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LERAY–SCHAUDER ALTERNATIVES FOR MAPS SATISFYING COUNTABLE COMPACTNESS CONDITIONS

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Abstract

In this paper we present Leray–Schauder alternatives for a general class of Mönch type maps.

1. Introduction.

Leray–Schauder type alternatives for compact, condensing, Mönch type maps have been discussed extensively in the literature; we refer the reader to [1, 2, 3, 7] and the references therein. In this paper we present coincidence theory of Leray–Schauder type for very general Mönch type maps using the idea of an essential map initially introduced by Granas [2]. The results in this paper generalizes the theory in the literature (see [1, 4] and the references therein).

In the remainder of this section we present Mönch type coincidence results from the literature [5]. By a space we mean a Hausdorff topological space. Let X and Y be spaces. For a multivalued map $G : X \to 2^Y$ (here 2^Y denotes the family of nonempty subsets of Y) we consider the upper inverse G^u defined by $G^u(A) = \{x \in X : G(x) \subseteq A\}$ and the lower inverse G^l defined by $G^l(A) = \{x \in X : G(x) \cap A \neq \emptyset\}$ (here $A \subseteq Y$); of course $G^u(A) \subseteq G^l(A)$. In this paper we will let G^{-1} denote G^u.

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In this paper we consider classes **A**, **B** and **C** of maps. Let X and E be spaces.

Definition 1.1. We say $G \in M(X,E)$ (respectively, $G \in MB(X,E)$, $G \in MC(X,E)$) if $G : X \to 2^E$ and $G \in \mathbf{A}(X,E)$ (respectively, $G \in \mathbf{B}(X,E)$, $G \in \mathbf{C}(X,E)$).

We now state two Mönch type coincidence theorems established in [5] (other results can also be found there and also in [6]).

Theorem 1.2. *Let X be a metrizable topological vector space and Y a space. Assume $\Phi : Y \to 2^X$, $F : Y \to 2^X$, $x_0 \in \Phi(Y)$ and suppose the following conditions hold:*

$$\Phi^{-1}\left(\overline{co}\left(\{x_0\} \cup F(Y)\right)\right) \subseteq Y; \quad \text{here} \;\; \Phi^{-1} = \Phi^u \tag{1.1}$$

$$\begin{cases} A \subseteq Y, \;\; A = \Phi^{-1}\left(\overline{co}\left(\{x_0\} \cup F(A)\right)\right), \;\; \text{for any} \\ \text{countable set} \;\; Q \subseteq A \;\; \text{we have a countable set} \\ M \subseteq X \;\; \text{with} \;\; M \subseteq \Phi(Q) \subseteq \overline{M} \end{cases} \tag{1.2}$$

$$\begin{cases} A \subseteq Y, \;\; A = \Phi^{-1}\left(\overline{co}\left(\{x_0\} \cup F(A)\right)\right) \;\; \text{with} \;\; C \subseteq A \\ \text{countable and} \;\; \Phi(C) \subseteq \overline{co}\left(\{x_0\} \cup F(C)\right), \\ \text{implies} \;\; \overline{co}\left(F(C)\right) \;\; \text{is compact} \end{cases} \tag{1.3}$$

and

$$\begin{cases} \text{for any nonempty set} \;\; A \subseteq Y, \;\; A = \Phi^{-1}\left(\overline{co}\left(\{x_0\} \cup F(A)\right)\right) \\ \text{with} \;\; \overline{co}\left(F(A)\right) \;\; \text{compact we have that} \\ F\,\Phi^{-1} \in MC(\overline{co}\left(F(A)\right), \overline{co}\left(F(A)\right)) \;\; \text{and there exists} \\ x \in \Phi^{-1}\left(\overline{co}\left(F(A)\right)\right) \;\; \text{with} \;\; F(x) \cap \Phi(x) \neq \emptyset. \end{cases} \tag{1.4}$$

Then there exists $x \in Y$ with $F(x) \cap \Phi(x) \neq \emptyset$.

Remark 1.3. (a). In Theorem 1.2, X metrizable can be replaced by any space with the following properties: (i). X is such that the closure of a subset Ω of X is compact if and only if Ω is sequentially compact, and (ii). for any convex set $D \subseteq X$ if $x \in \overline{D}$ then there exists a sequence $x_1, x_2, ...$ in D with x_n converging to x.
(b). In some applications we are interested in maps $\Theta : Y \to 2^X$ and $\Psi : Y \to 2^X$ where F (maybe Θ itself) is a selection of Θ and Φ (maybe Ψ itself) is a selection of Ψ; note F is a selection of Θ if $F(x) \subseteq \Theta(x)$ for $x \in Y$. Now assuming the conditions in Theorem 1.2 we know there exists a $x \in Y$ with $F(x) \cap \Phi(x) \neq \emptyset$, so as a result $\Theta(x) \cap \Psi(x) \neq \emptyset$.

(c). Note in (1.4) we could of course replace $F\,\Phi^{-1} \in MC(\overline{co}\,(F(A)), \overline{co}\,(F(A)))$ with $F\,\Phi^{-1} \in MC(\overline{co}\,(F(A)), F(A))$.
(d). If $\Phi : Y \to X$ is single valued then trivially (1.2) holds by taking $M = \Phi(Q)$. Of course for (1.2) we just need that Φ maps countable sets in Y to separable sets in X.

Theorem 1.4. *Let X be a metrizable topological vector space, Y a space, $\Phi : Y \to 2^X$, $F : Y \to 2^X$, $x_0 \in \Phi(Y)$ and suppose* (1.1) *and* (1.2) *hold. In addition assume the following conditions are satisfied:*

$$(1.5)\qquad \begin{cases} A \subseteq Y,\ \ A = \Phi^{-1}\left(\overline{co}\,(\{x_0\} \cup F(A))\right), \ \ \textit{for any} \\ \textit{countable set}\ \ N \subseteq A\ \ \textit{there exists a countable set} \\ P \subseteq A\ \ \textit{with}\ \ \overline{co}\,(\{x_0\} \cup F(N)) \subseteq \overline{\Phi(P)} \end{cases}$$

and

$$(1.6)\qquad \begin{cases} A \subseteq Y,\ \ A = \Phi^{-1}\left(\overline{co}\,(\{x_0\} \cup F(A))\right)\ \ \textit{with}\ \ C \subseteq A \\ \textit{countable and}\ \ \overline{\Phi(C)} = \overline{co}\,(\{x_0\} \cup F(C)), \\ \textit{implies}\ \ \overline{co}\,(F(C))\ \ \textit{is compact.} \end{cases}$$

Finally suppose (1.4) *holds. Then there exists $x \in Y$ with $F(x) \cap \Phi(x) \neq \emptyset$.*

Example 1.5. Suppose $\Phi : Y \to X$ is single valued and surjective and $F : Y \to 2^X$ maps countable sets in Y to separable sets in X (for example upper semicontinuous maps in metric spaces with separable values map separable sets to separable sets; see [8 pp. 345]). Then (1.5) holds. To see this note since $N \subseteq A$ that $\Phi^{-1}\left(\overline{co}\,(\{x_0\} \cup F(N))\right) \subseteq \Phi^{-1}\left(\overline{co}\,(\{x_0\} \cup F(A))\right) = A$ so $\{w \in Y : \Phi(w) \in \overline{co}\,(\{x_0\} \cup F(N))\} \subseteq A$. Now since Φ is surjective then

$$\overline{co}\,(\{x_0\} \cup F(N)) = \overline{co}\,(\{x_0\} \cup F(N)) \cap \Phi(Y) \subseteq \Phi(A);$$

to see this note if $x \in \overline{co}\,(\{x_0\} \cup F(N)) \cap \Phi(Y)$ then there exists $y \in Y$ with $x \in \overline{co}\,(\{x_0\} \cup F(N))$ and $x = \Phi(y)$, and note $\Phi(y)\,(= x) \in \overline{co}\,(\{x_0\} \cup F(N))$ so from the above $y \in A$ i.e. $x = \Phi(y)$, $y \in A$ i.e. $x \in \Phi(A)$. Thus $\overline{co}\,(\{x_0\} \cup F(N)) \subseteq \Phi(A)$. Now N is countable so $F(N)$ is separable and so we have [6] that $co\,(\{x_0\} \cup F(N))$ is separable. Thus there exists a countable set $Q_0 \subseteq X$ with $Q_0 \subseteq co\,(\{x_0\} \cup F(N)) \subseteq \overline{Q_0}$ and since $\overline{co}\,(\{x_0\} \cup F(N)) \subseteq \Phi(A)$ we have $Q_0 \subseteq \Phi(A)$. Thus there exists a countable set $P \subseteq A$ with $Q_0 \subseteq \Phi(P)$ and as a result $\overline{co}\,(\{x_0\} \cup F(N)) = \overline{Q_0} \subseteq \overline{\Phi(P)}$. Thus (1.5) holds.

2. Main results.

Let X be a Hausdorff topological vector space, Y a space and U an open subset of Y.

Definition 2.1. We say $F \in M(\overline{U}, X)$ (as in Section 1) if $F : \overline{U} \to 2^X$ and $F \in \mathbf{A}(\overline{U}, X)$; here $\overline{U}$ denotes the closure of U in Y.

In this section we will fix a $\Phi : \overline{U} \to 2^X$ (from the class $MB(\overline{U}, X)$).

Definition 2.2. (i). We say $F \in M^M(\overline{U}, X)$ if $F \in M(\overline{U}, X)$ and if $D \subseteq \overline{U}$ and $D \subseteq \Phi^{-1}\left(\overline{co}\left(\{0\} \cup F(D)\right)\right)$ with $C \subseteq D$ countable and $\Phi(C) \subseteq \overline{co}\left(\{0\} \cup F(C)\right)$ then $\overline{co}\left(F(C)\right)$ is compact.

(ii). We say $G \in M^{MM}(\Omega, X)$ (here $\Omega \subseteq Y$ and $\Phi : Y \to 2^X$) if $G \in M(\Omega, X)$ and if $D \subseteq \Omega$, $D = \Phi^{-1}\left(\overline{co}\left(\{0\} \cup G(D)\right)\right)$ with $C \subseteq D$ countable and $\Phi(C) \subseteq \overline{co}\left(\{0\} \cup G(C)\right)$ (or $\overline{\Phi(C)} = \overline{co}\left(\{0\} \cup G(C)\right)$) then $\overline{co}\left(G(C)\right)$ is compact.

Definition 2.3. We say $F \in M^M_{\partial U}(\overline{U}, X)$ if $F \in M^M(\overline{U}, X)$ and $F(x) \cap \Phi(x) = \emptyset$ for $x \in \partial U$; here ∂U denotes the boundary of U in Y.

Definition 2.4. Let $F \in M^M_{\partial U}(\overline{U}, X)$. We say $F : \overline{U} \to 2^X$ is Φ–essential in $M^M_{\partial U}(\overline{U}, X)$ if for any map $J \in M^M_{\partial U}(\overline{U}, X)$ with $J|_{\partial U} = F|_{\partial U}$ there exists an $x \in U$ with $J\,(x) \cap \Phi\,(x) \neq \emptyset$.

Remark 2.5. (i). Note if $F \in M^M_{\partial U}(\overline{U}, X)$ is Φ–essential in $M^M_{\partial U}(\overline{U}, X)$ then there exists an $x \in U$ with $F(x) \cap \Phi\,(x) \neq \emptyset$ (take $J = F$ in Definition 2.4).

(ii). In Definition 2.2 (and throughout the paper) we could replace $\{0\}$ with $\{x_0\}$ where $x_0 \in X$ is fixed.

We begin with a nonlinear alternative of Leray–Schauder type (a more general result will be presented in Theorem 2.14).

Theorem 2.6. *Let X be a Hausdorff topological vector space, Y a normal topological space, U an open subset of Y, $\Phi : \overline{U} \to 2^X$ and $F \in M^M(\overline{U}, X)$. Assume the following conditions hold:*

$$\left\{ \begin{array}{l} \text{the zero map (denoted by } 0) \text{ is in } M^M_{\partial U}(\overline{U}, X) \text{ and} \\ 0 \text{ is } \Phi-\text{essential in } M^M_{\partial U}(\overline{U}, X) \end{array} \right. \tag{2.1}$$

$$\Phi(x) \cap t\,F(x) = \emptyset \ \text{ for every } x \in \partial U \text{ and } t \in (0,1) \tag{2.2}$$

and

$$\left\{ \begin{array}{l} \mu\,F \in M(\overline{U}, X) \text{ for any continuous map} \\ \mu : \overline{U} \to [0,1] \text{ with } \mu(\partial U) = 0. \end{array} \right. \tag{2.3}$$

Let $\Omega = \{x \in \overline{U} : \ \Phi(x) \cap t\,F(x) \neq \emptyset \ \text{ for some } \ t \in [0,1]\}$ and we suppose

$$\Omega \ \text{ is closed.} \tag{2.4}$$

Then there exists an $x \in \overline{U}$ with $\Phi(x) \cap F(x) \neq \emptyset$.

Proof: Suppose $\Phi(x) \cap F(x) = \emptyset$ for $x \in \partial U$ (otherwise we are finished). Let Ω be as in the statement of Theorem 2.6 and note (2.1) (see Remark 2.5) guarantees that $\Omega \neq \emptyset$. Next note $\Omega \cap \partial U = \emptyset$ (see (2.2), $\Phi(x) \cap F(x) = \emptyset$ for $x \in \partial U$ is assumed at the beginning of the proof, and $0 \in M^M_{\partial U}(\overline{U}, X)$). Now since Y is a normal topological space then (see (2.4)) there exists a continuous map $\mu : \overline{U} \to [0,1]$ with $\mu(\partial U) = 0$ and $\mu(\Omega) = 1$. Define a map R by $R(x) = \mu(x)\, F(x)$. Note (2.3) guarantees that $R \in M(\overline{U}, X)$. We now claim

$$R \in M^M_{\partial U}(\overline{U}, X). \tag{2.5}$$

First we show $R \in M^M(\overline{U}, X)$. Let $D \subseteq \overline{U}$ and $D \subseteq \Phi^{-1}\left(\overline{co}\,(\{0\} \cup R(D))\right)$ with $C \subseteq D$ countable and $\Phi(C) \subseteq \overline{co}\,(\{0\} \cup R(C))$. Note $R(C) \subseteq co\,(\{0\} \cup F(C))$, $R(D) \subseteq co\,(\{0\} \cup F(D))$ so

$$\overline{co}\,(\{0\} \cup R(D)) \subseteq \overline{co}\,(\{0\} \cup co\,(\{0\} \cup F(D))) = \overline{co}\,(co\,(\{0\} \cup F(D))) = \overline{co}\,(\{0\} \cup F(D))$$

and $\overline{co}\,(\{0\} \cup R(C)) \subseteq \overline{co}\,(\{0\} \cup F(C))$. Thus

$$D \subseteq \Phi^{-1}\left(\overline{co}\,(\{0\} \cup R(D))\right) \subseteq \Phi^{-1}\left(\overline{co}\,(\{0\} \cup F(D))\right)$$

and

$$\Phi(C) \subseteq \overline{co}\,(\{0\} \cup R(C)) \subseteq \overline{co}\,(\{0\} \cup F(C)).$$

Then since $F \in M^M(\overline{U}, X)$ we have that $\overline{co}\,(F(C))$ is compact. Now since $\overline{co}\,(R(C)) \subseteq \overline{co}\,(co\,(\{0\} \cup F(C))) = \overline{co}\,(\{0\} \cup F(C))$ we have that $\overline{co}\,(R(C))$ is compact. Thus $R \in M^M(\overline{U}, X)$. Next notice $R(x) \cap \Phi(x) = \emptyset$ for $x \in \partial U$ since if $x \in \partial U$ then $R(x) = \{0\}$ (note $\mu(\partial U) = 0$) and $\Phi(x) \cap \{0\} = \emptyset$ (recall $0 \in M^M_{\partial U}(\overline{U}, X)$). Thus (2.5) is true.

Note $R|_{\partial U} = 0|_{\partial U}$, $R \in M^M_{\partial U}(\overline{U}, X)$ and since 0 is Φ–essential in $M^M_{\partial U}(\overline{U}, X)$ then there exists a $x \in U$ with $R(x) \cap \Phi(x) \neq \emptyset$. Thus $x \in \Omega$ so $\mu(x) = 1$ and as a result $F(x) \cap \Phi(x) \neq \emptyset$. □

Remark 2.7. (i). In (2.1) note if $0 \in M(\overline{U}, X)$ then $0 \in M^M(\overline{U}, X)$ since if $D \subseteq \overline{U}$, $D \subseteq \Phi^{-1}\left(\overline{co}\,(\{0\} \cup 0(D))\right)$ with $C \subseteq D$ countable and $\Phi(C) \subseteq \overline{co}\,(\{0\} \cup 0(C))$ then since $0(x) = \{0\}$ for $x \in C$ we have that $\overline{co}\,(0(C))$ is (trivially) compact.

(ii). Note in Theorem 2.6 if we replace Y a normal topological space with Y a completely regular topological space then the result in Theorem 2.6 is true provided we replace (2.4) with Ω is compact.

(iii). In Theorem 2.6 let $\Phi = i$ (the identity map) so $\Omega = \{x \in \overline{U} : x \in t\, F(x)$ for some $t \in [0,1]\}$. Let Y be any space with the property that the closure of a subset E of Y is compact if and only if E is sequentially compact. If Ω is closed then Ω is compact. To see this it is enough to show Ω is sequentially compact. Let $\{x_n\}_{n=1}^\infty$ be a sequence in Ω and let $C = \{x_n\}_{n=1}^\infty$. Now there exists a sequence $\{t_n\}_{n=1}^\infty$ in $[0,1]$ with $x_n \in t_n\, F(x_n)$. Now C is countable and $C \subseteq co\,(\{0\} \cup F(C))$ so $\Phi(C) = C \subseteq \overline{co}\,(\{0\} \cup F(C))$.

Then since $F \in M^M(\overline{U}, X)$ (take $D = \Omega$ and note $\Omega \subseteq co(\{0\} \cup F(\Omega))$ so $\Omega \subseteq \Phi^{-1}(\overline{co}(\{0\} \cup F(\Omega))))$ we have that $\overline{co}(F(C))$ is compact. As a result since $C \subseteq co(\{0\} \cup F(C))$ we have that $\overline{C}$ is compact so $C = \{x_n\}_{n=1}^{\infty}$ has a convergent subsequence. Thus Ω is sequentially compact.

(iv). Note (2.2) is called the Leray–Schauder condition.

We now present some results which guarantee (2.1). For our next result we need $\Phi : Y \to 2^X$.

Theorem 2.8. *Let X be a Hausdorff topological vector space, Y a space, U an open subset of Y, $\Phi : Y \to 2^X$, and assume the following conditions hold:*

$$0 \in M(\overline{U}, X) \ \textit{with} \ \{0\} \cap \Phi(x) = \emptyset \ \textit{for} \ x \in \partial U \ \ (\textit{i.e.} \ 0 \notin \Phi(\partial U)) \tag{2.6}$$

$$\textit{there is no} \ \ z \in Y \setminus \overline{U} \ \ \textit{with} \ \ \Phi(z) \cap \{0\} \neq \emptyset \tag{2.7}$$

$$\begin{cases} \textit{for any map} \ J \in M^M_{\partial U}(\overline{U}, X) \ \textit{with} \ J|_{\partial U} = 0|_{\partial U} \ \textit{and} \\ R(x) = \begin{cases} J(x), & x \in \overline{U} \\ \{0\}, & x \in Y \setminus \overline{U}, \end{cases} \\ \textit{we have that} \ R \in M(Y, X) \end{cases} \tag{2.8}$$

and

$$\begin{cases} \textit{for any map} \ H \in M^{MM}(Y, X) \ \textit{there exists} \\ x \in Y \ \textit{with} \ \Phi(x) \cap H(x) \neq \emptyset. \end{cases} \tag{2.9}$$

Then the zero map is Φ–essential in $M^M_{\partial U}(\overline{U}, X)$.

Remark 2.9. (i). Note in fact R in (2.8) is in $M^{MM}(Y, X)$ (see the proof below) so one could replace (2.9) with: there exists $x \in Y$ with $\Phi(x) \cap R(x) \neq \emptyset$.

(ii). Note Theorem 1.2 (or Theorem 1.4) give conditions to guarantee (2.9). One could also use other theorems in [5, 6] to guarantee (2.9) (we might have to change slightly the definition of M^M and M^{MM} if we use these other theorems).

Remark 2.10. Note (2.6) and as in Remark 2.7 note $0 \in M^M(\overline{U}, X)$.

Proof: Let $J \in M^M_{\partial U}(\overline{U}, X)$ with $J|_{\partial U} = 0|_{\partial U}$. We must show there exists a $x \in U$ with $\Phi(x) \cap J(x) \neq \emptyset$. Let R be as in (2.8) and note $R \in M(Y, X)$. We claim $R \in M^{MM}(Y, X)$. To see this let $D \subseteq Y$ and $D = \Phi^{-1}(\overline{co}(\{0\} \cup R(D)))$ with $C \subseteq D$ countable and $\Phi(C) \subseteq \overline{co}(\{0\} \cup R(C))$ (or $\Phi(C) = \overline{co}(\{0\} \cup R(C))$). First note $\overline{co}(\{0\} \cup R(D)) \subseteq \overline{co}(\{0\} \cup J(D \cap \overline{U}))$ so $D = \Phi^{-1}(\overline{co}(\{0\} \cup R(D))) \subseteq \Phi^{-1}\left(\overline{co}(\{0\} \cup J(D \cap \overline{U}))\right)$ and $\Phi(C) \subseteq \overline{co}(\{0\} \cup J(C \cap \overline{U}))$. As a result

$$D \cap \overline{U} \subseteq \Phi^{-1}\left(\overline{co}(\{0\} \cup J(D \cap \overline{U}))\right) \ \text{ and } \ \Phi(C \cap \overline{U}) \subseteq \overline{co}(\{0\} \cup J(C \cap \overline{U})); \tag{2.10}$$

note $C \cap \overline{U}$ is countable. Now since $J \in M^M(\overline{U}, X)$ we have (see (2.10)) that $\overline{co}\,(J(C \cap \overline{U}))$ is compact. Now since $\overline{co}\,(R(C)) \subseteq \overline{co}\,(\{0\} \cup J(C \cap \overline{U}))$ we have that $\overline{co}\,(R(C))$ is compact. Thus $R \in M^{MM}(Y, X)$.

Now (2.9) guarantees that there exists a $x \in Y$ with $\Phi(x) \cap R(x) \neq \emptyset$. There are two cases to consider, namely $x \in U$ and $x \in Y \setminus U$. If $x \in U$ then $\Phi(x) \cap J(x) \neq \emptyset$, and we are finished. If $x \in Y \setminus U$ then since $R(x) = \{0\}$ (note also $J|_{\partial U} = 0|_{\partial U}$) we have $\Phi(x) \cap \{0\} \neq \emptyset$, and this contradicts (2.7) (see also (2.6)). $\square$

We now give another example of a Φ–essential map when $X = Y$ (we present the result for a general map F and a particular case is when F is the zero map assuming $0 \in M(\overline{U}, Y)$ and $0 \notin \Phi(\partial U)$).

Theorem 2.11. *Let $X = Y$ be a Hausdorff topological vector space, U an open subset of Y, $\Phi : \overline{U} \to 2^Y$, $F \in M^M_{\partial U}(\overline{U}, Y)$ and assume the following conditions hold:*

$$\text{(2.11)} \qquad \begin{cases} \textit{there exists a retraction } \ r : Y \to \overline{U} \ \textit{ with} \\ r(B) \subseteq co\,(\{0\} \cup B) \ \textit{ for any subset } \ B \ \textit{ of } Y \end{cases}$$

$$\text{(2.12)} \qquad \begin{cases} \textit{for any map } \ J \in M^M_{\partial U}(\overline{U}, Y) \ \textit{ with } \ J|_{\partial U} = F|_{\partial U} \\ \textit{the map } \ r\,J \in M(\overline{U}, \overline{U}) \end{cases}$$

$$\text{(2.13)} \qquad \begin{cases} \textit{for any map } \ H \in M^{MM}(\overline{U}, \overline{U}) \ \textit{ there exists} \\ x \in \overline{U} \ \textit{ with } \ \Phi(x) \cap H(x) \neq \emptyset \end{cases}$$

and

$$\text{(2.14)} \qquad \begin{cases} \textit{for any map } \ J \in M^M_{\partial U}(\overline{U}, Y) \ \textit{ with } \ J|_{\partial U} = F|_{\partial U} \\ \textit{there is no } \ z \in Y \setminus U \ \textit{ and } \ y \in \overline{U} \ \textit{ with} \\ z \in J(y) \ \textit{ and } \ r(z) \in \Phi(y). \end{cases}$$

Then F is Φ–essential in $M^M_{\partial U}(\overline{U}, Y)$.

Proof: Let $J \in M^M_{\partial U}(\overline{U}, Y)$ with $J|_{\partial U} = F|_{\partial U}$. Let $H = r\,J$. Note $H \in M(\overline{U}, \overline{U})$. We claim $H \in M^{MM}(\overline{U}, \overline{U})$. To see this let $D \subseteq \overline{U}$ and $D = \Phi^{-1}\,(\overline{co}\,(\{0\} \cup H(D)))$ with $C \subseteq D$ countable and $\Phi(C) \subseteq \overline{co}\,(\{0\} \cup r\,J(C))$ (or $\overline{\Phi(C)} = \overline{co}\,(\{0\} \cup r\,J(C))$). Now from (2.11) we have

$$\overline{co}\,(\{0\} \cup r\,J(C)) \subseteq \overline{co}\,(\{0\} \cup co\,(\{0\} \cup J(C))) = \overline{co}\,(\{0\} \cup J(C))$$

and

$$\overline{co}\,(\{0\} \cup r\,J(D)) \subseteq \overline{co}\,(\{0\} \cup J(D))$$

so since $D = \Phi^{-1}\left(\overline{co}\left(\{0\} \cup r\, J(D)\right)\right)$ we have

$$D \subseteq \Phi^{-1}\left(\overline{co}\left(\{0\} \cup J(D)\right)\right) \quad \text{and} \quad \Phi(C) \subseteq \overline{co}\left(\{0\} \cup J(C)\right).$$

Now since $J \in M^M(\overline{U}, Y)$ we have that $\overline{co}\,(J(C))$ is compact. Now since $\overline{co}\,(r\, J(C)) \subseteq \overline{co}\,(co\,(\{0\} \cup J(C))) = \overline{co}\,(\{0\} \cup J(C))$ we have that $\overline{co}\,(r\, J(C))$ is compact. Thus $H = r\, J \in M^{MM}(\overline{U}, \overline{U})$.

Now (2.13) guarantees that there exists a $x \in \overline{U}$ with $\Phi(x) \cap r\, J(x) \neq \emptyset$. Then $r(y) \in \Phi(x)$ for some $y \in J(x)$. There are two cases to consider, namely $y \in U$ and $y \in Y \setminus U$. If $y \in U$ then $y = r(y) \in \Phi(x)$ and $y \in J(x)$ i.e. $\Phi(x) \cap J(x) \neq \emptyset$, and we are finished (note $x \in U$ since $J \in M^M_{\partial U}(\overline{U}, Y)$ so in particular $J(w) \cap \Phi(w) = \emptyset$ for $w \in \partial U$). If $y \in Y \setminus U$ then $y \in J(x)$, $r(y) \in \Phi(x)$, $x \in \overline{U}$ and this contradicts (2.14). □

Remark 2.12. Let Y be a locally convex Hausdorff topological vector space, U a convex subset of Y, $0 \in U$, $\Phi = i$ (the identity),

$$x \notin \lambda\, Fx \quad \text{for} \quad x \in \partial U \quad \text{and} \quad \lambda \in (0, 1] \tag{2.15}$$

and let

$$r(x) = \frac{x}{\max\{1, \mu(x)\}} \quad \text{for} \quad x \in Y;$$

here μ is the Minkowski functional on $\overline{U}$ (i.e. $\mu(x) = \inf\{\alpha > 0 : x \in \alpha\, \overline{U}\}$).

Note (2.11) is true. Now we show (2.14) holds. To see this suppose $J \in M^M_{\partial U}(\overline{U}, Y)$ with $J|_{\partial U} = F|_{\partial U}$ and assume there exists $z \in Y \setminus U$ and $y \in \overline{U}$ with $z \in J(y)$ and $r(z) \in \Phi(y)$ (i.e. $r(z) = y$). Now

$$y = r(z) = \frac{z}{\mu(z)} \quad \text{with} \quad \mu(z) \geq 1 \quad \text{since} \quad z \in Y \setminus U.$$

Then $y \in \lambda\, J(y)$ with $0 < \lambda = \frac{1}{\mu(z)} \leq 1$. Note $y = r(z) \in \partial U$ since $z \in Y \setminus U$, and so

$$y \in \lambda\, J(y) = \lambda\, F(y) \quad \text{since} \quad J|_{\partial U} = F|_{\partial U}.$$

This contradicts (2.15), so (2.14) is true.

In fact the argument in Theorem 2.11 establishes the following coincidence result.

Theorem 2.13. *Let $X = Y$ be a Hausdorff topological vector space, U an open subset of Y, $\Phi : \overline{U} \to 2^Y$, $F \in M^M(\overline{U}, Y)$ and assume* (2.11) *and* (2.13) *hold. In addition suppose the following conditions hold:*

$$r\, F \in M(\overline{U}, \overline{U}) \tag{2.16}$$

and

$$\text{(2.17)} \qquad \begin{cases} \textit{there is no } \ z \in Y \setminus U \ \textit{ and } \ y \in \overline{U} \ \textit{ with} \\ z \in F(y) \ \textit{ and } \ r(z) \in \Phi(y). \end{cases}$$

Then there exists a $x \in \overline{U}$ *with* $\Phi(x) \cap F(x) \neq \emptyset$.

Proof: To see this let $H = r\,F$ and as in Theorem 2.11 (with J=F) we see that $H \in M^{MM}(\overline{U}, \overline{U})$. Now (2.13) guarantees that there exists a $x \in \overline{U}$ with $\Phi(x) \cap r\,F(x) \neq \emptyset$. Then $r(y) \in \Phi(x)$ for some $y \in F(x)$. There are two cases to consider, namely $y \in \overline{U}$ and $y \in Y \setminus \overline{U}$. If $y \in \overline{U}$ then $y = r(y) \in \Phi(x)$ and $y \in F(x)$ and we are finished. If $y \in Y \setminus \overline{U}$ then $y \in F(x)$, $r(y) \in \Phi(x)$, $x \in \overline{U}$ and this contradicts (2.17). □

Our final two results are generalizations of Theorem 2.6.

Theorem 2.14. *Let* X *be a Hausdorff topological vector space,* Y *a normal topological space,* U *an open subset of* Y*,* $\Phi : \overline{U} \to 2^X$*,* $F \in M^M(\overline{U}, X)$ *and* $G \in M^M_{\partial U}(\overline{U}, X)$ *is* Φ*–essential in* $M^M_{\partial U}(\overline{U}, X)$*. Also assume there exists a map* $H : \overline{U} \times [0,1] \to 2^X$ *with* $H(\,.\,, \eta(\,.\,)) \in M^M(\overline{U}, X)$ *for any continuous function* $\eta : \overline{U} \to [0,1]$ *with* $\eta(\partial U) = 0$*,* $\Phi(x) \cap H_t(x) = \emptyset$ *for any* $x \in \partial U$ *and* $t \in (0,1)$ *(here* $H_t(x) = H(x,t)$*),* $H_1 = F$*,* $H_0 = G$ *and* $\Omega = \{x \in \overline{U} : \ \Phi(x) \cap H(x,t) \neq \emptyset \ \textit{ for some } \ t \in [0,1]\}$ *is closed. Then there exists a* $x \in \overline{U}$ *with* $\Phi(x) \cap F(x) \neq \emptyset$*.*

Proof: Suppose $\Phi(x) \cap F(x) = \emptyset$ for $x \in \partial U$ (otherwise we are finished). Let Ω be as in the statement of Theorem 2.14 and note $\Omega \neq \emptyset$ (note G is Φ–essential in $M^M_{\partial U}(\overline{U}, X)$, $H_0 = G$ and see Remark 2.5). Then there exists a continuous map $\mu : \overline{U} \to [0,1]$ with $\mu(\partial U) = 0$ and $\mu(\Omega) = 1$. Define the map R by $R(x) = H(x, \mu(x))$. Now $R \in M^M_{\partial U}(\overline{U}, X)$ with $R|_{\partial U} = G|_{\partial U}$ (note if $x \in \partial U$ then $R(x) = H(x,0) = G(x)$ and $R(x) \cap \Phi(x) = \Phi(x) \cap G(x) = \emptyset$). Since G is Φ–essential in $M^M_{\partial U}(\overline{U}, X)$ there exists $x \in U$ with $R(x) \cap \Phi(x) \neq \emptyset$ (i.e. $H_{\mu(x)}(x) \cap \Phi(x) \neq \emptyset$). Thus $x \in \Omega$ so $\mu(x) = 1$. As a result $H_1(x) \cap \Phi(x) \neq \emptyset$ i.e. $F(x) \cap \Phi(x) \neq \emptyset$. □

Remark 2.15. Note in Theorem 2.14 if we replace Y a normal topological space with Y a completely regular topological space then the result in Theorem 2.14 is true provided we replace Ω is closed with Ω is compact.

It is also possible to generalize slightly the result in Theorem 2.14 if one modifies slightly the assumptions.

Theorem 2.16. *Let* X *be a Hausdorff topological vector space,* Y *a normal topological space,* U *an open subset of* Y*,* $\Phi : \overline{U} \to 2^X$*,* $F \in M^M_{\partial U}(\overline{U}, X)$ *and* $G \in M^M_{\partial U}(\overline{U}, X)$ *is* Φ*–essential in* $M^M_{\partial U}(\overline{U}, X)$*. Also assume for any map* $J \in M^M_{\partial U}(\overline{U}, X)$ *with* $J|_{\partial U} = F|_{\partial U}$ *there exists a map* $H^J : \overline{U} \times [0,1] \to 2^X$

with $H^J(\,.\,,\eta(\,.\,)) \in M^M(\overline{U},X)$ *for any continuous function* $\eta : \overline{U} \to [0,1]$ *with* $\eta(\partial U) = 0$, $\Phi(x) \cap H^J_t(x) = \emptyset$ *for any* $x \in \partial U$ *and* $t \in (0,1)$ *(here* $H^J_t(x) = H^J(x,t)$*),* $H^J_1 = J$, $H^J_0 = G$ *and*

$$\Omega = \left\{x \in \overline{U} : \ \Phi(x) \cap H^J(x,t) \neq \emptyset \ \ \textit{for some} \ \ t \in [0,1]\right\}$$

is closed. Then F *is* Φ*–essential in* $M^M_{\partial U}(\overline{U},X)$.

Proof: Consider any map $J \in M^M_{\partial U}(\overline{U},X)$ with $J|_{\partial U} = F|_{\partial U}$. We must show there exists a $x \in U$ with $\Phi(x) \cap J(x) \neq \emptyset$. Choose the map H^J and the set Ω as in the statement of Theorem 2.16 and note $\Omega \neq \emptyset$ (note G is Φ–essential in $M^M_{\partial U}(\overline{U},X)$ and $H^J_0 = G$). Then there exists a continuous map $\mu : \overline{U} \to [0,1]$ with $\mu(\partial U) = 0$ and $\mu(\Omega) = 1$. Define the map R by $R(x) = H^J(x,\mu(x))$. Now $R \in M^M_{\partial U}(\overline{U},X)$ with $R|_{\partial U} = G|_{\partial U}$ (note if $x \in \partial U$ then $R(x) = H^J(x,0) = G(x)$ and $R(x) \cap \Phi(x) = \Phi(x) \cap G(x) = \emptyset$). Since G is Φ–essential in $M^M_{\partial U}(\overline{U},X)$ there exists $x \in U$ with $R(x) \cap \Phi(x) \neq \emptyset$ (i.e. $H^J_{\mu(x)}(x) \cap \Phi(x) \neq \emptyset$). Thus $x \in \Omega$ so $\mu(x) = 1$. As a result $H^J_1(x) \cap \Phi(x) \neq \emptyset$ i.e. $J(x) \cap \Phi(x) \neq \emptyset$. □

Remark 2.17. Note in Theorem 2.16 if we replace Y a normal topological space with Y a completely regular topological space then the result in Theorem 2.16 is true provided we replace Ω is closed with Ω is compact.

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