DOI: 10.2478/auom-2019-0028
An. Şt. Univ. Ovidius Constanţa **Vol. 27(2),2019, 213–232**

CUBIC INTUITIONISTIC STRUCTURES APPLIED TO IDEALS OF BCI-ALGEBRAS

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Abstract

In this paper, the notion of closed cubic intuitionistic ideals, cubic intuitionistic p-ideals and cubic intuitionistic a-ideals in BCI-algebras are introduced, and several related properties are investigated. Relations between cubic intuitionistic subalgebras, closed cubic intuitionistic ideals, cubic intuitionistic q-ideals, cubic intuitionistic p-ideals and cubic intuitionistic a-ideals are discussed. Conditions for a cubic intuitionistic ideal to be a cubic intuitionistic p-ideal are provided. Characterizations of a cubic intuitionistic a-ideal are considered. The cubic intuitionistic extension property for a cubic intuitionistic a-ideal is established.

1 Introduction

In 2012, Y.B. Jun et al. [1] introduced cubic sets, and then this notion is applied to several algebraic structures (see [2, 3, 4, 5, 7, 8, 10, 11, 12, 13]). In 2017, extending the concept of a cubic set, Y.B. Jun [6] introduced the notion of a cubic intuitionistic set. He introduced the notions of (left, right) internal cubic intuitionistic set, double left (right) internal cubic intuitionistic set, cross left (right) internal cubic intuitionistic set and (cross) external cubic intuitionistic set, and investigate related properties. He applied this theory to subalgebras and ideals in a BCK/BCI-algebra and obtained some useful

Key Words: Cubic intuitionistic subalgebra, closed cubic intuitionistic ideal, cubic intuitionistic p-ideal, cubic intuitionistic a-ideal, cubic intuitionistic q-ideal.
2010 Mathematics Subject Classification: 06F35, 03G25, 94D05.
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Received: 26.06.2018
Accepted: 08.10.2018

results. T. Senapati et al. [14] applied the notion of cubic intuitionistic set to q-ideals of a BCI-algebra, and provided relations between a cubic intuitionistic ideal, a cubic intuitionistic subalgebra and a cubic intuitionistic q-ideal.

This paper is a continuation of the paper [6] and [14]. The organization of our work is as follows. In Section 2, we cover some relevant preliminaries related to BCK/BCI-algebras and cubic intuitionistic sets. In Section 3, we introduce the notions of closed cubic intuitionistic ideals. We prove that every closed cubic intuitionistic ideal is a cubic intuitionistic subalgebra. Section 4, we first introduce the notion of cubic intuitionistic p-ideals and discuss their properties in details. Section 5 contains definition and results of cubic intuitionistic a-ideals. We discuss the relationship between a cubic intuitionistic q-ideal, cubic intuitionistic p-ideal, and a cubic intuitionistic a-ideal, and provide conditions for a cubic intuitionistic ideal to be a cubic intuitionistic a-ideal. We establish characterizations of a cubic intuitionistic a-ideal, and consider the cubic intuitionistic extension property for a cubic intuitionistic a-ideal. In Section 6, the conclusion and scope for future research are outlined and discussed.

2 Preliminaries

We assume that the reader is familiar with the classical results BCK/BCI-algebras, but to make this work more self-contained, we introduce basic notations used in the text and we briefly mention some of the concepts and results employed in the rest of the work.

By a BCI-algebra we mean an algebra X with a constant 0 and a binary operation "$*$" satisfying the following axioms for all $x, y, z \in X$:

(I) $((x * y) * (x * z)) * (z * y) = 0$

(II) $(x * (x * y)) * y = 0$

(III) $x * x = 0$

(IV) $x * y = 0$ and $y * x = 0$ imply $x = y$.

We can define a partial ordering "$\leq$" by $x \leq y$ if and only if $x * y = 0$.

If a BCI-algebra X satisfies $0 * x = 0$, for all $x \in X$, then we say that X is a BCK-algebra. Any BCK/BCI-algebra X satisfies the following axioms for all $x, y, z \in X$:

$(a1)$ $(x * y) * z = (x * z) * y$

$(a2)$ $((x * z) * (y * z)) * (x * y) = 0$

$(a3)$ $x * 0 = x$

$(a4)$ $x * y = 0 \Rightarrow (x * z) * (y * z) = 0, (z * y) * (z * x) = 0.$

A BCI-algebra X is to be a weakly BCK-algebra if $0 * x \leq x$ for all $x \in X$. A BCI-algebra X is called p-semisimple if its BCK-part is equal to $\{0\}$. In a p-semisimple BCI-algebra, the following conditions are valid for all $x, y, \in X$:

($a5$) $0 * (x * y) = y * x$
($a6$) $x * (x * y) = y$.
Throughout this paper, X always means a BCK/BCI-algebra without any specification.
A non-empty subset S of X is called a subalgebra of X if $x * y \in S$ for any $x, y \in S$. A non-empty subset I of X is called an ideal of X if it satisfies
(I_1) $0 \in I$ and
(I_2) $x * y \in I$ and $y \in I$ imply $x \in I$.
A non-empty subset I of X is said to be an q-ideal [4] of X if it satisfies (I_1) and (I_3) $x * (y * z) \in I$ and $y \in I$ imply $x * z \in I$, for all $x, y, z \in X$.
A non-empty subset I of X is called an a-ideal [4] of X if it satisfies (I_1) and
(I_4) $(x * z) * (0 * y) \in I$ and $z \in I$ imply $y * x \in I$, for all $x, y, z \in X$.

Given two closed subintervals $D_1 = [D_1^-, D_1^+]$ and $D_2 = [D_2^-, D_2^+]$ of $[0, 1]$, we define the order " $\ll$ " and " $\gg$ " as follows:

$$D_1 \ll D_2 \Leftrightarrow D_1^- \leq D_2^- \text{ and } D_1^+ \leq D_2^+$$
$$D_1 \gg D_2 \Leftrightarrow D_1^- \geq D_2^- \text{ and } D_1^+ \geq D_2^+.$$

We also define the refined maximum (briefly, rmax) and refined minimum (briefly, rmin) as

$$\text{rmax}\{D_1, D_2\} = [\max\{D_1^-, D_2^-\}, \max\{D_1^+, D_2^+\}]$$
$$\text{rmin}\{D_1, D_2\} = [\min\{D_1^-, D_2^-\}, \min\{D_1^+, D_2^+\}].$$

Denote by $D[0, 1]$ the set of all closed subintervals of $[0, 1]$. In this paper we use the interval-valued intuitionistic fuzzy set

$$A = \{\langle x, M_A(x), N_A(x)\rangle : x \in X\}$$

in which $M_A(x)$ and $N_A(x)$ are closed subintervals of $[0, 1]$ for all $x \in X$. Also, we use the notations $M_A^-(x)$ and $M_A^+(x)$ to mean the left end point and the right end point of the interval $M_A(x)$, respectively, and so we have $M_A(x) = [M_A^-(x), M_A^+(x)]$. For the sake of simplicity, we shall use the symbol $A(x) = \langle M_A(x), N_A(x)\rangle$ or $A = \langle M_A, N_A\rangle$ for the interval-valued intuitionistic fuzzy set $A = \{\langle x, M_A(x), N_A(x)\rangle : x \in X\}$.

Jun [6] defined the cubic intuitionistic set in the following way:

Definition 2.1. [6] *Let X be a nonempty set. By a cubic intuitionistic set in X we mean a structure $\widetilde{A} = \{\langle x, A(x), \lambda(x)\rangle : x \in X\}$ in which A is an interval-valued intuitionistic fuzzy set in X and λ is an intuitionistic fuzzy set in X.*

A cubic intuitionistic set $\widetilde{A} = \{\langle x, A(x), \lambda(x)\rangle : x \in X\}$ is simply denoted by $\widetilde{A} = \langle A, \lambda\rangle$.

3 Closed cubic intuitionistic ideals

In this section, we define closed cubic intuitionistic ideals of BCI-algebras and present some important properties. In what follows, we simply use X to denote a BCI-algebra unless otherwise specified.

Definition 3.1. [6] A cubic intuitionistic set $\widetilde{A} = \langle A, \lambda \rangle$ in X is called a cubic intuitionistic subalgebra of X over the binary operator $*$ if it satisfies the following conditions for all $x, y \in X$

(a) $M_A(x * y) \gg \text{rmin}\{M_A(x), M_A(y)\}$
(b) $N_A(x * y) \ll \text{rmax}\{N_A(x), N_A(y)\}$
(c) $\mu_\lambda(x * y) \leq \max\{\mu_\lambda(x), \mu_\lambda(y)\}$
(d) $\nu_\lambda(x * y) \geq \min\{\nu_\lambda(x), \nu_\lambda(y)\}$.

Definition 3.2. [6] A cubic intuitionistic set $\widetilde{A} = \langle A, \lambda \rangle$ in X is called a cubic intuitionistic ideal of X if it satisfies the following conditions for all $x, y \in X$:

(a) $M_A(0) \gg M_A(x)$ and $N_A(0) \ll N_A(x)$
(b) $\mu_\lambda(0) \leq \mu_\lambda(x)$ and $\nu_\lambda(0) \geq \nu_\lambda(x)$
(c) $M_A(x) \gg \text{rmin}\{M_A(x * y), M_A(y)\}$
(d) $N_A(x) \ll \text{rmax}\{N_A(x * y), N_A(y)\}$
(e) $\mu_\lambda(x) \leq \max\{\mu_\lambda(x * y), \mu_\lambda(y)\}$
(f) $\nu_\lambda(x) \geq \min\{\nu_\lambda(x * y), \nu_\lambda(y)\}$.

Definition 3.3. *A cubic intuitionistic ideal $\widetilde{A} = \langle A, \lambda \rangle$ of X is said to be closed if it satisfies $M_A(0*x) \gg M_A(x)$, $N_A(0*x) \ll N_A(x)$, $\mu_\lambda(0*x) \leq \mu_\lambda(x)$ and $\nu_\lambda(0 * x) \geq \nu_\lambda(x)$, for all $x \in X$.*

Example 3.4. *Let $X = \{0, a, b, c\}$ be a BCI-algebra with the following Cayley table:*

$*$	0	a	b	c
0	0	0	0	c
a	a	0	0	c
b	b	b	0	c
c	c	c	c	0

Define a cubic intuitionistic set $\widetilde{A} = \langle A, \lambda \rangle$ in X as follows:

X	$A = \langle M_A, N_A \rangle$	$\lambda = (\mu_\lambda, \nu_\lambda)$
0	$\langle [0.5, 0.6], [0.1, 0.3] \rangle$	$(0.2, 0.8)$
a	$\langle [0.3, 0.4], [0.3, 0.6] \rangle$	$(0.4, 0.5)$
b	$\langle [0.3, 0.4], [0.3, 0.6] \rangle$	$(0.4, 0.5)$
c	$\langle [0.3, 0.4], [0.3, 0.6] \rangle$	$(0.4, 0.5)$

It is easy to verify that $\widetilde{A} = \langle A, \lambda \rangle$ is a closed cubic intuitionistic ideal of X.

Theorem 3.5. *Every closed cubic intuitionistic ideal of X is a cubic intuitionistic subalgebra of X.*

Proof. Let $\widetilde{A} = \langle A, \lambda \rangle$ be a closed cubic intuitionistic ideal of a BCI-algebra X. Then $M_A(0 * x) \gg M_A(x)$, $N_A(0 * x) \ll N_A(x)$, $\mu_\lambda(0 * x) \leq \mu_\lambda(x)$ and $\nu_\lambda(0 * x) \geq \nu_\lambda(x)$, for all $x \in X$. It follows from Definition 3.2 and (a1) that

$$\begin{aligned}
M_A(x*y) &\gg \text{rmin}\{M_A((x*y)*x), M_A(x)\} \\
&= \text{rmin}\{M_A(0*y), M_A(x)\} \gg \text{rmin}\{M_A(x), M_A(y)\}, \\
N_A(x*y) &\ll \text{rmax}\{N_A((x*y)*x), N_A(x)\} \\
&= \text{rmax}\{N_A(0*y), N_A(x)\} \ll \text{rmax}\{N_A(x), N_A(y)\}, \\
\mu_\lambda(x*y) &\leq \max\{\mu_\lambda((x*y)*x), \mu_\lambda(x)\} \\
&= \max\{\mu_\lambda(0*y), \mu_\lambda(x)\} \leq \max\{\mu_\lambda(x), \mu_\lambda(y)\}, \\
\nu_\lambda(x*y) &\geq \min\{\nu_\lambda((x*y)*x), \nu_\lambda(x)\} \\
&= \min\{\nu_\lambda(0*y), \nu_\lambda(x)\} \geq \min\{\nu_\lambda(x), \nu_\lambda(y)\},
\end{aligned}$$

for all $x, y \in X$. Hence $\widetilde{A}$ is a cubic intuitionistic subalgebra of X. □

Theorem 3.6. *In a weakly BCK-algebra, every cubic intuitionistic ideal is closed.*

Proof. Let $\widetilde{A} = \langle A, \lambda \rangle$ be a cubic intuitionistic ideal of a weakly BCK-algebra X. For any $x \in X$, we get $M_A(0 * x) \gg \text{rmin}\{M_A((0 * x) * x), M_A(x)\} = \text{rmin}\{M_A(0), M_A(x)\} = M_A(x)$, $N_A(0*x) \ll \text{rmax}\{N_A((0*x)*x), N_A(x)\} = \text{rmax}\{N_A(0), N_A(x)\} = N_A(x)$, $\mu_\lambda(0 * x) \leq \max\{\mu_\lambda((0 * x) * x), \mu_\lambda(x)\} = \max\{\mu_\lambda(0), \mu_\lambda(x)\} = \mu_\lambda(x)$, and $\nu_\lambda(0 * x) \geq \min\{\nu_\lambda((0 * x) * x), \nu_\lambda(x)\} = \min\{\nu_\lambda(0), \nu_\lambda(x)\} = \nu_\lambda(x)$. Consequently, $\widetilde{A} = \langle A, \lambda \rangle$ is closed. □

Corollary 3.7. *In a weakly BCK-algebra, every cubic intuitionistic ideal is a cubic intuitionistic subalgebra.*

Theorem 3.8. *In a p-semisimple BCI-algebra, every cubic intuitionistic subalgebra is a closed cubic intuitionistic ideal.*

Proof. Let $\widetilde{A} = \langle A, \lambda \rangle$ be a cubic intuitionistic subalgebra of a p-semisimple BCI-algebra X. For any $x \in X$, we get $M_A(0) = M_A(x * x) \gg \text{rmin}\{M_A(x), M_A(x)\} = M_A(x)$, $N_A(0) = N_A(x * x) \ll \text{rmax}\{N_A(x), N_A(x)\} = N_A(x)$, $\mu_\lambda(0) = \mu_\lambda(x * x) \leq \max\{\mu_\lambda(x), \mu_\lambda(x)\} = \mu_\lambda(x)$, and $\nu_\lambda(0) = \nu_\lambda(x * x) \geq \min\{\nu_\lambda(x), \nu_\lambda(x)\} = \nu_\lambda(x)$. Thus $M_A(0 * x) \gg \text{rmin}\{M_A(0), M_A(x)\} = M_A(x)$, $N_A(0 * x) \ll \text{rmax}\{N_A(0), N_A(x)\} = N_A(x)$, $\mu_\lambda(0 * x) \leq \max\{\mu_\lambda(0),

$\mu_\lambda(x)\} = \mu_\lambda(x)$, and $\nu_\lambda(0 * x) \geq \min\{\nu_\lambda(0), \nu_\lambda(x)\} = \nu_\lambda(x)$. Let $x, y \in X$. Then

$$\begin{aligned}
M_A(x) &= M_A(y * (y * x)) \gg \text{rmin}\{M_A(y), M_A(y * x)\} \\
&= \text{rmin}\{M_A(y), M_A(0 * (x * y))\} \gg \text{rmin}\{M_A(x * y), M_A(y)\}, \\
N_A(x) &= N_A(y * (y * x)) \ll \text{rmax}\{N_A(y), N_A(y * x)\} \\
&= \text{rmax}\{N_A(y), N_A(0 * (x * y))\} \ll \text{rmax}\{N_A(x * y), N_A(y)\}, \\
\mu_\lambda(x) &= \mu_\lambda(y * (y * x)) \leq \max\{\mu_\lambda(y), \mu_\lambda(y * x)\} \\
&= \max\{\mu_\lambda(y), \mu_\lambda(0 * (x * y))\} \leq \max\{\mu_\lambda(x * y), \mu_\lambda(y)\}, \\
\nu_\lambda(x) &= \nu_\lambda(y * (y * x)) \geq \min\{\nu_\lambda(y), \nu_\lambda(y * x)\} \\
&= \min\{\nu_\lambda(y), \nu_\lambda(0 * (x * y))\} \geq \min\{\nu_\lambda(x * y), \nu_\lambda(y)\}.
\end{aligned}$$

Therefore $\widetilde{A} = \langle A, \lambda \rangle$ is a closed cubic intuitionistic ideal of X. □

Let $\widetilde{A} = \langle A, \lambda \rangle$ be a cubic intuitionistic set in a nonempty set X. Given $([s_1, t_1], [s_2, t_2]) \in D[0, 1] \times D[0, 1]$ and $[\theta_1, \theta_2] \in [0, 1] \times [0, 1]$, we consider the sets

$$\begin{aligned}
M_A[s_1, t_1] &= \{x \in X | M_A(x) \gg [s_1, t_1]\}, \\
N_A[s_2, t_2] &= \{x \in X | N_A(x) \ll [s_2, t_2]\}, \\
\mu_\lambda(\theta_1) &= \{x \in X | \mu_\lambda(x) \leq (\theta_1)\}, \\
\mu_\lambda(\theta_2) &= \{x \in X | \mu_\lambda(x) \geq (\theta_2)\}.
\end{aligned}$$

Theorem 3.9. *Let $\widetilde{A} = \langle A, \lambda \rangle$ be a cubic intuitionistic ideal of X, then the sets $M_A[s, t]$, $N_A[s, t]$, $\mu_\lambda(\theta)$ and $\nu_\lambda(\theta)$ are ideals of X for all $[s, t] \in D[0, 1]$ and $\theta \in [0, 1]$.*

Proof. Assume that $\widetilde{A} = \langle A, \lambda \rangle$ is a cubic intuitionistic ideal of X. For any $[s, t] \in D[0, 1]$ and $\theta \in [0, 1]$, let $x \in X$ be such that $x \in M_A[s, t] \cap N_A[s, t] \cap \mu_\lambda(\theta) \cap \nu_\lambda(\theta)$. Then $M_A(x) \gg [s, t]$, $N_A(x) \ll [s, t]$, $\mu_\lambda(x) \leq \theta$ and $\nu_\lambda(x) \geq \theta$. Now, $M_A(0) \gg M_A(x) \gg [s, t]$, $N_A(0) \ll N_A(x) \ll [s, t]$, $\mu_\lambda(0) \leq \mu_\lambda(x) \leq \theta$ and $\nu_\lambda(0) \geq \nu_\lambda(x) \geq \theta$. Thus $0 \in M_A[s, t] \cap N_A[s, t] \cap \mu_\lambda(\theta) \cap \nu_\lambda(\theta)$. Now, letting $x * y$, $y \in M_A[s, t] \cap N_A[s, t] \cap \mu_\lambda(\theta) \cap \nu_\lambda(\theta)$. This implies that

$$\begin{aligned}
M_A(x) &\gg \text{rmin}\{M_A(x * y), M_A(y)\} \gg \text{rmin}\{[s, t], [s, t]\} = [s, t], \\
N_A(x) &\ll \text{rmax}\{N_A(x * y), N_A(y)\} \ll \text{rmax}\{[s, t], [s, t]\} = [s, t], \\
\mu_\lambda(x) &\leq \max\{\mu_\lambda(x * y), \mu_\lambda(y)\} \leq \max\{\theta, \theta\} = \theta, \\
\nu_\lambda(x) &\geq \min\{\nu_\lambda(x * y), \nu_\lambda(y)\} \geq \min\{\theta, \theta\} = \theta.
\end{aligned}$$

Therefore, $x \in M_A[s, t] \cap N_A[s, t] \cap \mu_\lambda(\theta) \cap \nu_\lambda(\theta)$. Hence $M_A[s, t]$, $N_A[s, t]$, $\mu_\lambda(\theta)$ and $\nu_\lambda(\theta)$ are ideals of X. □

Theorem 3.10. *Let $\widetilde{A} = \langle A, \lambda \rangle$ be a cubic intuitionistic set in X such that the non-empty sets $M_A[s_1, t_1]$, $N_A[s_2, t_2]$, $\mu_\lambda(\theta_1)$ and $\nu_\lambda(\theta_2)$ are ideals of X for all $([s_1, t_1], [s_2, t_2]) \in D[0,1] \times D[0,1]$ and $(\theta_1, \theta_2) \in [0,1] \times [0,1]$. Then $\widetilde{A} = \langle A, \lambda \rangle$ is a cubic intuitionistic ideal of X.*

Proof. Suppose that for every $([s_1, t_1], [s_2, t_2]) \in D[0,1] \times D[0,1]$ and $(\theta_1, \theta_2) \in [0,1] \times [0,1]$, $M_A[s_1, t_1]$, $N_A[s_2, t_2]$, $\mu_\lambda(\theta_1)$ and $\nu_\lambda(\theta_2)$ are non-empty ideals of X. Assume that $M_A(0) \ll M_A(p)$, that is $[M_A^-(0), M_A^+(0)] \ll [M_A^-(p), M_A^+(p)]$ for some $p \in X$. If we take $\tilde{s}_p = \frac{1}{2}[M_A^-(0) + M_A^-(p)]$, $\tilde{t}_p = \frac{1}{2}[M_A^+(0) + M_A^+(p)]$, then $M_A(0) = [M_A^-(0), M_A^+(0)] \ll [\tilde{s}_p, \tilde{t}_p] \ll [M_A^-(p), M_A^+(p)] = M_A(p)$. Hence $0 \notin M_A[\tilde{s}_p, \tilde{t}_p]$. This is a contradiction, and so $M_A(0) \gg M_A(x)$ for all $x \in X$. Similarly $N_A(0) \ll N_A(x)$, $\mu_\lambda(0) \leq \mu_\lambda(x)$ and $\nu_\lambda(0) \geq \nu_\lambda(x)$ for all $x \in X$.

Now, let $p, q \in X$ be such that $M_A(p) \ll \text{rmin}\{M_A(p*q), M_A(q)\}$. Suppose that $M_A(p) = [p^-, p^+]$, $M_A(q) = [q^-, q^+]$ and $M_A(p*q) = [(p*q)^-, (p*q)^+]$. Assume that $\tilde{s}_0 = \frac{1}{2}(p^- + \min\{(p*q)^-, q^-\})$, $\tilde{t}_0 = \frac{1}{2}(p^+ + \min\{(p*q)^+, q^+\})$. Then $p^- \ll \tilde{s}_0 \ll \min\{(p*q)^-, q^-\}$ and $p^+ \ll \tilde{t}_0 \ll \min\{(p*q)^+, q^+\}$, which implies that

$$\begin{aligned} M_A(p) &= [p^-, p^+] \ll [\tilde{s}_0, \tilde{t}_0] \\ &\ll [\min\{(p*q)^-, q^-\}, \min\{(p*q)^+, q^+\}] \\ &= \text{rmin}\{M_A(p*q), M_A(q)\}. \end{aligned}$$

Thus $p \notin M_A[\tilde{s}_0, \tilde{t}_0]$ but $p*q$, $q \in M_A[\tilde{s}_0, \tilde{t}_0]$. This is a contradiction and hence M_A satisfies $M_A(x) \gg \text{rmin}\{M_A(x*y), M_A(y)\}$, for all $x, y, z \in X$. Similarly, we can prove that $N_A(x) \ll \text{rmax}\{N_A(x*y), N_A(y)\}$, $\mu_\lambda(x) \leq \max\{\mu_\lambda(x*y), \mu_\lambda(y)\}$ and $\nu_\lambda(x) \geq \min\{\nu_\lambda(x*y), \nu_\lambda(y)\}$, for all $x, y, z \in X$. Therefore, $\widetilde{A} = \langle A, \lambda \rangle$ forms a cubic intuitionistic ideal of X. □

4 Cubic intuitionistic p-ideals

Definition 4.1. A cubic intuitionistic set $\widetilde{A} = \langle A, \lambda \rangle$ in X is called a cubic intuitionistic p-ideal of X if it satisfies conditions (a) and (b) in Definition 3.2 and for all $x, y, z \in X$:

(a) $M_A(x) \gg \text{rmin}\{M_A((x*z)*(y*z)), M_A(y)\}$
(b) $N_A(x) \ll \text{rmax}\{N_A((x*z)*(y*z)), N_A(y)\}$
(c) $\mu_\lambda(x) \leq \max\{\mu_\lambda((x*z)*(y*z)), \mu_\lambda(y)\}$
(d) $\nu_\lambda(x) \geq \min\{\nu_\lambda((x*z)*(y*z)), \nu_\lambda(y)\}$.

We now illustrate the above definitions by using the following examples.

Example 4.2. *Let $X = \{0, a, b, c\}$ be a BCI-algebra with the following Cayley table:*

$*$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

Define a cubic intuitionistic set $\widetilde{A} = \langle A, \lambda \rangle$ in X as follows:

X	$A = \langle M_A, N_A \rangle$	$\lambda = (\mu_\lambda, \nu_\lambda)$
0	$\langle [0.6, 0.8], [0.1, 0.2] \rangle$	$(0.3, 0.7)$
a	$\langle [0.5, 0.7], [0.2, 0.3] \rangle$	$(0.4, 0.6)$
b	$\langle [0.2, 0.4], [0.3, 0.5] \rangle$	$(0.6, 0.3)$
c	$\langle [0.2, 0.4], [0.3, 0.5] \rangle$	$(0.6, 0.3)$

By routine calculation $\widetilde{A} = \langle A, \lambda \rangle$ is a cubic intuitionistic p-ideal of X.

Note that every cubic intuitionistic p-ideal of a BCI-algebra X is a cubic intuitionistic ideal of X by putting $z = 0$ in Definition 4.1 and using (a3). But, the converse is not true as seen in the following example.

Example 4.3. *Let $X = \{0, a, b, c, d\}$ be a BCI-algebra with the following Cayley table:*

$*$	0	a	b	c	d
0	0	0	d	c	b
a	a	0	d	c	b
b	b	b	0	d	c
c	c	c	b	0	d
d	d	d	c	b	0

Define a cubic intuitionistic set $\widetilde{A} = \langle A, \lambda \rangle$ in X as follows:

X	$A = \langle M_A, N_A \rangle$	$\lambda = (\mu_\lambda, \nu_\lambda)$
0	$\langle [0.6, 0.7], [0.1, 0.2] \rangle$	$(0.1, 0.9)$
a	$\langle [0.4, 0.5], [0.2, 0.3] \rangle$	$(0.3, 0.6)$
b	$\langle [0.2, 0.3], [0.4, 0.6] \rangle$	$(0.5, 0.4)$
c	$\langle [0.2, 0.3], [0.4, 0.6] \rangle$	$(0.5, 0.4)$
d	$\langle [0.2, 0.3], [0.4, 0.6] \rangle$	$(0.5, 0.4)$

Then $\widetilde{A} = \langle A, \lambda \rangle$ is a cubic intuitionistic ideal of X, but not a cubic intuitionistic p-ideal of X since $M_A(a) = [0.4, 0.5] \ll [0.6, 0.7] = \text{rmin}\{M_A((a * b) * (0 * b)), M_A(0)\}$, $N_A(a) = [0.2, 0.3] \gg [0.1, 0.2] = \text{rmax}\{N_A((a*b)*(0*b)), N_A(0)\}$, $\mu_\lambda(a) = 0.3 \nleq 0.1 = \max\{\mu_\lambda((a * b) * (0 * b)), \mu_\lambda(0)\}$ and $\nu_\lambda(a) = 0.6 \ngeq 0.9 = \min\{\mu_\lambda((a * b) * (0 * b)), \mu_\lambda(0)\}$.

Proposition 4.4. *Every cubic intuitionistic p-ideal* $\widetilde{A} = \langle A, \lambda \rangle$ *of* X *satisfies the following inequalities:* $M_A(x) \gg M_A(0*(0*x))$, $N_A(x) \ll N_A(0*(0*x))$, $\mu_\lambda(x) \leq \mu_\lambda(0*(0*x))$ *and* $\nu_\lambda(x) \geq \nu_\lambda(0*(0*x))$, *for all* $x \in X$.

Proof. It can be easily obtained by putting $z = x$ and $y = 0$ in Definition 4.1. □

Proposition 4.5. *Every cubic intuitionistic p-ideal* $\widetilde{A} = \langle A, \lambda \rangle$ *of* X *satisfies the following inequalities:* $M_A(x*y) \ll M_A((x*z)*(y*z))$, $N_A(x*y) \gg N_A((x*z)*(y*z))$, $\mu_\lambda(x*y) \geq \mu_\lambda((x*z)*(y*z))$ *and* $\nu_\lambda(x*y) \leq \nu_\lambda((x*z)*(y*z))$, *for all* $x, y, z \in X$.

Proof. Let $\widetilde{A} = \langle A, \lambda \rangle$ be a cubic intuitionistic p-ideal of X. Note that $(x*z)*(y*z) \leq x*y$, i.e., $((x*z)*(y*z))*(x*y) = 0$, for all $x, y, z \in X$. Since every cubic intuitionistic p-ideal of X is a cubic intuitionistic ideal of X, therefore

$$\begin{aligned}
M_A((x*z)*(y*z)) &\gg \text{rmin}\{M_A(((x*z)*(y*z))*(x*y)), M_A(x*y)\} \\
&= \text{rmin}\{M_A(0), M_A(x*y)\} = M_A(x*y), \\
N_A((x*z)*(y*z)) &\ll \text{rmax}\{N_A(((x*z)*(y*z))*(x*y)), N_A(x*y)\} \\
&= \text{rmax}\{N_A(0), N_A(x*y)\} = N_A(x*y), \\
\mu_\lambda((x*z)*(y*z)) &\leq \max\{\mu_\lambda(((x*z)*(y*z))*(x*y)), \mu_\lambda(x*y)\} \\
&= \max\{\mu_\lambda(0), \mu_\lambda(x*y)\} = \mu_\lambda(x*y), \\
\nu_\lambda((x*z)*(y*z)) &\geq \min\{\nu_\lambda(((x*z)*(y*z))*(x*y)), \nu_\lambda(x*y)\} \\
&= \min\{\nu_\lambda(0), \nu_\lambda(x*y)\} = \nu_\lambda(x*y),
\end{aligned}$$

for all $x, y, z \in X$. This completes the proof. □

We provide conditions for a cubic intuitionistic ideal to be a cubic intuitionistic p-ideal.

Theorem 4.6. *Let* $\widetilde{A} = \langle A, \lambda \rangle$ *be a cubic intuitionistic ideal of* X *that satisfies:* $M_A(x*y) \gg M_A((x*z)*(y*z))$, $N_A(x*y) \ll N_A((x*z)*(y*z))$, $\mu_\lambda(x*y) \leq \mu_\lambda((x*z)*(y*z))$ *and* $\nu_\lambda(x*y) \geq \nu_\lambda((x*z)*(y*z))$, *for all* $x, y, z \in X$. *Then* $\widetilde{A}$ *is a cubic intuitionistic p-ideal of* X.

Proof. For any $x, y, z \in X$, $M_A(x) \gg \text{rmin}\{M_A(x*y), M_A(y)\} = \text{rmin}\{M_A((x*z)*(y*z)), M_A(y)\}$, $N_A(x) \ll \text{rmax}\{N_A(x*y), N_A(y)\} = \text{rmax}\{N_A((x*z)*(y*z)), N_A(y)\}$, $\mu_\lambda(x) \leq \max\{\mu_\lambda(x*y), \mu_\lambda(y)\} = \max\{\mu_\lambda((x*z)*(y*z)), \mu_\lambda(y)\}$, $\nu_\lambda(x) \geq \min\{\nu_\lambda(x*y), \nu_\lambda(y)\} = \min\{\nu_\lambda((x*z)*(y*z)), \nu_\lambda(y)\}$. This completes the proof. □

Lemma 4.7. *Every cubic intuitionistic ideal* $\widetilde{A} = \langle A, \lambda \rangle$ *satisfies the following inequalities:* $M_A(0*(0*x)) \gg M_A(x)$, $N_A(0*(0*x)) \ll N_A(x)$, $\mu_\lambda(0*(0*x)) \leq \mu_\lambda(x)$ *and* $\nu_\lambda(0*(0*x)) \geq \nu_\lambda(x)$, *for all* $x \in X$.

Proof. Let $\widetilde{A} = \langle A, \lambda\rangle$ be a cubic intuitionistic ideal of X. For any $x \in X$, $M_A(x) = \text{rmin}\{M_A(0), M_A(x)\} = \text{rmin}\{M_A(0*(0*x)), M_A(x)\} \ll M_A(0*(0*x))$, $N_A(x) = \text{rmax}\{N_A(0),\ N_A(x)\} = \text{rmax}\{N_A(0*(0*x)), N_A(x)\} \gg N_A(0*(0*x))$, $\mu_\lambda(x) = \max\{\mu_\lambda(0), \mu_\lambda(x)\} = \max\{\mu_\lambda(0*(0*x)), \mu_\lambda(x)\} \geq \mu_\lambda(0*(0*x))$ and $\nu_\lambda(x) = \min\{\nu_\lambda(0), \nu_\lambda(x)\} = \min\{\nu_\lambda(0*(0*x)), \nu_\lambda(x)\} \leq \nu_\lambda(0*(0*x))$. This completes the proof. □

Lemma 4.8. [15] *Let X be a BCI-algebra. Then, for all $x, y, z \in X$:*

(a) $0*(0*((x*z)*(y*z))) = (0*y)*(0*x)$,

(b) $0*(0*(x*y)) = (0*y)*(0*x)$.

Theorem 4.9. *Let $\widetilde{A} = \langle A, \lambda\rangle$ be a cubic intuitionistic ideal of X that satisfies: $M_A(0*(0*x)) \ll M_A(x)$, $N_A(0*(0*x)) \gg N_A(x)$, $\mu_\lambda(0*(0*x)) \geq \mu_\lambda(x)$ and $\nu_\lambda(0*(0*x)) \leq \nu_\lambda(x)$, for all $x \in X$. Then $\widetilde{A} = \langle A, \lambda\rangle$ is a cubic intuitionistic p-ideal of X.*

Proof. Let $\widetilde{A} = \langle A, \lambda\rangle$ be a cubic intuitionistic ideal of X and $x, y, z \in X$. Using Lemmas 4.7 and 4.8, we get $M_A((x*z)*(y*z)) \ll M_A(0*(0*((x*z)*(y*z)))) = M_A((0*y)*(0*x)) = M_A(0*(0*(x*y))) \ll M_A(x*y)$, $N_A((x*z)*(y*z)) \gg N_A(0*(0*((x*z)*(y*z)))) = N_A((0*y)*(0*x)) = N_A(0*(0*(x*y))) \gg N_A(x*y)$, $\mu_\lambda((x*z)*(y*z)) \geq \mu_\lambda(0*(0*((x*z)*(y*z)))) = \mu_\lambda((0*y)*(0*x)) = \mu_\lambda(0*(0*(x*y))) \geq \mu_\lambda(x*y)$ and $\nu_\lambda((x*z)*(y*z)) \leq \nu_\lambda(0*(0*((x*z)*(y*z)))) = \nu_\lambda((0*y)*(0*x)) = \nu_\lambda(0*(0*(x*y))) \leq \nu_\lambda(x*y)$. It follows from Theorem 4.6 that $\widetilde{A} = \langle A, \lambda\rangle$ is a cubic intuitionistic p-ideal of X. □

5 Cubic intuitionistic a-ideals

Definition 5.1. [14] A cubic intuitionistic set $\widetilde{A} = \langle A, \lambda\rangle$ in X is called a cubic intuitionistic q-ideal of X if it satisfies conditions (a) and (b) in Definition 3.2 and for all $x, y, z \in X$:

(a) $M_A(x*z) \gg \text{rmin}\{M_A(x*(y*z)), M_A(y)\}$

(b) $N_A(x*z) \ll \text{rmax}\{N_A(x*(y*z)), N_A(y)\}$

(c) $\mu_\lambda(x*z) \leq \max\{\mu_\lambda(x*(y*z)), \mu_\lambda(y)\}$

(d) $\nu_\lambda(x*z) \geq \min\{\nu_\lambda(x*(y*z)), \nu_\lambda(y)\}$.

Definition 5.2. *A cubic intuitionistic set $\widetilde{A} = \langle A, \lambda\rangle$ in X is called a cubic intuitionistic a-ideal of X if it satisfies conditions (a) and (b) in Definition 3.2 and for all $x, y, z \in X$:*

(a) $M_A(y*x) \gg \text{rmin}\{M_A((x*z)*(0*y)), M_A(z)\}$

(b) $N_A(y*x) \ll \text{rmax}\{N_A((x*z)*(0*y)), N_A(z)\}$

(c) $\mu_\lambda(y*x) \leq \max\{\mu_\lambda((x*z)*(0*y)), \mu_\lambda(z)\}$

(d) $\nu_\lambda(y*x) \geq \min\{\nu_\lambda((x*z)*(0*y)), \nu_\lambda(z)\}$.

We now illustrate the above definitions by using the following examples.

Example 5.3. *Let* X=$\{0, a, b, c\}$ *be a* BCI*-algebra in Example 4.2 and* $\widetilde{A} = \langle A, \lambda\rangle$ *be a cubic intuitionistic set of* X *as follows:*

X	$A = \langle M_A, N_A\rangle$	$\lambda = (\mu_\lambda, \nu_\lambda)$
0	$\langle[0.7, 0.9], [0.0, 0.1]\rangle$	$(0.3, 0.7)$
a	$\langle[0.7, 0.9], [0.0, 0.1]\rangle$	$(0.3, 0.7)$
b	$\langle[0.5, 0.6], [0.3, 0.4]\rangle$	$(0.4, 0.5)$
c	$\langle[0.5, 0.6], [0.3, 0.4]\rangle$	$(0.4, 0.5)$

It is easy to verify that $\widetilde{A} = \langle A, \lambda\rangle$ *is a cubic intuitionistic a-ideal of* X*.*

Theorem 5.4. *Every cubic intuitionistic a-ideal of* X *is both a closed cubic intuitionistic ideal and a cubic intuitionistic subalgebra of* X*.*

Proof. Let $\widetilde{A} = \langle A, \lambda\rangle$ be a cubic intuitionistic a-ideal of X. Putting $z = y = 0$ in Definition 5.2 and using Definition 3.3, we have

$$\begin{aligned} M_A(0 * x) &\gg \text{rmin}\{M_A((x * 0) * (0 * 0)), M_A(0)\} = M_A(x), \\ N_A(0 * x) &\ll \text{rmax}\{N_A((x * 0) * (0 * 0)), N_A(0)\} = N_A(x), \\ \mu_\lambda(0 * x) &\leq \max\{\mu_\lambda((x * 0) * (0 * 0)), \mu_\lambda(0)\} = \mu_\lambda(x)\}, \\ \nu_\lambda(0 * x) &\geq \min\{\nu_\lambda((x * 0) * (0 * 0)), \nu_\lambda(0)\} = \nu_\lambda(x), \end{aligned} \tag{1}$$

for all $x \in X$. If we take $x = z = 0$ in Definition 5.2 and using Definition 3.3, then

$$\begin{aligned} M_A(y) &\gg \text{rmin}\{M_A(0 * (0 * y)), M_A(0)\} = M_A(0 * (0 * y)), \\ N_A(y) &\ll \text{rmax}\{N_A(0 * (0 * y)), N_A(0)\} = N_A(0 * (0 * y)), \\ \mu_\lambda(y) &\leq \max\{\mu_\lambda(0 * (0 * y)), \mu_\lambda(0)\} = \mu_\lambda(0 * (0 * y))\}, \\ \nu_\lambda(y) &\geq \min\{\nu_\lambda(0 * (0 * y)), \nu_\lambda(0)\} = \nu_\lambda(0 * (0 * y)), \end{aligned}$$

for all $y \in X$. It follows from (1) that $M_A(x) \gg M_A(0*x)$, $N_A(x) \ll N_A(0*x)$, $\mu_\lambda(x) \leq \mu_\lambda(0 * x)$ and $\nu_\lambda(x) \geq \nu_\lambda(0 * x)$, for all $x \in X$. From Definition 5.2, we obtain $M_A(x) \gg M_A(0 * x) \gg \text{rmin}\{M_A((x * z) * (0 * 0)), M_A(z)\} = \text{rmin}\{M_A(x * z), M_A(z)\}$, $N_A(x) \ll N_A(0 * x) \ll \text{rmax}\{N_A((x * z) * (0 * 0)), N_A(z)\} = \text{rmax}\{N_A(x * z), N_A(z)\}$, $\mu_\lambda(x) \leq \mu_\lambda(0 * x) \leq \max\{\mu_\lambda((x * 0) * (0 * 0)), \mu_\lambda(0)\} = \max\{\mu_\lambda(x * z), \mu_\lambda(z)\}$, $\nu_\lambda(x) \geq \nu_\lambda(0 * x) \geq \min\{\nu_\lambda((x * 0) * (0 * 0)), \nu_\lambda(0)\} = \min\{\nu_\lambda(x * 0), \nu_\lambda(0)\}$, for all $x, z \in X$. Hence $\widetilde{A} = \langle A, \lambda\rangle$ is a closed cubic intuitionistic ideal of X. □

From Theorem 3.5, we know that every closed cubic intuitionistic ideal of X is a cubic intuitionistic subalgebra of X, therefore every cubic intuitionistic a-ideal of X is a cubic intuitionistic subalgebra of X.

The converse of Theorem 5.4 is not true in general as seen in the following example.

Example 5.5. *Let $X = \{0, a, b, c, d\}$ be a BCI-algebra with the following Cayley table:*

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	0	0
b	b	b	0	0	0
c	c	c	c	0	0
d	d	c	d	a	0

Define a cubic intuitionistic set $\widetilde{A} = \langle A, \lambda \rangle$ in X as follows:

X	$A = \langle M_A, N_A \rangle$	$\lambda = (\mu_\lambda, \nu_\lambda)$
0	$\langle [0.5, 0.7], [0.1, 0.2] \rangle$	$(0.1, 0.8)$
a	$\langle [0.4, 0.5], [0.2, 0.3] \rangle$	$(0.5, 0.4)$
b	$\langle [0.5, 0.7], [0.1, 0.2] \rangle$	$(0.1, 0.8)$
c	$\langle [0.4, 0.5], [0.2, 0.3] \rangle$	$(0.5, 0.4)$
d	$\langle [0.4, 0.5], [0.2, 0.3] \rangle$	$(0.5, 0.4)$

Then $\widetilde{A} = \langle A, \lambda \rangle$ is both a cubic intuitionistic ideal and a cubic intuitionistic subalgebra of X, but not a cubic intuitionistic a-ideal of X since $M_A(a * b) = [0.4, 0.5] \ll [0.5, 0.7] = \text{rmin}\{M_A((b * 0) * (0 * a)), M_A(0)\}$, $N_A(a * b) = [0.2, 0.3] \gg [0.1, 0.2] = \text{rmax}\{N_A((b * 0) * (0 * a)), N_A(0)\}$, $\mu_\lambda(a * b) = 0.5 \nleq 0.1 = \max\{\mu_\lambda((b * 0) * (0 * a)), \mu_\lambda(0)\}$ and $\nu_\lambda(a * b) = 0.4 \ngeq 0.8 = \min\{\mu_\lambda((b * 0) * (0 * a)), \mu_\lambda(0)\}$.

Proposition 5.6. [6] *Let $\widetilde{A} = \langle A, \lambda \rangle$ be a cubic intuitionistic ideal of X. If the ineqality $x * y \leq z$ holds in X, then $M_A(x) \gg \text{rmin}\{M_A(y), M_A(z)\}$, $N_A(x) \ll \text{rmax}\{N_A(y), N_A(z)\}$, $\mu_\lambda(x) \leq \max\{\mu_\lambda(y), \mu_\lambda(z)\}$ and $\nu_\lambda(x) \geq \min\{\mu_\lambda(y), \mu_\lambda(z)\}$.*

The characterizations of cubic intuitionistic a-ideal are given by the following theorem.

Theorem 5.7. *If $\widetilde{A} = \langle A, \lambda \rangle$ is a cubic intuitionistic ideal of X, then the following assertions are equivalent:*
(i) $\widetilde{A} = \langle A, \lambda \rangle$ is a cubic intuitionistic a-ideal of X,
(ii) $\widetilde{A} = \langle A, \lambda \rangle$ satisfies the inequalities: $M_A(y * (x * z)) \gg M_A((x * z) * (0 * y))$, $N_A(y * (x * z)) \ll N_A((x * z) * (0 * y))$, $\mu_\lambda(y * (x * z)) \leq \mu_\lambda((x * z) * (0 * y))$ and $\nu_\lambda(y * (x * z)) \geq \nu_\lambda((x * z) * (0 * y))$, for all $x, y, z \in X$,
(iii) $\widetilde{A} = \langle A, \lambda \rangle$ satisfies the inequalities: $M_A(y * x) \gg M_A(x * (0 * y))$,

$N_A(y * x) \ll N_A(x * (0 * y))$, $\mu_\lambda(y * x) \leq \mu_\lambda(x * (0 * y))$ *and* $\nu_\lambda(y * x) \geq \nu_\lambda(x * (0 * y))$, *for all* $x, y \in X$.

Proof. $(i) \Rightarrow (ii)$ Let $\widetilde{A} = \langle A, \lambda \rangle$ be a cubic intuitionistic a-ideal of X. Then, for all $x, y, z \in X$, we have $M_A(y * (x * z)) \gg \text{rmin}\{M_A(((x * z) * 0) * (0 * y)), M_A(0)\} = M_A(((x * z) * 0) * (0 * y)) = M_A((x * z) * (0 * y))$, $N_A(y * (x * z)) \ll \text{rmax}\{N_A(((x * z) * 0) * (0 * y)), N_A(0)\} = N_A(((x * z) * 0) * (0 * y)) = N_A((x * z) * (0 * y))$, $\mu_\lambda(y * (x * z)) \leq \max\{\mu_\lambda(((x * z) * 0) * (0 * y)), \mu_\lambda(0)\} = \mu_\lambda(((x * z) * 0) * (0 * y)) = \mu_\lambda((x * z) * (0 * y))$ and $\nu_\lambda(y * (x * z)) \geq \min\{\nu_\lambda(((x * z) * 0) * (0 * y)), \nu_\lambda(0)\} = \mu_\lambda(((x * z) * 0) * (0 * y)) = \nu_\lambda((x * z) * (0 * y))$. Therefore, (ii) is satisfied.
$(ii) \Rightarrow (iii)$ Assume that (ii) is satisfied. (iii) is induced by taking $z = 0$ in (ii) and using (a3).
$(iii) \Rightarrow (i)$ Suppose that $\widetilde{A} = \langle A, \lambda \rangle$ satisfies (iii). Note that for all $x, y, z \in X$,

$$(x * (0 * y)) * ((x * z) * (0 * y)) \leq x * (x * z) \leq x.$$

It follows from (iii) and Proposition 5.6 that $M_A(y * x) \gg M_A(x * (0 * y)) \gg \text{rmin}\{M_A((x * z) * (0 * y), M_A(x)\}$, $N_A(y * x) \ll N_A(x * (0 * y)) \ll \text{rmax}\{N_A((x * z) * (0 * y), N_A(x)\}$, $\mu_\lambda(y * x) \leq \mu_\lambda(x * (0 * y)) \leq \max\{\mu_\lambda((x * z) * (0 * y), \mu_\lambda(x)\}$ and $\nu_\lambda(y * x) \geq \nu_\lambda(x * (0 * y)) \geq \min\{\nu_\lambda((x * z) * (0 * y), \nu_\lambda(x)\}$, for all $x, y, z \in X$. Therefore, $\widetilde{A} = \langle A, \lambda \rangle$ is a cubic intuitionistic a-ideal of X. Hence, the assertion (i) holds. The proof is complete. □

Lemma 5.8. [4] *A subset I of X is an a-ideal of X if and only if it is an ideal of X which satisfies the implication: $x * (0 * y) \in I \Rightarrow y * x \in I$, for all $x, y \in X$.*

Theorem 5.9. *For a cubic intuitionistic set $\widetilde{A} = \langle A, \lambda \rangle$ in X, the following are equivalent:*
(i) $\widetilde{A} = \langle A, \lambda \rangle$ is a cubic intuitionistic a-ideal of X,
(ii) Every non-empty sets $M_A[s, t]$, $N_A[s, t]$, $\mu_\lambda(\theta)$ and $\nu_\lambda(\theta)$ are a-ideals of X, for all $[s, t] \in D[0, 1]$ and $\theta \in [0, 1]$.

Proof. Assume that $\widetilde{A} = \langle A, \lambda \rangle$ is a cubic intuitionistic a-ideal of X. Then $\widetilde{A} = \langle A, \lambda \rangle$ is a cubic intuitionistic ideal of X by Theorem 5.4. By using Theorem 3.9, we get the non-empty sets $M_A[s, t]$, $N_A[s, t]$, $\mu_\lambda(\theta)$ and $\nu_\lambda(\theta)$ are ideals of X, for all $[s, t] \in D[0, 1]$ and $\theta \in [0, 1]$. Let $x, y \in X$ be such that $x * (0 * y) \in M_A[s, t] \cap N_A[s, t] \cap \mu_\lambda(\theta) \cap \nu_\lambda(\theta)$. Then $M_A(x * (0 * y)) \gg [s, t]$, $N_A(x * (0 * y)) \ll [s, t]$, $\mu_\lambda(x * (0 * y)) \leq \theta$ and $\nu_\lambda(x * (0 * y)) \geq \theta$. It follows from assertion (iii) of Theorem 5.7 that $M_A(y * x) \gg M_A(x * (0 * y)) \gg [s, t]$, $N_A(y * x) \ll N_A(x * (0 * y)) \ll [s, t]$, $\mu_\lambda(y * x) \leq \mu_\lambda(x * (0 * y)) \leq \theta$ and $\nu_\lambda(y * x) \geq \nu_\lambda(x * (0 * y)) \geq \theta$, so that $y * x \in M_A[s, t] \cap N_A[s, t] \cap \mu_\lambda(\theta) \cap \nu_\lambda(\theta)$.

Using Lemma 5.8, we conclude that $M_A[s,t]$, $N_A[s,t]$, $\mu_\lambda(\theta)$ and $\nu_\lambda(\theta)$ are a-ideals of X.

Conversely, suppose that every non-empty sets $M_A[s,t]$, $N_A[s,t]$, $\mu_\lambda(\theta)$ and $\nu_\lambda(\theta)$ are a-ideals of X, for all $[s,t] \in D[0,1]$ and $\theta \in [0,1]$. Since any a-ideal is an ideal (see [4]), it follows from Theorem 3.10 that $\widetilde{A} = \langle A, \lambda\rangle$ is a cubic intuitionistic ideal of X. Assume that assertion (iii) of Theorem 5.7 is not true. Then there exist $p, q \in X$ such that $M_A(q * p) \ll M_A(p * (0 * q))$, $N_A(q*p) \gg N_A(p*(0*q))$, $\mu_\lambda(q*p) > \mu_\lambda(p*(0*q))$ and $\nu_\lambda(q*p) < \nu_\lambda(p*(0*q))$. Thus $M_A(q*p) \ll [s_0, t_0] \ll M_A(p*(0*q))$, $N_A(q*p) \gg [s_0, t_0] \gg N_A(p*(0*q))$, $\mu_\lambda(q * p) > r_0 \geq \mu_\lambda(p * (0 * q))$ and $\nu_\lambda(q * p) < r_0 \leq \nu_\lambda(p * (0 * q))$, for some $[s_0, t_0] \in D[0,1]$ and $r_0 \in [0,1]$. It follows that $p * (0 * q) \in M_A[s,t] \cap N_A[s,t] \cap \mu_\lambda(\theta) \cap \nu_\lambda(\theta)$ but $q * p \notin M_A[s,t] \cap N_A[s,t] \cap \mu_\lambda(\theta) \cap \nu_\lambda(\theta)$. Therefore assertion (iii) of Theorem 5.7 is true, which implies from Theorem 5.7 that $\widetilde{A} = \langle A, \lambda\rangle$ is a cubic intuitionistic a-ideal of X. □

The sets $\{x \in X : M_A(x) = M_A(0)\}$, $\{x \in X : N_A(x) = N_A(0)\}$, $\{x \in X : \mu_\lambda(x) = \mu_\lambda(0)\}$ and $\{x \in X : \nu_\lambda(x) = \nu_\lambda(0)\}$ are denoted by T_{M_A}, T_{N_A}, T_{μ_λ} and T_{ν_λ} respectively.

Theorem 5.10. *Let $\widetilde{A} = \langle A, \lambda\rangle$ be a cubic intuitionistic a-ideal of X. Then the sets T_{M_A}, T_{N_A}, T_{μ_λ} and T_{ν_λ} are a-ideals of X.*

Proof. Let $\widetilde{A} = \langle A, \lambda\rangle$ be a cubic intuitionistic a-ideal of X. Then it is obvious that $0 \in T_{M_A} \cap T_{N_A} \cap T_{\mu_\lambda} \cap T_{\nu_\lambda}$. Let $x, y, z \in X$ be such that $(x*z)*(0*y), z \in T_{M_A} \cap T_{N_A} \cap T_{\mu_\lambda} \cap T_{\nu_\lambda}$. Then $M_A((x * z) * (0 * y)) = M_A(0) = M_A(z)$, $N_A((x * z) * (0 * y)) = N_A(0) = N_A(z)$, $\mu_\lambda((x * z) * (0 * y)) = \mu_\lambda(0) = \mu_\lambda(z)$ and $\nu_\lambda((x*z)*(0*y)) = \nu_\lambda(0) = \nu_\lambda(z)$. Thus $M_A(y*x) \gg \text{rmin}\{M_A((x*z)*(0*y)), M_A(y)\} = \text{rmin}\{M_A(0), M_A(0)\} = M_A(0)$, $N_A(y*x) \ll \text{rmax}\{N_A((x*z)*(0*y)), N_A(y)\} = \text{rmax}\{N_A(0), N_A(0)\} = N_A(0)$, $\mu_\lambda(y*x) \leq \max\{\mu_\lambda((x*z) * (0 * y)), \mu_\lambda(y)\} = \max\{\mu_\lambda(0), \mu_\lambda(0)\} = \mu_\lambda(0)$, $\nu_\lambda(y * x) \geq \min\{\nu_\lambda((x * z) * (0 * y)), \nu_\lambda(y)\} = \min\{\nu_\lambda(0), \nu_\lambda(0)\} = \nu_\lambda(0)$. Since $\widetilde{A} = \langle A, \lambda\rangle$ is a cubic intuitionistic a-ideal of X, we have $M_A(y * x) = M_A(0)$, $N_A(y * x) = N_A(0)$, $\mu_\lambda(y * x) = \mu_\lambda(0)$ and $\nu_\lambda(y * x) = \nu_\lambda(0)$ i.e., $y * x \in T_{M_A} \cap T_{N_A} \cap T_{\mu_\lambda} \cap T_{\nu_\lambda}$. Hence, the sets T_{M_A}, T_{N_A}, T_{μ_λ} and T_{ν_λ} are a-ideals of X. □

Theorem 5.11. *Every cubic intuitionistic a-ideal is a cubic intuitionistic p-ideal.*

Proof. Let $\widetilde{A} = \langle A, \lambda\rangle$ be a cubic intuitionistic a-ideal of X. Then $\widetilde{A} = \langle A, \lambda\rangle$ is a cubic intuitionistic ideal of X by Theorem 5.4. If we take $x = z = 0$ in assertion (ii) of Theorem 5.7, then $M_A(y) \gg M_A(0 * (0 * y))$, $N_A(y) \ll N_A(0*(0*y))$, $\mu_\lambda(y) \leq \mu_\lambda(0*(0*y))$ and $\nu_\lambda(y) \geq \nu_\lambda(0*(0*y))$, for all $y \in X$.

Hence, by Theorem 4.9, we conclude that $\widetilde{A} = \langle A, \lambda\rangle$ is a cubic intuitionistic p-ideal of X. □

The converse of Theorem 5.11 is not true in general as seen in the following example.

Example 5.12. *Let $X = \{0, a, b\}$ be a BCI-algebra with the following Cayley table:*

$*$	0	a	b
0	0	b	a
a	a	0	b
b	b	a	0

Define a cubic intuitionistic set $\widetilde{A} = \langle A, \lambda\rangle$ in X as follows:

X	$A = \langle M_A, N_A\rangle$	$\lambda = (\mu_\lambda, \nu_\lambda)$
0	$\langle[0.6, 0.8], [0.1, 0.2]\rangle$	$(0.3, 0.7)$
a	$\langle[0.2, 0.4], [0.3, 0.5]\rangle$	$(0.4, 0.5)$
b	$\langle[0.2, 0.4], [0.3, 0.5]\rangle$	$(0.4, 0.5)$

Then $\widetilde{A} = \langle A, \lambda\rangle$ is a cubic intuitionistic p-ideal of X but not a cubic intuitionistic a-ideal of X, since $M_A(b*a) = [0.2, 0.4] \ll [0.6, 0.8] = \text{rmin}\{M_A((a*0)*(0*b)), M_A(0)\}$, $N_A(b*a) = [0.3, 0.5] \gg [0.1, 0.2] = \text{rmax}\{N_A((a*0)*(0*b)), N_A(0)\}$, $\mu_\lambda(b*a) = 0.4 \nleq 0.3 = \max\{\mu_\lambda((a*0)*(0*b)), \mu_\lambda(0)\}$ and $\nu_\lambda(b*a) = 0.5 \ngeq 0.7 = \min\{\mu_\lambda((a*0)*(0*b)), \mu_\lambda(0)\}$.

Theorem 5.13. [14] *If $\widetilde{A} = \langle A, \lambda\rangle$ is a cubic intuitionistic ideal of X, then the following assertions are equivalent:*
(i) $\widetilde{A} = \langle A, \lambda\rangle$ is a cubic intuitionistic q-ideal of X,
(ii) $M_A(x*y) \gg M_A(x*(0*y))$, $N_A(x*y) \ll N_A(x*(0*y))$, $\mu_\lambda(x*y) \leq \mu_\lambda(x*(0*y))$ and $\nu_\lambda(x*y) \geq \nu_\lambda(x*(0*y))$, for all $x, y \in X$, for all $x, y, z \in X$.

Theorem 5.14. *Every cubic intuitionistic a-ideal is a cubic intuitionistic q-ideal.*

Proof. Let $\widetilde{A} = \langle A, \lambda\rangle$ be a cubic intuitionistic a-ideal of X. Then $\widetilde{A} = \langle A, \lambda\rangle$ is a cubic intuitionistic ideal of X by Theorem 5.4. Note that

$$\begin{aligned} & (0*(0*(y*(0*x))))*(x*(0*y)) \\ = \; & ((0*(0*y))*(0*(0*(0*x))))*(x*(0*y)) \\ = \; & ((0*(0*y))*(0*x))*(x*(0*y)) \\ \leq \; & (x*(0*y))*(x*(0*y)) = 0 \end{aligned}$$

for all $x, y \in X$. By using Theorem 5.11, we get $\widetilde{A} = \langle A, \lambda \rangle$ is a cubic intuitionistic p-ideal of X. It follows from assertion (iii) of Theorem 5.7, Proposition 4.5 and Proposition 5.6 that

$$\begin{aligned}
M_A(x * y) &\gg M_A(y * (0 * x)) \gg M_A(0 * (0 * (y * (0 * x)))) \\
&\gg \operatorname{rmin}\{M_A(x * (0 * y)), M_A(0)\} = M_A(x * (0 * y)), \\
N_A(x * y) &\ll N_A(y * (0 * x)) \ll N_A(0 * (0 * (y * (0 * x)))) \\
&\ll \operatorname{rmax}\{N_A(x * (0 * y)), N_A(0)\} = N_A(x * (0 * y)), \\
\mu_\lambda(x * y) &\leq \mu_\lambda(y * (0 * x)) \leq \mu_\lambda(0 * (0 * (y * (0 * x)))) \\
&\leq \max\{\mu_\lambda(x * (0 * y)), \mu_\lambda(0)\} = \mu_\lambda(x * (0 * y)), \\
\nu_\lambda(x * y) &\geq \nu_\lambda(y * (0 * x)) \geq \nu_\lambda(0 * (0 * (y * (0 * x)))) \\
&\geq \min\{\nu_\lambda(x * (0 * y)), \nu_\lambda(0)\} = \nu_\lambda(x * (0 * y)),
\end{aligned}$$

for all $x, y \in X$. Using Theorem 5.13, we conclude that $\widetilde{A} = \langle A, \lambda \rangle$ is a cubic intuitionistic q-ideal of X. □

The converse of Theorem 5.14 is not true in general as seen in the following example.

Example 5.15. *Let $X = \{0, a, b\}$ be a BCI-algebra with the following Cayley table:*

$*$	0	a	b
0	0	0	b
a	a	0	b
b	b	b	0

Define a cubic intuitionistic set $\widetilde{A} = \langle A, \lambda \rangle$ in X as follows:

X	$A = \langle M_A, N_A \rangle$	$\lambda = (\mu_\lambda, \nu_\lambda)$
0	$\langle [0.5, 0.7], [0.2, 0.3] \rangle$	$(0.2, 0.7)$
a	$\langle [0.1, 0.3], [0.4, 0.5] \rangle$	$(0.3, 0.6)$
b	$\langle [0.1, 0.3], [0.4, 0.5] \rangle$	$(0.3, 0.6)$

Then $\widetilde{A} = \langle A, \lambda \rangle$ is a cubic intuitionistic q-ideal of X but not a cubic intuitionistic a-ideal of X, since $M_A(a*0) = [0.1, 0.3] \ll [0.5, 0.7] = \operatorname{rmin}\{M_A((0*0)*(0*a)), M_A(0)\}$, $N_A(a*0) = [0.4, 0.5] \gg [0.2, 0.3] = \operatorname{rmax}\{N_A((0*0)*(0*a)), N_A(0)\}$, $\mu_\lambda(a*0) = 0.3 \nleq 0.2 = \max\{\mu_\lambda((0*0)*(0*a)), \mu_\lambda(0)\}$ and $\nu_\lambda(a*0) = 0.6 \ngeq 0.7 = \min\{\mu_\lambda((0*0)*(0*a)), \mu_\lambda(0)\}$.

Lemma 5.16. [6] *Let $\widetilde{A} = \langle A, \lambda \rangle$ be a cubic intuitionistic ideal of X. If for all $x, y \in X$, the inequality $x \leq y$ holds in X, then $M_A(x) \gg M_A(y)$, $N_A(x) \ll N_A(y)$, $\mu_\lambda(x) \leq \mu_\lambda(y)$ and $\nu_\lambda(x) \geq \nu_\lambda(y)$.*

Theorem 5.17. *For a cubic intuitionistic set $\widetilde{A} = \langle A, \lambda \rangle$ in X, the following are equivalent:*
(i) $\widetilde{A} = \langle A, \lambda \rangle$ is a cubic intuitionistic a-ideal of X.
(ii) $\widetilde{A} = \langle A, \lambda \rangle$ is both a cubic intuitionistic p-ideal and a cubic intuitionistic q-ideal of X.

Proof. Combining Theorem 5.11 and Theorem 5.14, we get every cubic intuitionistic a-ideal is both a cubic intuitionistic p-ideal and a cubic intuitionistic q-ideal.

Conversely, suppose that $\widetilde{A} = \langle A, \lambda \rangle$ be both a cubic intuitionistic p-ideal and a cubic intuitionistic q-ideal. Note that $\widetilde{A} = \langle A, \lambda \rangle$ is a cubic intuitionistic ideal of X (see [14]). Taking $z = y$ in Definition 5.1, we have $M_A(x*y) \gg \text{rmin}\{M_A(x), M_A(y)\}$, $N_A(x*y) \ll \text{rmax}\{N_A(x), N_A(y)\}$, $\mu_\lambda(x*y) \leq \max\{\mu_\lambda(x), \mu_\lambda(y)\}$ and $\nu_\lambda(x*y) \geq \min\{\nu_\lambda(x), \nu_\lambda(y)\}$, for all $x, y \in X$. Hence $\widetilde{A} = \langle A, \lambda \rangle$ is a cubic intuitionistic subalgebra of X, and so $\widetilde{A} = \langle A, \lambda \rangle$ is a closed cubic intuitionistic ideal of X. Using Theorem 5.13, we get $M_A(x*y) \gg M_A(x*(0*y))$, $N_A(x*y) \ll N_A(x*(0*y))$, $\mu_\lambda(x*y) \leq \mu_\lambda(x*(0*y))$ and $\nu_\lambda(x*y) \geq \nu_\lambda(x*(0*y))$, for all $x, y \in X$. Since $0*(y*x) \leq x*y$ for all $x, y \in X$, it follows from Lemma 5.16 and above inequalities that

$$\begin{aligned} M_A(0*(y*x)) &\gg M_A(x*y) \gg M_A(x*(0*y)), \\ N_A(0*(y*x)) &\ll N_A(x*y) \ll N_A(x*(0*y)), \\ \mu_\lambda(0*(y*x)) &\leq \mu_\lambda(x*y) \leq \mu_\lambda(x*(0*y)), \\ \nu_\lambda(0*(y*x)) &\geq \nu_\lambda(x*y) \geq \nu_\lambda(x*(0*y)), \end{aligned} \tag{2}$$

for all $x, y \in X$. Using Proposition 4.4, Definition 3.3 and (2), we have

$$\begin{aligned} M_A(y*x) &\gg M_A(0*(0*(y*x))) \gg M_A(0*(y*x)) \gg M_A(x*(0*y)), \\ N_A(y*x) &\ll N_A(0*(0*(y*x))) \ll N_A(0*(y*x)) \ll N_A(x*(0*y)), \\ \mu_\lambda(y*x) &\leq \mu_\lambda(0*(0*(y*x))) \leq \mu_\lambda(0*(y*x)) \leq \mu_\lambda(x*(0*y)), \\ \nu_\lambda(y*x) &\geq \nu_\lambda(0*(0*(y*x))) \geq \nu_\lambda(0*(y*x)) \geq \nu_\lambda(x*(0*y)), \end{aligned}$$

for all $x, y \in X$. It follows from Theorem 5.7 that $\widetilde{A}$ is a cubic intuitionistic a-ideal of X. □

Theorem 5.18. *(Cubic intuitionistic extension property for a cubic intuitionistic a-ideal) Let $\widetilde{A} = \langle A, \lambda \rangle$ and $\widetilde{B} = \langle B, \vartheta \rangle$ be cubic intuitionistic ideals of X such that $\widetilde{A} \precsim \widetilde{B}$ and $M_A(0) = M_B(0)$, $N_A(0) = N_B(0)$, $\mu_\lambda(0) = \mu_\vartheta(0)$ and $\nu_\lambda(0) = \nu_\vartheta(0)$. If $\widetilde{A} = \langle A, \lambda \rangle$ is a cubic intuitionistic a-ideal of X, then so is $\widetilde{B} = \langle B, \vartheta \rangle$.*

Proof. Suppose that $\widetilde{A} = \langle A, \lambda\rangle$ is a cubic intuitionistic a-ideal of X. Then $\widetilde{A}$ is both a cubic intuitionistic p-ideal and cubic intuitionistic q-ideal of X by Theorem 5.11 and Theorem 5.14. Using Theorem 5.13, (a1) and (III), we have

$$\begin{aligned}
M_B((x*y)*(x*(0*y))) &= M_B((x*(x*(0*y)))*y)\\
&\gg M_A((x*(x*(0*y)))*y)\\
&\gg M_A((x*(x*(0*y)))*(0*y))\\
&= M_A((x*(0*y))*(x*(0*y)))\\
&= M_A(0) = M_B(0) \gg M_B(x*(0*y)),\\
\mu_\vartheta((x*y)*(x*(0*y))) &= \mu_\vartheta((x*(x*(0*y)))*y)\\
&\leq \mu_\lambda((x*(x*(0*y)))*y)\\
&\leq \mu_\lambda((x*(x*(0*y)))*(0*y))\\
&= \mu_\lambda((x*(0*y))*(x*(0*y)))\\
&= \mu_\lambda(0) = \mu_\vartheta(0) \leq \mu_\vartheta(x*(0*y)),
\end{aligned}$$

for all $x, y \in X$. Similarly $N_B((x*y)*(x*(0*y))) \ll N_B(x*(0*y))$ and $\nu_\vartheta((x*y)*(x*(0*y))) \geq \nu_\vartheta(x*(0*y))$, for all $x, y \in X$. Since $\widetilde{B} = \langle B, \vartheta\rangle$ is a cubic intuitionistic ideal, it follows that $M_B(x*y) \gg \text{rmin}\{M_B((x*y)*(x*(0*y))), M_B(x*(0*y))\} = M_B(x*(0*y))$, $N_B(x*y) \ll \text{rmax}\{N_B((x*y)*(x*(0*y))), N_B(x*(0*y))\} = N_B(x*(0*y))$, $\mu_\vartheta(x*y) \leq \max\{\mu_\vartheta((x*y)*(x*(0*y))), \mu_\vartheta(x*(0*y))\} = \mu_\vartheta(x*(0*y))$, $\nu_\vartheta(x*y) \geq \min\{\nu_\vartheta((x*y)*(x*(0*y))), \nu_\vartheta(x*(0*y))\} = \nu_\vartheta(x*(0*y))$, for all $x, y \in X$. Therefore $\widetilde{B} = \langle B, \vartheta\rangle$ is a cubic intuitionistic q-ideal of X by Theorem 5.13. Since $\widetilde{A} = \langle A, \lambda\rangle$ is a cubic intuitionistic p-ideal of X, it follows from Proposition 4.4 that

$$\begin{aligned}
M_B(x*(0*(0*x))) &\gg M_A(x*(0*(0*x)))\\
&\gg M_A(0*(0*(x*(0*(0*x)))))\\
&= M_A(0) = M_B(0) \gg M_B(0*(0*x)),\\
N_B(x*(0*(0*x))) &\ll N_A(x*(0*(0*x)))\\
&\ll N_A(0*(0*(x*(0*(0*x)))))\\
&= N_A(0) = N_B(0) \ll N_B(0*(0*x)),\\
\mu_\vartheta(x*(0*(0*x))) &\leq \mu_\lambda(x*(0*(0*x)))\\
&\leq \mu_\lambda(0*(0*(x*(0*(0*x)))))\\
&= \mu_\lambda(0) = \mu_\vartheta(0) \leq \mu_\vartheta(0*(0*x)),\\
\nu_\vartheta(x*(0*(0*x))) &\geq \nu_\lambda(x*(0*(0*x)))\\
&\geq \nu_\lambda(0*(0*(x*(0*(0*x)))))\\
&= \nu_\lambda(0) = \nu_\vartheta(0) \geq \nu_\vartheta(0*(0*x)),
\end{aligned}$$

for all $x \in X$. Hence $M_B(x) \gg \text{rmin}\{M_B(x*(0*(0*x))), M_B(0*(0*x))\} = M_B(0*(0*x))$, $N_B(x) \ll \text{rmax}\{N_B(x*(0*(0*x))), N_B(0*(0*x))\} = N_B(0*(0*x))$, $\mu_\vartheta(x) \leq \max\{\mu_\vartheta(x*(0*(0*x))), \mu_\vartheta(0*(0*x))\} = \mu_\vartheta(0*(0*x))$ and $\nu_\vartheta(x) \geq \min\{\nu_\vartheta(x*(0*(0*x))), \nu_\vartheta(0*(0*x))\} = \nu_\vartheta(0*(0*x))$, for all $x \in X$. Using Theorem 4.9, we conclude that $\widetilde{B}$ is a cubic intuitionistic p-ideal of X. Therefore $\widetilde{B} = \langle B, \vartheta \rangle$ is a cubic intuitionistic a-ideal of X by Theorem 5.17. □

6 Conclusions and Future Work

Recently, Y. B. Jun [6] has studied a novel extension of cubic sets and its applications in BCK/BCI-algebras. T. Senapati et al. [14] studied cubic intuitionistic q-ideals of a BCI-algebra. In this paper, we have applied this new notion cubic intuitionistic set to closed ideals, p-ideals, a-ideals of a BCI-algebra and investigated some of their related properties in details. In our opinion, these definitions and main results can be similarly extended to some other algebraic systems such as lattices and Lie algebras.

In our future study of cubic intuitionistic structure of BCI-algebra, the following topics will be further studied and considered:
(i) to find the product of cubic intuitionistic subalgebras, ideals and q-ideals of a BCI-algebra,
(ii) to find cubic intuitionistic (positive implicative, implicative and commutative) ideals of a BCI-algebra,
(iii) to get relationship between cubic intuitionistic (positive implicative, implicative and commutative) ideals of a BCI-algebra.

Acknowledgment: The third author was partially supported by the research grant 0064-1439-S, Deanship of Scientific Research (DRS), University of Tabuk, Tabuk-71491, Saudi Arabia.

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