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Proper classes generated by τ- closed submodules

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Abstract

The main object of this paper is to study relative homological aspects as well as further properties of τ-closed submodules. A submodule N of a module M is said to be τ-closed (or τ-pure) provided that M/N is τ-torsion-free, where τ stands for an idempotent radical. Whereas the well-known proper class $\mathcal{C}losed$ ($\mathcal{P}ure$) of closed (pure) short exact sequences, the class $\tau-\mathcal{C}losed$ of τ-closed short exact sequences need not be a proper class. We describe the smallest proper class $\langle\tau - \mathcal{C}losed\rangle$ containing $\tau - \mathcal{C}losed$, through τ-closed submodules. We show that the smallest proper class $\langle\tau - \mathcal{C}losed\rangle$ is the proper classes projectively generated by the class of τ-torsion modules and coprojectively generated by the class of τ-torsion-free modules. Also, we consider the relations between the proper class $\langle\tau - \mathcal{C}losed\rangle$ and some of well-known proper classes, such as $\mathcal{C}losed, \mathcal{P}ure$.

1 Introduction

Closed submodules plays an important role in ring and module theory and relative homological algebra. Several relative notions of module theory by using the concept of closed submodule have defined. Researchers have defined different generalizations of closed submodules by following two distinct ways.

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Following closed subgroup characterizations, some researchers introduced and studied neat, coneat and s-pure submodules as different generalizations of closed submodule (for instance see [5, 6, 14, 18, 21, 23]). Also, by using torsion theories, τ-closed submodule were introduced and studied as generalization of closed submodule, where τ stands for an idempotent radical (for instance see [2, 3, 11, 15, 16, 19, 24, 26, 27]). Compared with the closed, neat, coneat and s-pure submodules, in homological aspect, τ-closed submodule has some deficiency. Because, while the class of closed (respectively, neat, coneat and s-pure) short exact sequences is a proper class, the class of τ-closed short exact sequences need not be a proper class. It is known that closed submodules are neat, and neat submodules are closed over C-rings([8, 10.10]). Moreover, over commutative ring R, closed, neat, coneat and s-pure submodules are the same if and only if R is noetherian distributive ([12]). Another problem related with τ-closed submodule is that it is not known when closed submodules are τ-closed and τ-closed submodules are closed. In this paper we study the proper class of short exact sequences determined by τ-closed submodules in the category of left R-modules. Finally, we consider the relations between the proper class $\langle \tau - \mathcal{C}losed \rangle$ and some of well-known proper classes, such as $\mathcal{C}losed, \mathcal{P}ure$.

Throughout this paper, R will denote an associative ring with identity, R-Mod will be the category of unitary left R-modules, and all modules and module homomorphisms will belong to R-Mod. After this introductory section, this paper is divided into two sections. In Section 2, we recall some torsion theoretic concepts and then give some properties of proper class. In Section 3, we first show that the class $\tau - \mathcal{C}losed$ of τ-closed short exact sequences need not be a proper class, and then we consider the smallest proper class $\mathcal{P}_\tau$ containing $\tau - \mathcal{C}losed$, that is, the intersection of all proper classes containing it. We show that $\mathcal{P}_\tau$ coincides with the proper classes projectively generated by the class of τ-torsion modules and coprojectively generated by the class of τ-torsion-free modules. We describe the class $\mathcal{P}_\tau$ in terms of τ-closed submodules: we show that A is a $\mathcal{P}_\tau$-submodule of B if and only if there is a submodule C of B such that $A \cap S = 0$ and $B/(A \oplus S)$ is τ-torsion-free. For a hereditary torsion theory τ, we show that; (1) every τ closed submodule is closed if and only if every $\mathcal{P}_\tau$-submodule is closed if and only if every singular module is τ-torsion module, (2) if $\tau(R_R) = 0$, then $\mathcal{P}_\sigma = \mathcal{P}_\tau$ if and only if R is a right C_τ, (3) if $\tau(R_R) = 0$, then $\mathcal{P}_\tau = \mathcal{N}eat$ if and only if each τ-torsion module is semisimple, (4). if R is a commutative Noetherian ring, then every τ-closed submodule is pure if and only if $R \cong A \times B$, wherein A is τ-torsion ring and B is hereditary C_τ ring. In particular, over commutative Noetherian ring R, every $\mathcal{S}$-closed submodule is pure if and only if $R \cong A \times B$, wherein A is Goldie torsion ring and B is hereditary ring. We refer the reader to [8, 17, 25] for the undefined notions used in the text.

2 Preliminaries

We refer the reader to [10, 15] for more detailed about torsion theories. A torsion theory $\tau := (\mathbb{T}_\tau, \mathbb{F}_\tau)$ for the category of left R-modules consists of two classes $\mathbb{T}_\tau$ and $\mathbb{F}_\tau$, the τ-torsion class and the τ-torsion-free class, respectively, such that $Hom(T, F) = 0$ whenever $T \in \mathbb{T}_\tau$ and $F \in \mathbb{F}_\tau$, the class $\mathbb{T}_\tau$ is closed under factor modules, extensions and arbitrary direct sums, the class $\mathbb{F}_\tau$ is closed under submodules, extensions and arbitrary direct products. Modules in $\mathbb{T}_\tau$ will be called τ-torsion and modules in $\mathbb{F}_\tau$ will be called τ-torsion-free modules. For each module M, the τ-torsion submodule of M, denoted by $\tau(M)$, is defined to be the sum of all τ-torsion submodules of M and there exists an exact sequence $0 \to \tau(M) \to M \to F \to 0$ such that $M/\tau(M) \cong F \in \mathbb{F}_\tau$. A torsion theory τ is called hereditary if $\mathbb{T}_\tau$ is closed under submodules. A submodule N of a module M is called τ-closed (or τ-pure), if the factor module M/N belongs to $\mathbb{F}_\tau$.

For a module M, the singular submodule $Z(M)$ consists of all elements $a \in M$, the annihilator left ideal $(0 : a) = \{r \in R : ra = 0\}$ of which is essential in R. Goldie's torsion theory for the category R-Mod is the hereditary torsion theory $\sigma := (\mathbb{T}_\sigma, \mathbb{F}_\sigma)$, where $\mathbb{T}_\sigma = \{M \in R-Mod : Z(M/Z(M)) = M/Z(M)\}$ and $\mathbb{F}_\sigma = \{M \in R - Mod : Z(M) = 0\}$. If the ring R is σ-torsion-free, $Z(R) = 0$, then R is called non-singular. Note that a submodule N of a module M is s-closed if and only if it is σ-closed in M (see [16, 26]). A module M is called semiartinian if every non-zero homomorphic image of M contains a simple submodule, that is, $\mathrm{Soc}(M/N) \neq 0$ for every submodule $N \lneqq M$. Dickson's torsion theory for the category R-Mod is the hereditary torsion theory $\rho := (\mathbb{T}_\rho, \mathbb{F}_\rho)$, where $\mathbb{T}_\rho = \{M \in R - Mod :\ \text{M is semiartinian}\}$ and $\mathbb{F}_\rho = \{M \in R - Mod : \mathrm{Soc}(M) = 0\}$.

Proper classes (or purities) have offered rich topics of research because of their important role played in category theory. Proper classes were introduced by Buchsbaum in [4] for an exact category. Buchsbaum introduced certain axioms on a class of monomorphisms that are necessary to study relative homology algebra. This lead to the notion of proper classes of monomorphisms and short exact sequences.

Definition 2.1. Let $\mathcal{P}$ be a class of short exact sequences of modules and module homomorphisms

$$\mathbb{E} : 0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0.$$

We suppose that $\mathcal{P}$ is closed under isomorphisms. If $\mathbb{E} \in \mathcal{P}$, we say that $\mathbb{E}$ is $\mathcal{P}$ exact, f is a $\mathcal{P}$-*monomorphism*, g is $\mathcal{P}$-*epimorphism* and $\operatorname{Im} f$ is $\mathcal{P}$ submodule of B.

The class $\mathcal{P}$ is said to be *proper* (in the sense of Buchsbaum) if it satisfies the following conditions (see, for example, [8]):

P-1) If a short exact sequence $\mathbb{E}$ is in $\mathcal{P}$, then $\mathcal{P}$ contains every short exact sequence isomorphic to $\mathbb{E}$.

P-2) $\mathcal{P}$ contains all splitting short exact sequences.

P-3) The composite of two $\mathcal{P}$-monomorphisms is a $\mathcal{P}$-monomorphism if this composite is defined. The composite of two $\mathcal{P}$-epimorphisms is a $\mathcal{P}$-epimorphism if this composite is defined.

P-4) If g and f are monomorphisms and $g \circ f$ is a $\mathcal{P}$-monomorphism, then f is a $\mathcal{P}$-monomorphism. If g and f are epimorphisms and $g \circ f$ is a $\mathcal{P}$-epimorphism, then g is a $\mathcal{P}$-epimorphism.

The smallest proper class of modules consists of all splitting short exact sequences of modules which we denote by $\mathcal{S}plit$. The largest proper class of modules consists of all short exact sequences of modules which we denote by $\mathcal{A}bs$.

The intersection of all proper classes containing the class $\mathcal{P}$ is clearly a proper class, denoted by $\langle\mathcal{P}\rangle$. The class $\langle\mathcal{P}\rangle$ is the smallest proper class containing $\mathcal{P}$, called the proper class *generated* by $\mathcal{P}$. A module M is called $\mathcal{P}$-*projective* if it is projective with respect to all short exact sequences in $\mathcal{P}$, that is, $\operatorname{Hom}(M, E)$ is exact for every E in $\mathcal{P}$. Notice that the proper class $\langle\mathcal{P}\rangle$ has the same projective modules as $\mathcal{P}$ (see [22]). A module M is called $\mathcal{P}$-*coprojective* if every short exact sequence of the form $0 \to A \to B \to M \to 0$ is in $\mathcal{P}$. For a given class $\mathcal{M}$ of modules, denote by $\overline{k}(\mathcal{M})$ the smallest proper class for which each $M \in \mathcal{M}$ is $\overline{k}(\mathcal{M})$-coprojective; it is called the proper class *coprojectively generated* by $\mathcal{M}$. The largest proper class $\mathcal{P}$ for which each $M \in \mathcal{M}$ is $\mathcal{P}$-projective is called the proper class *projectively generated* by $\mathcal{M}$. See [25] and [20] for further details on proper classes. A well-known example of proper classes is the class $\mathcal{C}losed$ of short exact sequences determined by closed submodules. Another important example is $\mathcal{P}ure$, the class of all pure short exact sequences in the sense of Cohn [9], that is, the class of all short exact sequences $\mathbb{E}$ such that $\operatorname{Hom}(P, \mathbb{E})$ is exact for every finitely presented left R-module P.

3 Proper Classes Relative to Torsion Theories

Let $\tau = (\mathbb{T}_\tau, \mathbb{F}_\tau)$ be a torsion theory. Recall that a submodule N of a module M is called τ-closed (or τ-pure) if the factor module M/N is τ-torsion-free. Denote by $\tau - \mathcal{C}losed$ the class of all short exact sequences $0 \to A \xrightarrow{f} B \to C \to 0$ such that $\operatorname{Im} f$ is τ-closed in B. In contrast to the class $\mathcal{C}losed$, the class $\tau - \mathcal{C}losed$ is not a proper class, in general.

Example 3.1. Let M be a module which is not τ-torsion-free. We consider the short exact sequence $0 \to 0 \to M \to M \to 0$. This sequence splits. The zero submodule 0 is not $\tau-$closed submodule of M because $M/0 \cong M$ is not $\tau-$torsion-free. So the sequence is not a $\tau-$closed exact sequence. Therefore, by the condition $P-2)$ of proper class, we can say that $\tau-$closed does not form a proper class.

Note that the class $\tau - \mathcal{C}losed$ satisfies all conditions in Definition 2.1 except $P-2)$, in general. Now we give some conditions for the class $\tau - \mathcal{C}losed$ to be proper.

Proposition 3.2. *The following are equivalent for a ring R .*

(1) $\tau - \mathcal{C}losed$ is a proper class.

(2) Every left R-module is τ-torsion-free.

(3) Every injective left R-module module is τ-torsion-free.

(4) $\mathcal{A}bs = \tau - \mathcal{C}losed$

Proof. (1) $\Rightarrow$ (2) Consider the short exact sequence $0 \to 0 \to M \to M \to 0$ for a module M. The sequence splits, so by hypothesis, it is τ-closed exact, that is, the zero submodule 0 is τ-closed in M. So M is τ-torsion-free. (2) $\Rightarrow$ (4), (2) $\Rightarrow$ (3) and (4) $\Rightarrow$ (1) are clear.

(3) $\Rightarrow$ (2) This follows by the facts that every module is a submodule of an injective module and any submodule of τ-torsion-free module is τ-torsion-free. □

Remark 3.3. (1) $\sigma - \mathcal{C}losed$ is a proper class if and only if R is a semisimple ring (see [13, Proposition 3.2]).

(2) $\rho - \mathcal{C}losed$ is not a proper class over any ring R. Because, if it were, simple modules would be zero, and so this case is not possible.

Kepka introduced new kind of proper classes in [20]. Following Kepka, we introduced extended τ-closed submodules such as: A submodule A of a module B is called *extended τ-closed* in B if there is a submodule S in B such that $S \cap A = 0$ and $B/(S \oplus A)$ is τ-torsion-free. τ-closed submodules are extended τ-closed, but the converse is not true in general (see Example 3.1). Denote by $\mathcal{P}_\tau$ the class of all short exact sequences $E : 0 \to A \xrightarrow{f} B \to C \to 0$ such that $\operatorname{Im} f$ is extended τ-closed in B. It is known that the class $\mathcal{P}_\tau$ forms a proper class (see [20, Theorem 2.1]). Now we show that $\mathcal{P}_\tau$ is the smallest proper class generated by the class $\tau - \mathcal{C}losed$.

By mimicking the proof of [13, Proposition 3.3-3.4], we prove the following two result. We have included full proofs of these results for completeness.

Proposition 3.4. $\langle \tau - \mathcal{C}losed \rangle = \mathcal{P}_\tau$.

Proof. Since $\langle \tau - \mathcal{C}losed \rangle$ is the smallest proper class generated by the class $\tau - \mathcal{C}losed$ and $\tau - \mathcal{C}losed \subseteq \mathcal{P}_\tau$, we have $\langle \tau - \mathcal{C}losed \rangle \subseteq \mathcal{P}_\tau$. Conversely, let $0 \to A \xrightarrow{f} B \to C \to 0$ be a $\mathcal{P}_\tau$ exact sequence. Then there is a submodule S in B such that $S \cap A = 0$ and $B/(S \oplus A)$ is τ-torsion-free. Consider the following commutative diagram, α is a $\mathcal{S}plit$-monomorphism, and so an $\langle \tau - \mathcal{C}losed \rangle$-monomorphism by $P-2)$ of Definition 2.1. Furthermore, since $B/(S \oplus A)$ is τ-torsion-free, β is an $\langle \tau - \mathcal{C}losed \rangle$-monomorphism. By $P-3)$ of Definition 2.1, we see that $f = \beta\alpha$ is also an $\langle \tau - \mathcal{C}losed \rangle$-monomorphism.

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A \arrow[r, "\alpha"] \arrow[d, equal] & S \oplus A \arrow[r] \arrow[d, "\beta"] & (S \oplus A)/A \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r, "f"] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
 & & B/(S \oplus A) \arrow[r, equal] \arrow[d] & B/(S \oplus A) \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

$\square$

Proposition 3.5. *A submodule A of a module B is $\mathcal{P}_\tau$-submodule in B if and only if* $\operatorname{Hom}(T, B) \to \operatorname{Hom}(T, B/A) \to 0$ *is exact for every τ-torsion module T.*

Proof. $(\Rightarrow)$ Let $E : 0 \to A \hookrightarrow B \xrightarrow{\gamma} B/A \to 0$ be an $\mathcal{P}_\tau$ exact sequence and let $f : T \to B/A$ be a homomorphism with T a τ-torsion module. We will show that there is a homomorphism $\varphi : T \to B$ such that $\gamma\varphi = f$. Since E is $\mathcal{P}_\tau$ exact sequence, then $f^*(E) : 0 \to A \to X \to T \to 0$ is an $\mathcal{P}_\tau$ exact by the properties of a proper class. So there is a submodule S of X such that $A \cap S = 0$ and $X/(S \oplus A)$ is τ-torsion-free. Now, we consider the exact sequence, $0 \to (S \oplus A)/A \to T \to X/(S \oplus A) \to 0$. Since $X/(S \oplus A)$ is homomorphic image of a T τ-torsion module and a torsion class closed under homomorphic image, $X/(S \oplus A)$ is τ-torsion. So X would be $S \oplus A$. Now consider the following pushout commutative diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
f^*(E) : 0 \arrow[r] & A \arrow[r, "\alpha"] \arrow[d, equal] & X \arrow[r, "\pi"] \arrow[d, "\beta"] & T \arrow[r] \arrow[d, "f"] \arrow[dl, dashed, "\varphi"'] & 0 \\
E : 0 \arrow[r] & A \arrow[r, "\theta"'] & B \arrow[r, "\gamma"'] \arrow[d] & B/A \arrow[r] \arrow[d, "g"] & 0 \\
 & & B/(S \oplus A) \arrow[r, equal] \arrow[d] & B/(S \oplus A) \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

Since $f^*(E)$ splits, there is a homomorphism $\pi^{-1} : T \to X$ such that $\pi\pi^{-1} = id$. So if we take a homomorphism $\varphi = \beta\pi^{-1} : T \to B$, then $\gamma\varphi = \gamma(\beta\pi^{-1}) = f$ as desired. ($\Leftarrow$) Assume that $\tau(B/A) = T/A$ for some submodule $T \subseteq B$. Since T/A is a τ-torsion module, by hypothesis, it is projective relatively to the short exact sequence $E : 0 \to A \hookrightarrow B \to B/A \to 0$. So the inclusion map $T/A \hookrightarrow B/A$ can be extended to $h : T/A \to B$ with h necessarily monic. Therefore there exists a submodule $S \cong T/A$ of B such that $S \cap A = 0$, and $B/(S \oplus A) \cong B/T \cong (B/A)/(T/A)$ is τ-torsion-free. Hence A is an $\mathcal{P}_\tau$-submodule of B. $\square$

A submodule K of a module L is called τ-complement if there exists a submodule $U \subseteq L$ such that $K \cap U = 0$ and $\tau(L/K) \cong U$ (see [1]). A submodule A of a module B is $\mathcal{P}_\tau$-submodule in B if and only if it is τ-complement by Proposition 3.5

Corollary 3.6. *A ring R is a right τ-torsion if and only if every right R-module is τ-torsion if and only if $\mathcal{P}_\tau = \mathcal{S}plit$.*

Proposition 3.7. $\overline{k}(\mathbb{F}_\tau) = \mathcal{P}_\tau$.

Proof. Since all τ-torsion-free modules are $\mathcal{P}_\tau$-coprojective, we see that $\overline{k}(\mathbb{F}_\tau) \subseteq \mathcal{P}_\tau$. For the converse, let $0 \to A \xrightarrow{f} B \to C \to 0$ be an $\mathcal{P}_\tau$ exact sequence. Then there is a submodule $S \subseteq B$ such that $S \cap A = 0$ and $B/(S \oplus A)$ is τ-torsion-free. Consider the diagram in the proof of Proposition 3.4, since $B/(S \oplus A)$ is τ-torsion-free, β is a $\overline{k}(\mathbb{F}_\tau)$-monomorphism. Thus $f = \beta\alpha$ is also a $\overline{k}(\mathbb{F}_\tau)$-monomorphism. $\square$

Proposition 3.8. *$\mathcal{P}_\tau = \mathcal{S}plit$ if and only if every τ-torsion-free module is projective.*

Proof. Let C be a τ-torsion-free module. We consider the short exact sequence $0 \to A \to P \to C \to 0$ with P projective. Since $C \cong P/A$ is τ-torsion-free, the sequence is an $\mathcal{P}_\tau$ exact, and thus it splits. Since C is a projective as a direct summand of P, C is a projective module. To converse, let $0 \to A \xrightarrow{f} B \to C \to 0$ be an $\mathcal{P}_\tau$ exact sequence. Then there is a submodule S of B such that $S \cap A = 0$ and $B/(S \oplus A)$ is τ-torsion-free. By hypothesis, $B/(S \oplus A)$ is a projective module. Now, consider the diagram in the proof of Proposition 3.4, β is $\mathcal{S}plit$-monomorphisms and thus $f = \beta\alpha$ is also a $\mathcal{S}plit$-monomorphism. $\square$

In the rest of the paper τ is assumed to be hereditary.

Theorem 3.9. *The following conditions are equivalent for a torsion theory τ.*

(1) Every τ-closed submodule is closed.

(2) Every $\mathcal{P}_\tau$-submodule is closed.

(3) Every singular module is τ-torsion module.

(4) Every τ-closed submodule is s-closed.

(5) Every τ-closed left ideal is s-closed.

Proof. (1) $\Rightarrow$ (2) Let A be a $\mathcal{P}_\tau$-submodule of a module B. Then there is a submodule S in B such that $S \cap A = 0$ and $B/(S \oplus A)$ is τ-torsion-free. Consider the diagram in the proof of Proposition 3.4. Since $B/(S \oplus A)$ is τ-torsion-free, by our assumption, β is a $\mathcal{C}losed$-monomorphism. α is a $\mathcal{S}plit$-monomorphisms, and so $f = \beta\alpha$ is a $\mathcal{C}losed$-monomorphism. (2) $\Rightarrow$ (3) Let M be a singular module with $\tau(M) \neq M$. Remind that a module M is singular if there exists a free module F and an essential submodule A of F such that $M \cong F/A$. By our hypothesis, there is a proper submodule X of F such that $X/A \cong \tau(F/A) \neq F/A$. Since A is an essential submodule of F, X is also an essential submodule of F. But $F/X \cong M/\tau(M)$, and so X is a closed submodule of F by our hypothesis, a contradiction. (3) $\Rightarrow$ (4) It is enough to show that every τ-torsion-free module is nonsingular by [24, Lemma 2.3]. Let M be a τ-torsion-free module such that $\sigma(M) \neq 0$. Then $\sigma(M)$ is a τ-torsion submodule of τ-torsion-free module M by (3). But τ-torsion-free modules are closed under submodules, and so $\sigma(M) = 0$. (4) $\Rightarrow$ (1) This follows by [24, Lemma 2.3]. (4) $\Rightarrow$ (5) is obvious. (5) $\Rightarrow$ (3) It is sufficient to show that R/I is τ-torsion for every essential left ideal I of R. Assume contrary that $\tau(R/I) \neq R/I$. Then there is a left ideal J of R such that $J/I = \tau(R/I)$. By essentiality of I, J is also essential in R, and hence R/J is singular. But, by (5), R/J is nonsingular since it is isomorphic to a τ-torsion-free module $(R/I)/\tau(R/I)$, a contradiction. □

The rings R such that every cyclic singular left R-module has a nonzero socle are called C-rings, i.e. every singular left R-module is semiartinian ([8, 10.10]). As a generalization of right C-rings, we call a ring R is left C_τ if it satisifies one of the equivalent conditions of Theorem 3.9.

Remark 3.10. Every ring is a right (or left) C_σ ring. A ring R is right C_ρ if and only if R is right C-ring. Left perfect rings, right semiartinian rings are examples of right C-rings.

Proposition 3.11. *A closed submodule of a τ-torsion-free module is τ-closed.*

Proof. Let B be a closed submodule of a τ-torsion-free module A. Assume that $\tau(A/B) \neq 0$. Then there is a submodule C of A such that C/B is a τ-torsion module. By proper class properties, B is closed in C. Now let $c \in C/B$. Then, $Rc/(Rc \cap B) \cong Rc + B/B \subset C/B$ is a τ-torsion. Since $Rc \subset C$ and C is τ-torsion-free, $Rc \cap B \neq 0$. Otherwise, $Rc/(Rc \cap B) = Rc$ is τ-torsion. Therefore, B is essential in C. But B is closed in C, yields a controdiction. □

The next result follows by [24, Lemma 2.3] and Proposition 3.11.

Corollary 3.12. *Let R be a ring and assume that $\tau(_RR) = 0$. Then, $\mathcal{P}_\sigma \subseteq \mathcal{P}_\tau$.*

By Corollary 3.12 and Theorem 3.9, we have the following.

Corollary 3.13. *Let R be a ring with $\tau(_RR) = 0$. $\mathcal{P}_\sigma = \mathcal{P}_\tau$ if and only if R is a left C_τ.*

Corollary 3.14. *Let R be a ring with $Soc(_RR) = 0$. $\mathcal{P}_\sigma = \mathcal{P}_\rho$ if and only if R is a left C-ring.*

It is well known that a subgroup A of an abelian group B is closed if and only if the sequence $\mathrm{Hom}(S, B) \to \mathrm{Hom}(S, B/A) \to 0$ is exact for each simple abelian group S ([18]). Inspired by this characterization of closed abelian subgroup the notion of a neat submodule is defined in [23]. A submodule A of a module B is called neat in B if the sequence $\mathrm{Hom}(S,B) \to \mathrm{Hom}(S,B/A) \to 0$ is exact for each simple module S. Closed submodules are neat, but the converse is true exactly for C-rings. As the proper class $\mathcal{C}losed$, the class $\mathcal{N}eat$ of neat exact sequences is also a proper class.

Lemma 3.15. *$\mathcal{N}eat = \mathcal{P}_\tau$ if and only if each τ-torsion module T can be represented as $T = S \oplus P$, where S is a semisimple module and P is a projective module.*

Proof. Let T be a τ-torsion module, and let $0 \to A \to B \to B/A \to 0$ be a neat exact sequence. By hypothesis, the sequence is an $\mathcal{P}_\tau$ sequence. So, by Proposition 3.5, the sequence $\mathrm{Hom}(T,B) \to \mathrm{Hom}(T,B/A) \to 0$ is exact, equivalently, T is $\mathcal{N}eat$-projective. Hence, T is a direct sum of a projective module and a semisimple module by [14, Theorem 2.6]. Conversely, by hypothesis, every τ-torsion module T is $\mathcal{N}eat$-projective, and thus every neat exact sequence is $\mathcal{P}_\tau$ sequence by Proposition 3.5. □

Corollary 3.16. *Let R be a ring with $\tau({}_RR) = 0$. Then, $\mathcal{P}_\tau = \mathcal{N}eat$ if and only if each τ-torsion module is semisimple.*

Corollary 3.17. *Let R be a ring with $Soc({}_RR) = 0$. Then, $\mathcal{P}_\rho = \mathcal{N}eat$ if and only if every semiartinian left R-module is semisimple.*

A ring R is called a left SI-ring if every singular left R-module is injective, i.e. R is left nonsingular and every singular left R-module is semisimple.

Corollary 3.18. [13, Corollary 3.10] *Let R be a left nonsingular ring. Then, $\mathcal{P}_\sigma = \mathcal{N}eat$ if and only if R is a left SI-ring.*

Theorem 3.19. *Let R be a commutative Noetherian ring. The following statements are equivalent:*

(1) Every τ-closed submodule is pure.

(2) $\mathcal{P}_\tau \subseteq \mathcal{P}ure$

(3) $R \cong A \times B$, wherein A is τ-torsion ring and B is hereditary C_τ ring.

Proof. (1) $\Leftrightarrow$ (2) This follows by the fact that $\mathcal{P}_\tau$ is the smallest proper class containing $\tau - \mathcal{C}losed$. (1) $\Rightarrow$ (3) Assume that every τ-closed submodule is pure. Then $R/\tau(R)$ is a flat module, and by Noethernity of R, it is a projective module. Therefore $\tau(R)$ is direct summand of R, i.e. $R \cong A \times B$, where $A \cong \tau(R)$, and $\tau(B) = 0$. W.l.o.g, we can assume R is τ-torsion ring or $\tau(R) = 0$. In the later case, let I be an ideal of R. Since $\tau(R) = 0$, we have $\tau(I) = 0$. Then, I is flat by (2). But R is Noetherian, and so I is finitely generated. Then I is projective, and so R is hereditary. Note that hereditary noetherian rins are C-rings ([8, 10.15 (3)]). This would mean that $\mathcal{N}eat = \mathcal{C}losed$. On the other hand, by Noethernity of R, every simple module is pure-projective. Then, $\mathcal{P}_\tau \subseteq \mathcal{P}ure \subseteq \mathcal{N}eat = \mathcal{C}losed$ and, by Theorem 3.9, R is a C_τ ring. (3) $\Rightarrow$ (1) Assume that $R \cong A \times B$, wherein A is τ-torsion ring and B is hereditary C_τ ring. Let M be a τ-torsion-free module. It is enough to show that M is a flat module. $M = MA \oplus MB$, where MA is an A-module and MB is a B-module. Note that, since $\tau(M) = 0$, $\tau(MA) = \tau(MB) = 0$. Consider the short exact sequence $0 \to K \to P \to MB \to 0$, where P is a projective B-module. Since B is C_τ ring, by Theorem 3.9, K is a closed submodule of P. Then MB is nonsingular B-module by [24, Lemma 2.3]. By [16, Proposition 2.3], MB is a flat B-module. Since A is τ-torsion ring, every A-module is τ-torsion, and this would means that $MA = 0$. Therefore, $M = MA \oplus MB = MB$ is a flat module, as desired.

□

Corollary 3.20. *Let R be a commutative Noetherian ring. The following statements are equivalent:*

(1) Every s-closed submodule is pure.

(2) $\mathcal{P}_\sigma \subseteq \mathcal{P}ure$

(3) $R \cong A \times B$, wherein A is Goldie torsion ring and B is hereditary ring.

It is known that, for a right nonsingular ring R, every nonsingular right R-module is projective if and only if R is Artinian hereditary serial ([7, Theorem 4.2]).

Theorem 3.21. *Let R be a commutative Noetherian ring. The following statements are equivalent:*

(1) Every τ-closed submodule is a direct summand.

(2) $\mathcal{P}_\tau = \mathcal{S}plit$

(3) $R \cong A \times B$, wherein A is τ-torsion ring and B is Artinian hereditary serial C_τ ring.

Proof. (1) $\Leftrightarrow$ (2) This follows by the fact that $\mathcal{P}_\tau$ is the smallest proper class containing $\tau - \mathcal{C}losed$. (2) $\Rightarrow$ (3) Following proof of (1) $\Rightarrow$ (3) in the Theorem 3.19, we have $R \cong A \times B$, wherein A is τ-torsion ring and B is hereditary C_τ ring with $\tau(B) = 0$. By Corollary 3.13 and (2), $\mathcal{P}_\sigma = \mathcal{P}_\tau = \mathcal{S}plit$, and this means that every nonsingular module is projective. Therefore, by [7, Theorem

4.2], B is also Artinian serial. (3) $\Rightarrow$ (1) Assume that $R \cong A \times B$, wherein A is τ-torsion ring and B is Artinian hereditary serial C_τ ring. Let M be a τ-torsion-free module. $M = MA \oplus MB$, where MA is an A-module and MB is a B-module. Note that, since $\tau(M) = 0$, $\tau(MA) = \tau(MB) = 0$. Consider the short exact sequence $0 \to K \to P \to MB \to 0$, where P is a projective B-module. Since B is C_τ ring, by Theorem 3.9, K is a closed submodule of P. Then MB is nonsingular B-module by [24, Lemma 2.3]. By [7, Theorem 4.2], MB is a projective B-module. Recall that A is τ-torsion ring, and so every A-module is τ-torsion. This would means that $MA = 0$. Therefore, $M = MA \oplus MB = MB$ is a projective module. Then, by Proposition 3.8, $\mathcal{P}_\tau = \mathcal{S}plit$. □

Note that a Noetherian semiartinian ring is Artinian, and an Artinian ring is C_ρ ring.

Corollary 3.22. *Let R be a commutative Noetherian ring. The following statements are equivalent:*

(1) Every ρ-closed submodule is a direct summand.

(2) R is an Artinian ring.

Proof. (1) $\Rightarrow$ (2) By Theorem 3.22, R is an Artinian ring.
(2) $\Rightarrow$ (1) Since R is Artinian ring, it is τ-torsion ring. Then the claim follows by Corollary 3.6. □

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