investigated f-biharmonic and bi-f-harmonic submanifolds of $\bar{M}^{2n+1}(f_1, \cdots, f_6)$. Section 2 is concerned with some preliminaries. Section 3 is devoted to the study of f-biharmonic submanifolds of $\bar{M}^{2n+1}(f_1, \cdots, f_6)$. We have obtained a necessary and sufficient condition of a submanifold of $\bar{M}^{2n+1}(f_1, \cdots, f_6)$ to become a f-biharmonic submanifold (see, Theorem 3.1). We also considered the necessary and sufficient condition of some particular classes of submanifolds of $\bar{M}^{2n+1}(f_1, \cdots, f_6)$ to be f-biharmonic submanifolds (see, Corollary 3.1). In [17], Roth and Upadhyay studied f-biharmonic submanifolds of generalized Sasakian-space-forms and obtained the necessary and sufficient condition of a submanifold of generalized Sasakian-space-form to be f-biharmonic submanifold (see, Theorem 3.10 of [17]), which follows from Theorem 3.1 of present paper (see, Corollary 3.2).

As we know that a hypersurface is a manifold immersed in an ambient manifold with co-dimension 1. The hypersurface is called f-biharmonic hypersurface if the isometric immersion is f-biharmonic map. In section 4, we have obtained a necessary and sufficient condition of a hypersurface of $\bar{M}^{2n+1}(f_1, \cdots, f_6)$ to be a f-biharmonic hypersurface. Also in section 5 we have studied bi-f-harmonic submanifolds and obtained a necessary and sufficient condition of a submanifold of $\bar{M}^{2n+1}(f_1, \cdots, f_6)$ to become a bi-f-harmonic submanifold. We also considered some particular classes of such submanifolds.

2 Preliminaries

An $(2n+1)$-dimensional smooth manifold $\bar{M}^{2n+1}$ is said to be an almost contact metric manifold [2] if it admits a $(1,1)$ tensor field ϕ, a vector field ξ, an 1-form η and a Riemannian metric g which satisfy

$$\phi\xi = 0, \quad \eta(\phi X) = 0, \quad \phi^2 X = -X + \eta(X)\xi, \tag{2.1}$$

$$g(\phi X, Y) = -g(X, \phi Y), \quad \eta(X) = g(X, \xi), \quad \eta(\xi) = 1, \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{2.3}$$

for all vector fields X, Y on $\bar{M}$ and such manifold is denoted by $\bar{M}^{2n+1}(\phi, \xi, \eta, g)$. An $\bar{M}^{2n+1}(\phi, \xi, \eta, g)$ is called a contact metric manifold [2] if $d\eta(X, Y) = g(X, \phi Y)$.

Given a contact metric manifold $\bar{M}^{2n+1}(\phi, \xi, \eta, g)$, we define a $(1,1)$ tensor field h by $h = \frac{1}{2}\pounds_{\xi}\phi$, where $\pounds$ denotes the Lie differentiation. Then h is symmetric and satisfies $h\phi = -\phi h$. Thus, if λ is an eigenvalue of h with eigenvector X, $-\lambda$ is also an eigenvalue with eigenvector ϕX. Also we have

$Tr.\ h = Tr.\ \phi h = 0$ and $h\xi = 0$. Moreover, if $\bar{\nabla}$ denotes the Riemannian connection of g, then the following relation holds:

$$\bar{\nabla}_X \xi = -\phi X - \phi h X. \tag{2.4}$$

If the characteristic vector field ξ belongs to the (k,μ)-nullity distribution in a contact metric manifold then such type of manifold is called (k,μ)-contact metric manifold [3]. In a (k,μ)-contact metric manifold the following relations hold [3]:

$$h^2 = (k-1)\phi^2, \qquad k \leq 1, \tag{2.5}$$

$$(\bar{\nabla}_X \phi)(Y) = g(X + hX, Y)\xi - \eta(Y)(X + hX), \tag{2.6}$$

From (2.4), it follows that

$$(\bar{\nabla}_X \eta)(Y) = g(X + hX, \phi Y). \tag{2.7}$$

Again a contact metric manifold $\bar{M}^{2n+1}(\phi,\xi,\eta,g)$ is said to be a generalized (k,μ)-space [4] if its curvature tensor $\bar{R}$ satisfies the [14]

$$\bar{R}(X,Y)\xi = k\{\eta(Y)X - \eta(X)Y\} + \mu\{\eta(Y)hX - \eta(X)hY\}$$

for some smooth functions k and μ on $\bar{M}^{2n+1}(\phi,\xi,\eta,g)$, which are independent of the choice of vector fields X and Y. If k and μ are constants then the manifold M is called a (k,μ)-space, where full classification was given in [4].

Let M be an m-dimensional submanifold of $\bar{M}^{2n+1}(f_1,\cdots,f_6)$ and let ∇ and $\nabla^{\perp}$ be the induced connection on the tangent bundle TM and the normal bundle $T^{\perp}M$ of M respectively. Then the Gauss and Weingarten formulae are given by

$$\bar{\nabla}_X Y = \nabla_X Y + \sigma(X,Y) \tag{2.8}$$

and

$$\bar{\nabla}_X V = -A_V X + \nabla^{\perp}_X V \tag{2.9}$$

for all $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$, where σ and A_V are second fundamental form and shape operator respectively for the immersion of M into $\bar{M}$ and they are related by [19]

$$g(\sigma(X,Y),V) = g(A_V X, Y). \tag{2.10}$$

From (2.8) and (2.9), we have the Gauss equation as

$$\begin{aligned}\bar{R}(X,Y,Z,W) &= R(X,Y,Z,W) \\ &\quad - g\big(\sigma(X,W),\sigma(Y,Z)\big) + g\big(\sigma(X,Z),\sigma(Y,W)\big),\end{aligned} \tag{2.11}$$

where R is the curvature tensor of M with respect to induced connection ∇. For any $X \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$, we can write

$$\phi X = PX + FX \quad \text{and} \quad \phi V = tV + sV, \tag{2.12}$$

where PX and FX are the tangential and normal components of ϕX and tV and sV are the tangential and normal components of ϕV, respectively.

A submanifold M of $\bar{M}^{2n+1}(f_1, \cdots, f_6)$ is said to be invariant (resp., anti-invariant) if the structure vector field ξ is tangent to M at every point of M and $\phi X \subset TM$ (resp., $\phi X \subset T^{\perp}M$) for any vector field X tangent to M at every point of M, i.e. $F = 0$ (resp., $P = 0$) at every point of M.

Proposition 2.1. *[18] In an m-dimensional invariant submanifold of $\bar{M}^{2n+1}(f_1, \cdots, f_6)$, we have*

$$||P||^2 = 0, \quad Tr.\ h^T = Tr.\ (\phi h)^T = 0, \quad \|(\phi h)^T\|^2 = \|h^T\|^2. \tag{2.13}$$

3 f-biharmonic submanifolds of $\bar{M}^{2n+1}(f_1, \cdots, f_6)$

This section consists with necessary and sufficient condition of f-biharmonic submanifolds of $\bar{M}^{2n+1}(f_1, \cdots, f_6)$.

Theorem 3.1. *A submanifold M^m of $\bar{M}^{2n+1}(f_1, \cdots, f_6)$ is f-biharmonic if and only if the following relations hold:*

$$\begin{aligned} &-\Delta^{\perp} H + Tr.\ (\sigma(e_i, A_H e_i)) + \frac{\Delta f}{f} H + 2\nabla^{\perp}_{\overrightarrow{\nabla}(lnf)} H \\ &= m f_1 H - 3 f_2 F t H - f_3\{|\xi^T|^2 H + m\eta(H)\xi^{\perp}\} + f_4\{m(hH)^{\perp} + (Tr.\ h^T)H\} \\ &- f_5\{(h(hH)^T)^{\perp} - (Tr.\ h^T)(hH)^{\perp} + (Tr.\ (\phi h)^T)(\phi hH)^{\perp} - (\phi h(\phi hH)^T)^{\perp}\} \\ &- f_6\{|\xi^T|^2 (hH)^{\perp} + (Tr.\ h^T)\eta(H)\xi^{\perp} - \eta(hH)\xi^{\perp}\}, \end{aligned} \tag{3.1}$$

$$\begin{aligned} &\frac{m}{2} grad|H|^2 + 2Tr.\ A_{\nabla^{\perp} H}(e_i) + 2A_H \overrightarrow{\nabla}(lnf) \\ &= -6 f_2 P t H - 2(m-1) f_3 \eta(H)\xi^T + 2(m-1) f_4 (hH)^T - 2 f_5\{(h(hH)^T)^T \\ &-(Tr.\ h^T)(hH)^T + (Tr.\ (\phi h)^T)(\phi hH)^T - (\phi h(\phi hH)^T)^T\} - 2 f_6\{|\xi^T|^2 (hH)^T \\ &+(Tr.\ h^T)\eta(H)\xi^T - \eta(hH)\xi^T\}, \end{aligned} \tag{3.2}$$

where $h^T, (\phi h)^T, \xi^T$ are tangential components and $h^{\perp}, (\phi h)^{\perp}, \xi^{\perp}$ are normal components of $h, (\phi h), \xi$ respectively.

Proof. The equation of f-biharmonicity are defined in ([12], [17]). After projection of the equation $\tau_{2,f}(\psi)=0$ on both tangent and normal bundles, we get the following equations

$$-\Delta^{\perp}H + Tr.\ (\sigma(e_i, A_He_i)) + \frac{\Delta f}{f}H + 2\nabla^{\perp}_{\overrightarrow{\nabla}(lnf)}H + (Tr.\ \bar{R}(e_i,H)e_i)^{\perp} = 0, \quad (3.3)$$

$$\frac{m}{2}\overrightarrow{\nabla}|H|^2 + 2Tr.\ A_{\nabla^{\perp}H}(e_i) + 2(Tr.\ \bar{R}(e_i,H)e_i)^T + 2A_H\overrightarrow{\nabla}(lnf) = 0. \quad (3.4)$$

Let us consider $\{e_1, e_2, \cdots, e_m\}$ be an orthonormal basis of $\Gamma(TM)$. Replacing $X=Z=e_i$ and $Y=H$ in (1.1) we get,

$$\begin{aligned}
&\bar{R}(e_i,H)e_i \qquad (3.5)\\
&= f_1\{g(H,e_i)e_i - g(e_i,e_i)H\} + f_2\{g(e_i,\phi e_i)\phi H - g(H,\phi e_i)\phi e_i\\
&+2g(e_i,\phi H)\phi e_i\} + f_3\{\eta(e_i)\eta(e_i)H - \eta(H)\eta(e_i)e_i + g(e_i,e_i)\eta(H)\xi\\
&-g(H,e_i)\eta(e_i)\xi\} + f_4\{g(H,e_i)he_i - g(e_i,e_i)hH + g(hH,e_i)e_i - g(he_i,e_i)H\}\\
&+f_5\{g(hH,e_i)he_i - g(he_i,e_i)hH + g(\phi he_i,e_i)\phi hH - g(\phi hH,e_i)\phi he_i\}\\
&+f_6\{\eta(e_i)\eta(e_i)hH - \eta(H)\eta(e_i)he_i + g(he_i,e_i)\eta(H)\xi - g(hH,e_i)\eta(e_i)\xi\}.
\end{aligned}$$

In $\bar{M}^{2n+1}(f_1,\cdots,f_6)$, we have

$$Tr.\ (P)=0,\ Tr.\ (F)=0,\ g(H,Fe_i) = -g(tH,e_i). \quad (3.6)$$

Taking contraction of (3.5) and using (3.6) we get

$$\begin{aligned}
&Tr.\ \bar{R}(e_i,H)e_i \qquad (3.7)\\
&= -mf_1H + 3f_2(PtH + FtH) + f_3\{|\xi^T|^2H - \eta(H)\xi^T\\
&+m\eta(H)\xi\} + f_4\{-mhH + (hH)^T - (Tr.\ h^T)H\}\\
&+f_5\{h(hH)^T - (Tr.\ h^T)hH + (Tr.\ (\phi h)^T)(\phi hH) - \phi h(\phi hH)^T\}\\
&+f_6\{|\xi^T|^2hH + (Tr.\ h^T)\eta(H)\xi - \eta(hH)\xi\}.
\end{aligned}$$

Substituting the tangential and normal parts of $Tr.\ \bar{R}(e_i,H)e_i$ from (3.7) in (3.3) and (3.4) respectively, we get (3.1) and (3.2). This proves the theorem. □

Corollary 3.1. *Let M^m be a submanifold of $\bar{M}^{2n+1}(f_1,\cdots,f_6)$. Then we have the following:*

1. M **is invariant:** *M is f-biharmonic if and only if*

$$\begin{aligned}
&-\Delta^{\perp}H + Tr.\ (\sigma(e_i,A_He_i)) + \frac{\Delta f}{f}H + 2\nabla^{\perp}_{\overrightarrow{\nabla}(lnf)}H \qquad (3.8)\\
&= mf_1H - f_3\{|\xi^T|^2H + m\eta(H)\xi^{\perp}\} + f_4m(hH)^{\perp}\\
&-f_5\{(h(hH)^T)^{\perp} - (\phi h(\phi hH)^T)^{\perp}\}\\
&-f_6\{|\xi^T|^2(hH)^{\perp} - \eta(hH)\xi^{\perp}\},
\end{aligned}$$

$$\frac{m}{2}\overrightarrow{\nabla}|H|^2 + 2Tr.\ A_{\nabla^\perp H}(e_i) + 2A_H\overrightarrow{\nabla}(lnf) \tag{3.9}$$
$$= -6f_2PtH - 2(m-1)f_3\eta(H)\xi^T + 2(m-1)f_4(hH)^T$$
$$-2f_5\{(h(hH)^T)^T - (\phi h(\phi hH)^T)^T\} - 2f_6\{|\xi^T|^2(hH)^T$$
$$-\eta(hH)\xi^T\}.$$

2. M **is anti-invariant:** *M is f-biharmonic if and only if*

$$-\Delta^\perp H + Tr.\ (\sigma(e_i, A_He_i)) + \frac{\Delta f}{f}H + 2\nabla^\perp_{\overrightarrow{\nabla}(lnf)}H \tag{3.10}$$
$$= mf_1H - 3f_2FtH - f_3\{|\xi^T|^2H + m\eta(H)\xi^\perp\}$$
$$+f_4\{m(hH)^\perp + (Tr.\ h^T)H\} - f_5\{(h(hH)^T)^\perp$$
$$-(Tr.\ h^T)(hH)^\perp + (Tr.\ (\phi h)^T)(\phi hH)^\perp - (\phi h(\phi hH)^T)^\perp\}$$
$$-f_6\{|\xi^T|^2(hH)^\perp + (Tr.\ h^T)\eta(H)\xi^\perp - \eta(hH)\xi^\perp\},$$

$$\frac{m}{2}\overrightarrow{\nabla}|H|^2 + 2Tr.\ A_{\nabla^\perp H}(e_i) + 2A_H\overrightarrow{\nabla}(lnf) \tag{3.11}$$
$$= -2(m-1)f_3\eta(H)\xi^T + 2(m-1)f_4(hH)^T - 2f_5\{(h(hH)^T)^T$$
$$-(Tr.\ h^T)(hH)^T + (Tr.\ (\phi h)^T)(\phi hH)^T - (\phi h(\phi hH)^T)^T\}$$
$$-2f_6\{|\xi^T|^2(hH)^T + (Tr.\ h^T)\eta(H)\xi^T - \eta(hH)\xi^T\}.$$

3. $\xi \in T^\perp M$**:** *M is f-biharmonic if and only if*

$$-\Delta^\perp H + Tr.\ (\sigma(e_i, A_He_i)) + \frac{\Delta f}{f}H + 2\nabla^\perp_{\overrightarrow{\nabla}(lnf)}H \tag{3.12}$$
$$= mf_1H - 3f_2FtH - f_3m\eta(H)\xi + f_4\{m(hH)^\perp + (Tr.\ h^T)H\}$$
$$-f_5\{(h(hH)^T)^\perp - (Tr.\ h^T)(hH)^\perp + (Tr.\ (\phi h)^T)(\phi hH)^\perp$$
$$-(\phi h(\phi hH)^T)^\perp\} - f_6\{(Tr.\ h^T)\eta(H)\xi - \eta(hH)\xi\},$$

$$\frac{m}{2}\overrightarrow{\nabla}|H|^2 + 2Tr.\ A_{\nabla^\perp H}(e_i) + 2A_H\overrightarrow{\nabla}(lnf) \tag{3.13}$$
$$= 2(m-1)f_4(hH)^T - 2f_5\{(h(hH)^T)^T - (Tr.\ h^T)(hH)^T$$
$$+(Tr.\ (\phi h)^T)(\phi hH)^T - (\phi h(\phi hH)^T)^T\}.$$

4. $\xi \in TM$**:** *M is f-biharmonic if and only if*

$$-\Delta^\perp H + Tr.\ (\sigma(e_i, A_He_i)) + \frac{\Delta f}{f}H + 2\nabla^\perp_{\overrightarrow{\nabla}(lnf)}H \tag{3.14}$$
$$= mf_1H - 3f_2FtH - f_3H + f_4\{m(hH)^\perp + (Tr.\ h^T)H\}$$
$$-f_5\{(h(hH)^T)^\perp - (Tr.\ h^T)(hH)^\perp + (Tr.\ (\phi h)^T)(\phi hH)^\perp$$
$$-(\phi h(\phi hH)^T)^\perp\} - f_6(hH)^\perp,$$

$$\begin{aligned}&\frac{m}{2}\overrightarrow{\nabla}|H|^2 + 2Tr.\ A_{\nabla^\perp H}(e_i) + 2A_H\overrightarrow{\nabla}(lnf)\\&= -6f_2PtH + 2(m-1)f_4(hH)^T - 2f_5\{(h(hH)^T)^T\\&-(Tr.\ h^T)(hH)^T + (Tr.\ (\phi h)^T)(\phi hH)^T - (\phi h(\phi hH)^T)^T\}\\&-2f_6\{(hH)^T - \eta(hH)\xi\}.\end{aligned} \tag{3.15}$$

Proof. The proof is a direct consequence of the Theorem 3.1 and using the following facts:

1. Proposition 2.1 and $F = 0$ as M is invariant.
2. $P = 0$ as M is anti-invariant.
3. $\xi^T = 0$ and $\xi^\perp = \xi$ if ξ is normal. Since ξ is normal then M is anti-invariant and so $P = 0$.
4. $\xi^T = \xi$, $|\xi^T|^2 = 1$ and $\eta(H) = 0$ as ξ is tangent.

□

Corollary 3.2. (Theorem 3.10 of [17]) *A submanifold M^m of a generalized Sasakian-space-form is f-biharmonic if and only if the following relations hold:*

$$\begin{aligned}&-\Delta^\perp H + Tr.\ (\sigma(e_i, A_He_i)) + \frac{\Delta f}{f}H + 2\nabla^\perp_{\overrightarrow{\nabla}(lnf)}H\\&= mf_1H - 3f_2FtH - f_3\{|\xi^T|^2H + m\eta(H)\xi^\perp\},\end{aligned} \tag{3.16}$$

$$\begin{aligned}&\frac{m}{2}\overrightarrow{\nabla}|H|^2 + 2Tr.\ A_{\nabla^\perp H}(e_i) + 2A_H\overrightarrow{\nabla}(lnf)\\&= -6f_2PtH - 2(m-1)f_3\eta(H)\xi^T.\end{aligned} \tag{3.17}$$

4 f-biharmonic hypersurfaces of $\bar{M}^{2n+1}(f_1,\cdots,f_6)$

This section deals with f-biharmonic hypersurface of $\bar{M}^{2n+1}(f_1,\cdots,f_6)$ and obtained the following:

Proposition 4.1. *Let M^{2n} be a hypersurface of $\bar{M}^{2n+1}(f_1,\cdots,f_6)$ with non-zero constant mean curvature H and $\xi \in \Gamma(TM)$. Then M is proper f-biharmonic if and only if*

$$|\sigma|^2 + \frac{\Delta f}{f} = 2nf_1 + 3f_2 - f_3 + f_4Tr.\ h^T$$

or equivalently

$$\begin{aligned} scal_M &= 2n(2n-2)f_1 + 6(n-1)f_2 - (4n-1)f_3 \\ &+ (4n-3)Tr.\ h^T + f_5\{(Tr.\ h^T)^2 - \|h^T\|^2 + \|(\phi h)^T\|^2 \\ &- (Tr.\ (\phi h)^T)^2\} - 2f_6 Tr.\ h^T + 4n^2H^2 + \frac{\Delta f}{f}H, \end{aligned} \tag{4.1}$$

where $scal_M$ is the scalar curvature of M.

Proof. If M^{2n} is a hypersurface of $\bar{M}^{2n+1}(f_1, \cdots, f_6)$, then ϕH is tangent and from (2.12), we get $sH = 0$.
Since $\xi \in \Gamma(TM)$, then $\eta(H) = 0$. Now we have

$$-H + \eta(H)\xi = \phi^2 H = PtH + FtH$$

i.e.,

$$PtH = 0 \text{ and } FtH = -H. \tag{4.2}$$

Also we have hH and ϕhH are tangent. Then from Theorem 3.1 we have

$$Tr.\ (\sigma(e_i, A_H e_i)) = \{2nf_1 + 3f_2 - f_3 + f_4(Tr.\ h^T) - \frac{\Delta f}{f}\}H. \tag{4.3}$$

Since $Tr.\ (\sigma(e_i, A_H e_i)) = |\sigma|^2 H$, from (4.3) we have

$$|\sigma|^2 H = \{2nf_1 + 3f_2 - f_3 + f_4(Tr.\ h^T) - \frac{\Delta f}{f}\}H. \tag{4.4}$$

Since H is non-zero constant mean curvature, we have from (4.4) that

$$|\sigma|^2 = 2nf_1 + 3f_2 - f_3 + f_4 Tr.\ h^T - \frac{\Delta f}{f}. \tag{4.5}$$

From (1.1) and (2.11) we have

$$\begin{aligned} scal_M &= \sum_{i,j=1}^{2n} g(\bar{R}(e_i, e_j)e_j, e_i) - |\sigma|^2 + 4n^2H^2 \\ &= 2n(2n-1)f_1 + 3(2n-1)f_2 - 2(2n-1)f_3 \\ &+ 2(2n-1)f_4 Tr.\ h^T + f_5\{(Tr.\ h^T)^2 - \|h^T\|^2 \\ &+ \|(\phi h)^T\|^2 - (Tr.\ (\phi h)^T)^2\} - 2f_6 Tr.\ h^T - |\sigma|^2 + 4n^2H^2. \end{aligned} \tag{4.6}$$

Using (4.5) in (4.6) we have (4.1). This proves the proposition. □

Remark 4.1. There exists no proper f-biharmonic hypersurface with non-zero constant mean curvature H of $\bar{M}^{2n+1}(f_1, \cdots, f_6)$ such that ξ is tangent to M and $2nf_1 + 3f_2 - f_3 + f_4 Tr.\ h^T - \frac{\Delta f}{f} \leq 0$.

5 Bi-f-harmonic submanifolds of $\bar{M}^{2n+1}(f_1, \cdots, f_6)$

This section deals with bi-f-harmonic submanifolds of $\bar{M}^{2n+1}(f_1, \cdots, f_6)$ and obtained the following:

Theorem 5.1. *Let M^m be a submanifold of $\bar{M}^{2n+1}(f_1, \cdots, f_6)$ with a positive $f \in C^\infty(M^m)$. Then M^m is bi-f-harmonic if and only if the following relations hold:*

$$\begin{aligned}
& mf^2\nabla^\perp H + mf^2 Tr.\ \sigma(e_i, A_H e_i) - mf(\Delta f)H - 3mf\nabla^\perp_{\overrightarrow{\nabla} f} H \\
& -fTr.\ \sigma(e_i, \nabla\overrightarrow{\nabla} f) - fTr.\ \nabla^\perp\ \sigma(e_i, \overrightarrow{\nabla}\ f) - m|\overrightarrow{\nabla} f|^2 H - \sigma(\overrightarrow{\nabla} f, \overrightarrow{\nabla} f) \\
& = mf^2\Big[mf_1 H - 3f_2 FtH - f_3\{|\xi^T|^2 H + m\eta(H)\xi^\perp\} + f_4\{m(hH)^\perp \\
& +(Tr.\ h^T)H\} - f_5\{(h(hH)^T)^\perp - (Tr.\ h^T)(hH)^\perp + (Tr.\ (\phi h)^T)(\phi hH)^\perp \\
& -(\phi h(\phi hH)^T)^\perp\} - f_6\{|\xi^T|^2(hH)^\perp + (Tr.\ h^T)\eta(H)\xi^\perp \\
& -\eta(hH)\xi^\perp\}\Big] - f\Big[3f_2 FP\overrightarrow{\nabla} f + f_3(m-1)\eta(\overrightarrow{\nabla} f)\xi^\perp + f_4(1-m)(h(\overrightarrow{\nabla} f))^\perp \\
& +f_5\{(h(h\overrightarrow{\nabla} f)^T)^\perp - (Tr.\ h^T)(h\overrightarrow{\nabla} f)^\perp + (Tr.\ (\phi h)^T)(\phi h\overrightarrow{\nabla} f)^\perp \\
& -(\phi h(\phi h\overrightarrow{\nabla} f)^T)^\perp\} + f_6\{|\xi^T|^2(h\overrightarrow{\nabla} f)^\perp + (Tr.\ h^T)\eta(\overrightarrow{\nabla} f)\xi^\perp - \eta(h\overrightarrow{\nabla} f)\xi^\perp\}
\end{aligned} \tag{5.1}$$

and

$$\begin{aligned}
& \frac{m^2 f^2}{2}\overrightarrow{\nabla}|H|^2 + 2mf^2 Tr.\ (A_{\nabla^\perp H} e_i) + 3mfA_H\overrightarrow{\nabla} f \\
& +f\ Ric_M(\overrightarrow{\nabla} f) + f\overrightarrow{\nabla}(\Delta f) + fTr.(A_{\sigma(e_i, \overrightarrow{\nabla} f)} e_i) - \frac{1}{2}\overrightarrow{\nabla}(|\overrightarrow{\nabla} f|^2) \\
& = mf^2\Big[-6f_2 PtH - 2(m-1)f_3\eta(H)\xi^T + 2(m-1)f_4(hH)^T \\
& -f_5\{(h(hH)^T)^T - (Tr.\ h^T)(hH)^T + (Tr.\ (\phi h)^T)(\phi hH)^T \\
& -(\phi h(\phi hH)^T)^T\} - 2f_6\{|\xi^T|^2(hH)^T + (Tr.\ h^T)\eta(H)\xi^T \\
& -\eta(hH)\xi^T\}\Big] - f\Big[-f_1(m-1)\overrightarrow{\nabla} f + 3f_2 P^2\overrightarrow{\nabla} f \\
& +f_3\{|\xi^T|^2\overrightarrow{\nabla} f + (m-2)\eta(\overrightarrow{\nabla} f)\xi^T + f_4\{(2-m)(h(\overrightarrow{\nabla} f))^T \\
& -(Tr.\ h^T)\overrightarrow{\nabla} f\} + f_5\{(h(h\overrightarrow{\nabla} f)^T)^T - (Tr.\ h^T)(h\overrightarrow{\nabla} f)^T \\
& +Tr.\ (\phi h)^T(\phi h\overrightarrow{\nabla} f)^T - (\phi h(\phi h\overrightarrow{\nabla} f)^T)^T\} + f_6\{|\xi^T|^2(h\overrightarrow{\nabla} f)^T \\
& +Tr.\ h^T\eta(\overrightarrow{\nabla} f)\xi^T - \eta(h\overrightarrow{\nabla} f)\xi^T\}\Big]
\end{aligned} \tag{5.2}$$

where $h^T, (\phi h)^T, \xi^T, (h(\overrightarrow{\nabla} f))^T, (\phi h(\overrightarrow{\nabla} f))^T$ are tangential components and $h^\perp, (\phi h)^\perp, \xi^\perp, (h(\overrightarrow{\nabla} f))^\perp, (\phi h(\overrightarrow{\nabla} f))^\perp$ are normal components of $h, (\phi h)$, $\xi, (h(\overrightarrow{\nabla} f)), (\phi h(\overrightarrow{\nabla} f))$ respectively.

Proof. The equation of bi-f-harmonicity are given by [17]

$$\begin{aligned} & mf^2\nabla^{\perp}H + mf^2 Tr.\ \sigma(e_i, A_H e_i) - mf(\Delta f)H - 3mf\nabla^{\perp}_{\overrightarrow{\nabla} f}H \\ & -fTr.\ \sigma(e_i, \nabla\overrightarrow{\nabla} f) - fTr.\ \nabla^{\perp}\ \sigma(e_i, \overrightarrow{\nabla}\ f) - m|\overrightarrow{\nabla} f|^2 H - \sigma(\overrightarrow{\nabla} f, \overrightarrow{\nabla} f) \\ & = -mf^2 Tr.\ (\bar{R}(e_i, H)e_i)^{\perp} - fTr.\ (\bar{R}(e_i, \overrightarrow{\nabla} f)e_i)^{\perp} \end{aligned} \tag{5.3}$$

and

$$\begin{aligned} & \frac{m^2 f^2}{2}\overrightarrow{\nabla}|H|^2 + 2mf^2 Tr.\ (A_{\nabla^{\perp}H}e_i) + 3mfA_H\overrightarrow{\nabla} f \\ & +f\ Ric_M(\overrightarrow{\nabla} f) + f\overrightarrow{\nabla}(\Delta f) + fTr.(A_{\sigma(e_i,\overrightarrow{\nabla} f)}e_i) - \frac{1}{2}\overrightarrow{\nabla}(|\overrightarrow{\nabla} f|^2) \\ & = -2mf^2 Tr.\ (\bar{R}(e_i, H)e_i)^T - fTr.\ (\bar{R}(e_i, \overrightarrow{\nabla} f)e_i)^T. \end{aligned} \tag{5.4}$$

Putting $X = Z = e_i$ and $Y = \overrightarrow{\nabla} f$ in (1.1) and then taking summation over $i = 1, \cdots, m$ we have

$$\begin{aligned} & Tr.\ \bar{R}(e_i, \overrightarrow{\nabla} f)e_i \\ & = -f_1(m-1)\overrightarrow{\nabla} f + 3f_2\{P^2\overrightarrow{\nabla} f + FP\overrightarrow{\nabla} f\} + f_3\{|\xi^T|^2\overrightarrow{\nabla} f \\ & -\eta(\overrightarrow{\nabla} f)\xi^T + (m-1)\eta(\overrightarrow{\nabla} f)\xi\} + f_4\{(1-m)(h(\overrightarrow{\nabla} f)) + (h\overrightarrow{\nabla} f)^T \\ & -(Tr.\ h^T)\overrightarrow{\nabla} f\} + f_5\{(h(h\overrightarrow{\nabla} f)^T) - (Tr.\ h^T)(h\overrightarrow{\nabla} f) \\ & +(Tr.\ (\phi h)^T)(\phi h\overrightarrow{\nabla} f)\} - (\phi h(\phi h\overrightarrow{\nabla} f)^T)\} + f_6\{|\xi^T|^2(h\overrightarrow{\nabla} f) \\ & +(Tr.\ h^T)\eta(\overrightarrow{\nabla} f)\xi - \eta(h\overrightarrow{\nabla} f)\xi\}. \end{aligned} \tag{5.5}$$

Substituting the tangential components of (3.7) and (5.5) in (5.3) we get (5.1) and normal components of (3.7) and (5.5) in (5.4) we get (5.2). □

Corollary 5.1. *Let M^m be a submanifold of $\bar{M}^{2n+1}(f_1, \cdots, f_6)$. Then we have the following:*
(i) M **is invariant:** *M is bi-f-harmonic if and only if*

$$\begin{aligned} & mf^2\nabla^{\perp}H + mf^2 Tr.\ \sigma(e_i, A_H e_i) - mf(\Delta f)H - 3mf\nabla^{\perp}_{\overrightarrow{\nabla}\ f}H \\ & -fTr.\ \sigma(e_i, \nabla\overrightarrow{\nabla} f) - fTr.\ \nabla^{\perp}\ \sigma(e_i, \overrightarrow{\nabla}\ f) - m|\overrightarrow{\nabla} f|^2 H - \sigma(\overrightarrow{\nabla} f, \overrightarrow{\nabla} f) \\ & = mf^2\Big[mf_1 H - f_3\{|\xi^T|^2 H + m\eta(H)\xi^{\perp}\} + f_4 m(hH)^{\perp} \\ & -f_5\{(h(hH)^T)^{\perp} - (\phi h(\phi hH)^T)^{\perp}\} - f_6\{|\xi^T|^2(hH)^{\perp} \\ & -\eta(hH)\xi^{\perp}\}\Big] - f\Big[3f_2 FP\overrightarrow{\nabla} f + f_3(m-1)\eta(\overrightarrow{\nabla} f)\xi^{\perp} \\ & +f_4(1-m)(h(\overrightarrow{\nabla} f))^{\perp} + f_5\{(h(h\overrightarrow{\nabla} f)^T)^{\perp} - (\phi h(\phi h\overrightarrow{\nabla} f)^T)^{\perp}\} \\ & +f_6\{|\xi^T|^2(h\overrightarrow{\nabla} f)^{\perp} - \eta(h\overrightarrow{\nabla} f)\xi^{\perp}\}\Big], \end{aligned} \tag{5.6}$$

$$\frac{m^2f^2}{2}\overrightarrow{\nabla}|H|^2 + 2mf^2Tr.\ (A_{\nabla^{\perp}H}e_i) + 3mfA_H\overrightarrow{\nabla}f \tag{5.7}$$
$$+f\ Ric_M(\overrightarrow{\nabla}f) + f\overrightarrow{\nabla}(\Delta f) + fTr.(A_{\sigma(e_i,\overrightarrow{\nabla}f)}e_i) - \frac{1}{2}\overrightarrow{\nabla}(|\overrightarrow{\nabla}f|^2)$$
$$= mf^2\Big[-6f_2PtH - 2(m-1)f_3\eta(H)\xi^T + 2(m-1)f_4(hH)^T$$
$$-2f_5\{(h(hH)^T)^T - (\phi h(\phi hH)^T)^T\} - 2f_6\{|\xi^T|^2(hH)^T$$
$$-\eta(hH)\xi^T\}\Big] - f\Big[-f_1(m-1)\overrightarrow{\nabla}f + 3f_2P^2\overrightarrow{\nabla}f$$
$$+f_3\{|\xi^T|^2\overrightarrow{\nabla}f + (m-2)\eta(\overrightarrow{\nabla}f)\xi^T + f_4\{(2-m)(h(\overrightarrow{\nabla}f))^T$$
$$+f_5\{(h(h\overrightarrow{\nabla}f)^T)^T - (\phi h(\phi h\overrightarrow{\nabla}f)^T)^T\} + f_6\{|\xi^T|^2(h\overrightarrow{\nabla}f)^T$$
$$-\eta(h\overrightarrow{\nabla}f)\xi^T\}\Big].$$

(ii) M **is anti-invariant:** M *is bi-*f*-harmonic if and only if*

$$mf^2\nabla^{\perp}H + mf^2Tr.\ \sigma(e_i,A_He_i) - mf(\Delta f)H - 3mf\nabla^{\perp}_{\overrightarrow{\nabla}f}H \tag{5.8}$$
$$-fTr.\ \sigma(e_i,\nabla\overrightarrow{\nabla}f) - fTr.\ \nabla^{\perp}\ \sigma(e_i,\overrightarrow{\nabla}\ f) - m|\overrightarrow{\nabla}f|^2H - \sigma(\overrightarrow{\nabla}f,\overrightarrow{\nabla}f)$$
$$= mf^2\Big[mf_1H - 3f_2FtH - f_3\{|\xi^T|^2H + m\eta(H)\xi^{\perp}\} + f_4\{m(hH)^{\perp}$$
$$+(Tr.\ h^T)H\} - f_5\{(h(hH)^T)^{\perp} - (Tr.\ h^T)(hH)^{\perp} + (Tr.\ (\phi h)^T)(\phi hH)^{\perp}$$
$$-(\phi h(\phi hH)^T)^{\perp}\} - f_6\{|\xi^T|^2(hH)^{\perp} + (Tr.\ h^T)\eta(H)\xi^{\perp}$$
$$-\eta(hH)\xi^{\perp}\}\Big] - f\Big[f_3(m-1)\eta(\overrightarrow{\nabla}f)\xi^{\perp} + f_4(1-m)(h(\overrightarrow{\nabla}f))^{\perp}$$
$$+f_5\{(h(h\overrightarrow{\nabla}f)^T)^{\perp} - (Tr.\ h^T)(h\overrightarrow{\nabla}f)^{\perp} + (Tr.\ (\phi h)^T)(\phi h\overrightarrow{\nabla}f)^{\perp}$$
$$-(\phi h(\phi h\overrightarrow{\nabla}f)^T)^{\perp}\} + f_6\{|\xi^T|^2(h\overrightarrow{\nabla}f)^{\perp} + (Tr.\ h^T)\eta(\overrightarrow{\nabla}f)\xi^{\perp}$$
$$-\eta(h\overrightarrow{\nabla}f)\xi^{\perp}\},$$

$$\frac{m^2f^2}{2}\overrightarrow{\nabla}|H|^2 + 2mf^2Tr.\ (A_{\nabla^{\perp}H}e_i) + 3mfA_H\overrightarrow{\nabla}f \tag{5.9}$$
$$+f\ Ric_M(\overrightarrow{\nabla}f) + f\overrightarrow{\nabla}(\Delta f) + fTr.(A_{\sigma(e_i,\overrightarrow{\nabla}f)}e_i) - \frac{1}{2}\overrightarrow{\nabla}(|\overrightarrow{\nabla}f|^2)$$
$$= mf^2\Big[-2(m-1)f_3\eta(H)\xi^T + 2(m-1)f_4(hH)^T - 2f_5\{(h(hH)^T)^T$$
$$-(Tr.\ h^T)(hH)^T + (Tr.\ (\phi h)^T)(\phi hH)^T - (\phi h(\phi hH)^T)^T\}$$
$$-2f_6\{|\xi^T|^2(hH)^T + (Tr.\ h^T)\eta(H)\xi^T - \eta(hH)\xi^T\}\Big]$$

$$-f\Big[-f_1(m-1)\overrightarrow{\nabla}f+3f_2P^2\overrightarrow{\nabla}f+f_3\{|\xi^T|^2\overrightarrow{\nabla}f$$
$$+(m-2)\eta(\overrightarrow{\nabla}f)\xi^T+f_4\{(2-m)(h(\overrightarrow{\nabla}f))^T-(Tr.\ h^T)\overrightarrow{\nabla}f\}$$
$$+f_5\{(h(h\overrightarrow{\nabla}f)^T)^T-(Tr.\ h^T)(h\overrightarrow{\nabla}f)^T+(Tr.\ (\phi h)^T)(\phi h\overrightarrow{\nabla}f)^T$$
$$-(\phi h(\phi h\overrightarrow{\nabla}f)^T)^T\}+f_6\{|\xi^T|^2(h\overrightarrow{\nabla}f)^T+(Tr.\ h^T)\eta(\overrightarrow{\nabla}f)\xi^T$$
$$-\eta(h\overrightarrow{\nabla}f)\xi^T\}\Big].$$

(iii) $\xi\in T^{\perp}M$**:** *M is bi-f-harmonic if and only if*

$$mf^2\nabla^{\perp}H+mf^2Tr.\ \sigma(e_i,A_He_i)-mf(\Delta f)H-3mf\nabla^{\perp}_{\overrightarrow{\nabla}f}H \tag{5.10}$$
$$-fTr.\ \sigma(e_i,\nabla\overrightarrow{\nabla}f)-fTr.\ \nabla^{\perp}\ \sigma(e_i,\overrightarrow{\nabla}\ f)-m|\overrightarrow{\nabla}f|^2H$$
$$=-\sigma(\overrightarrow{\nabla}f,\overrightarrow{\nabla}f)+mf^2\Big[mf_1H-3f_2FtH-f_3m\eta(H)\xi+f_4\{m(hH)^{\perp}$$
$$+(Tr.\ h^T)H\}-f_5\{(h(hH)^T)^{\perp}-(Tr.\ h^T)(hH)^{\perp}+(Tr.\ (\phi h)^T)(\phi hH)^{\perp}$$
$$-(\phi h(\phi hH)^T)^{\perp}\}-f_6\{(Tr.\ h^T)\eta(H)\xi-\eta(hH)\xi\}\Big]-f\Big[3f_2FP\overrightarrow{\nabla}f$$
$$+f_4(1-m)(h(\overrightarrow{\nabla}f))^{\perp}+f_5\{(h(h\overrightarrow{\nabla}f)^T)^{\perp}-(Tr.\ h^T)(h\overrightarrow{\nabla}f)^{\perp}$$
$$+(Tr.\ (\phi h)^T)(\phi h\overrightarrow{\nabla}f)^{\perp}-(\phi h(\phi h\overrightarrow{\nabla}f)^T)^{\perp}\}-f_6\eta(h\overrightarrow{\nabla}f)\xi,$$

$$\frac{m^2f^2}{2}\overrightarrow{\nabla}|H|^2+2mf^2Tr.\ (A_{\nabla^{\perp}H}e_i)+3mfA_H\overrightarrow{\nabla}f \tag{5.11}$$
$$+f\ Ric_M(\overrightarrow{\nabla}f)+f\overrightarrow{\nabla}(\Delta f)+fTr.(A_{\sigma(e_i,\overrightarrow{\nabla}f)}e_i)-\frac{1}{2}\overrightarrow{\nabla}(|\overrightarrow{\nabla}f|^2)$$
$$=mf^2\Big[2(m-1)f_4(hH)^T-2f_5\{(h(hH)^T)^T-(Tr.\ h^T)(hH)^T$$
$$+(Tr.\ (\phi h)^T)(\phi hH)^T-(\phi h(\phi hH)^T)^T\}\Big]-f\Big[-f_1(m-1)\overrightarrow{\nabla}f$$
$$+3f_2P^2\overrightarrow{\nabla}f+f_4\{(2-m)(h(\overrightarrow{\nabla}f)^T)^T-(Tr.\ h^T)\overrightarrow{\nabla}f\}+f_5\{(h(h\overrightarrow{\nabla}f)^T)^T$$
$$-(Tr.\ h^T)(h\overrightarrow{\nabla}f)^T+(Tr.\ (\phi h)^T)(\phi h\overrightarrow{\nabla}f)^T-(\phi h(\phi h\overrightarrow{\nabla}f)^T)^T\}\Big].$$

(iv) $\xi \in TM$: *M is bi-f-harmonic if and only if*

$$
\begin{aligned}
& mf^2\nabla^{\perp}H + mf^2Tr.\ \sigma(e_i, A_He_i) - mf(\Delta f)H - 3mf\nabla^{\perp}_{\overrightarrow{\nabla} f}H \\
& -fTr.\ \sigma(e_i, \nabla\overrightarrow{\nabla} f) - fTr.\ \nabla^{\perp}\ \sigma(e_i, \overrightarrow{\nabla}\ f) - m|\overrightarrow{\nabla} f|^2H - \sigma(\overrightarrow{\nabla} f, \overrightarrow{\nabla} f) \\
& = mf^2\Big[mf_1H - 3f_2FtH - f_3H + f_4\{m(hH)^{\perp} + (Tr.\ h^T)H\} \\
& -f_5\{(h(hH)^T)^{\perp} - (Tr.\ h^T)(hH)^{\perp} + (Tr.\ (\phi h)^T)(\phi hH)^{\perp} \\
& -(\phi h(\phi hH)^T)^{\perp}\} - f_6(hH)^{\perp}\Big] - f\Big[3f_2FP\overrightarrow{\nabla} f + f_4(1-m)(h(\overrightarrow{\nabla} f))^{\perp} \\
& +f_5\{(h(h\overrightarrow{\nabla} f)^T)^{\perp} - (Tr.\ h^T)(h\overrightarrow{\nabla} f)^{\perp} + (Tr.\ (\phi h)^T)(\phi h\overrightarrow{\nabla} f)^{\perp} \\
& -(\phi h(\phi h\overrightarrow{\nabla} f)^T)^{\perp}\} + f_6\{(h\overrightarrow{\nabla} f)^{\perp}\Big],
\end{aligned}
\tag{5.12}
$$

$$
\begin{aligned}
& \frac{m^2f^2}{2}\overrightarrow{\nabla}|H|^2 + 2m^2f^2Tr.\ (A_{\nabla^{\perp}H}e_i) + 3mfA_H\overrightarrow{\nabla} f \\
& +f\ Ric_M(\overrightarrow{\nabla} f) + f\overrightarrow{\nabla}(\Delta f) + fTr.(A_{\sigma(e_i,\overrightarrow{\nabla} f)}e_i) - \frac{1}{2}\overrightarrow{\nabla}(|\overrightarrow{\nabla} f|^2) \\
& = mf^2\Big[-6f_2PtH + 2(m-1)f_4(hH)^T - 2f_5\{(h(hH)^T)^T \\
& -(Tr.\ h^T)(hH)^T + (Tr.\ (\phi h)^T)(\phi hH)^T - (\phi h(\phi hH)^T)^T\} \\
& -2f_6\{(hH)^T - \eta(hH)\xi\}\Big] - f\Big[-f_1(m-1)\overrightarrow{\nabla} f + 3f_2P^2\overrightarrow{\nabla} f \\
& +f_3\{\overrightarrow{\nabla} f + (m-2)\eta(\overrightarrow{\nabla} f)\xi^T\} + f_4\{(2-m)(h(\overrightarrow{\nabla} f))^T \\
& -(Tr.\ h^T)\overrightarrow{\nabla} f\} + f_5\{(h(h\overrightarrow{\nabla} f)^T)^T - (Tr.\ h^T)(h\overrightarrow{\nabla} f)^T \\
& +(Tr.\ (\phi h)^T)(\phi h\overrightarrow{\nabla} f)^T - (\phi h(\phi h\overrightarrow{\nabla} f)^T)^T\} + f_6\{(h\overrightarrow{\nabla} f)^T \\
& +(Tr.\ h^T)\eta(\overrightarrow{\nabla} f)\xi - \eta(h\overrightarrow{\nabla} f)\xi\}\Big].
\end{aligned}
\tag{5.13}
$$

Proof. The proof is a direct consequence of the Theorem 5.1 and using the following facts:
(i) Proposition 2.1 and $F = 0$ as M is invariant.
(ii) $P = 0$ as M is anti-invariant.
(iii) $\xi^T = 0$ and $\xi^{\perp} = \xi$ if ξ is normal. Since ξ is normal then M is anti-invariant and so $P = 0$ and $\eta(\overrightarrow{\nabla} f) = 0$.
(iv) $\xi^T = \xi$, $|\xi^T|^2 = 1$ and ξ is tangent. □

Acknowledgement: The third author gratefully acknowledges to the CSIR(File No.: 09/025(0221)/2017-EMR-I), Government of India for financial support providing Junior Research Fellowship (JRF).

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