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Second Hankel determinant for a class of analytic functions defined by q-derivative operator

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Abstract

In this paper, we obtain the estimates for the second Hankel determinant for a class of analytic functions defined by q-derivative operator and subordinate to an analytic function.

1 Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$.

Let $\mathcal{P}$ denote the class of analytic functions in $\mathbb{U}$ satisfying $p(0) = 1$ and $\Re p(z) > 0$.

We say that an analytic function f is *subordinate* to an analytic function g, denoted $f \prec g$, if there exists an analytic self map w in $\mathbb{U}$ with $w(0) = 0$ such that $f(z) = g(w(z)), z \in \mathbb{U}$. Furthermore, if g is univalent in $\mathbb{U}$, then the subordination $f \prec g$ is equivalent to $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$.

Ma and Minda [17] defined two classes of analytic functions:

$$\mathcal{S}^*(\phi) = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \phi(z),\ z \in \mathbb{U} \right\}$$

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$$\mathcal{C}(\phi) = \left\{ f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} \prec \phi(z),\ z \in \mathbb{U} \right\}$$

where $\phi \in \mathcal{P}$ with $\phi'(0) > 0$ and such that ϕ maps $\mathbb{U}$ onto a starlike region with respect to 1 and symmetric with respect to the real axis.

Many subclasses of starlike and convex functions are contained in $\mathcal{S}^*(\phi)$ and $\mathcal{C}(\phi)$. For example, for the function ϕ given by $\phi_\alpha(z) = (1 + (1 - 2\alpha)z)/(1 - z), 0 \leq \alpha < 1$, the class $\mathcal{S}^*(\alpha) = \mathcal{S}^*(\phi_\alpha)$ is the familiar class of starlike functions of order α and the class $\mathcal{C}(\alpha) = \mathcal{C}(\phi_\alpha)$ is the class of convex functions of order α. The classes $\mathcal{S}^* = \mathcal{S}^*(0)$ and $\mathcal{C} = \mathcal{C}(0)$ are the well-known classes of starlike and convex functions, respectively.

Following the definitions of Ma and Minda starlike and convex functions, various classes of analytic functions defined by subordination were investigated (e.g. see [2], [25], [30]).

For a function $f \in \mathcal{A}$ given by (1.1) and $q \in \mathbb{N} = \{1, 2, \ldots\}$, the *qth Hankel determinant* is defined by

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \ldots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \ldots & a_{n+q} \\ \vdots & \vdots & \ldots & \vdots \\ a_{n+q-1} & a_{n+q} & \ldots & a_{n+2q-2} \end{vmatrix} \quad (a_1 = 1).$$

Note that the well-known Fekete-Szegö functional $a_3 - \mu a_2^2$ (see [8]), where μ is a real or complex number, is a generalized form of the Hankel determinant $H_2(1) = a_3 - a_2^2$. Upper bounds for the coefficient functional $|a_3 - \mu a_2^2|$ for various subclasses of univalent functions have been obtained by many authors. See, for example, the recent results in [1], [4], [24], [29]. The second Hankel determinant $H_2(2)$ is given by $H_2(2) = a_2 a_4 - a_3^2$.

Pommerenke [22], [23] and later Hayman [10] investigated the Hankel determinant of areally mean p-valent or univalent functions. The same problem was also considered in [20]. Recently, the bounds for the second or third Hankel determinants for different subclasses of univalent or multivalent functions have been investigated by many authors (e.g. see [6], [26], [28], [32], [33]). Hankel determinants for various classes of bi-univalent functions have been also considered (e.g. see [5], [21]).

The q-calculus operator theory is used in many areas of applied sciences such as fractional calculus, optimal control, quantum mechanics. The q-difference operator and the Jackson q-integral were first developed by Jackson [11], [12]. Recently, certain classes of analytic functions defined by q-derivative operators have been also investigated in [9], [13], [18], [27] etc.

For $q \in (0, 1)$ and for $n \in \mathbb{N}$, the *q-analogue* of n, or *q-integer number* n, is defined by

$$[n]_q = \frac{1-q^n}{1-q} = 1 + q + q^2 + \ldots + q^{n-1}. \tag{1.2}$$

It is obvious that $\lim\limits_{q\to 1^-}[n]_q = n$.

Let $q \in (0,1)$ and $f \in \mathcal{A}$. The *q-derivative* or *q-difference operator* of f is defined by (e.g. see [3], [12])

$$\mathcal{D}_q f(z) = \begin{cases} \dfrac{f(qz)-f(z)}{(q-1)z} & , z \neq 0 \\ f'(0) & , z = 0. \end{cases} \tag{1.3}$$

Note that $\lim\limits_{q\to 1^-} \mathcal{D}_q f(z) = f'(z)$.

If $f(z) = z^n$ then

$$\mathcal{D}_q f(z) = \mathcal{D}_q(z^n) = \frac{1-q^n}{1-q} z^{n-1} = [n]_q z^{n-1} \tag{1.4}$$

and $\lim\limits_{q\to 1^-} \mathcal{D}_q(z^n) = \lim\limits_{q\to 1^-}[n]_q z^{n-1} = nz^{n-1}$.

Let $f \in \mathcal{A}$ be given by (1.1). In view of (1.3) and (1.4), we have

$$\mathcal{D}_q f(z) = 1 + \sum_{n=2}^{\infty}[n]_q a_n z^{n-1}. \tag{1.5}$$

A well-known result due to Marx-Strohhäcker [19], [31] states that if $f \in \mathcal{C}$ then $\Re\sqrt{f'(z)} > 1/2$. Motivated by this implication and using the q-difference operator, we define the following class of analytic functions via subordination.

Definition 1.1. *Let $\phi : \mathbb{U} \to \mathbb{C}$ be analytic and let $q \in (0,1)$. A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{S}Q_q(\phi)$ if it satisfies the subordination*

$$\sqrt{\mathcal{D}_q f(z)} \prec \phi(z), \quad z \in \mathbb{U}. \tag{1.6}$$

Note that for $\phi(z) = 1/(1-z)$ and $q \to 1^-$ the subordination (1.6) becomes

$$\Re\sqrt{f'(z)} > \frac{1}{2}, \quad z \in \mathbb{U}. \tag{1.7}$$

For the particular case when ϕ is given by $\phi_\alpha(z) = (1+(1-2\alpha)z)/(1-z), 0 \leq \alpha < 1$, the class $\mathcal{S}Q_q(\alpha) = \mathcal{S}Q_q(\phi_\alpha)$ consists of functions $f \in \mathcal{A}$ which satisfy the inequality

$$\sqrt{\mathcal{D}_q f(z)} > \alpha, \quad z \in \mathbb{U}. \tag{1.8}$$

Recently, Lee et al. [14] obtained bounds for the second Hankel determinant $H_2(2)$ of functions belonging to the classes $\mathcal{S}^*(\phi)$, $\mathcal{C}(\phi)$ and other related classes defined by subordination.

In this paper, making use of the same technique as in [14], we find upper bounds for the second Hankel determinant $H_2(2)$ for the function class $\mathcal{S}Q_q(\phi)$.

2 Second Hankel determinant for the class $\mathcal{S}Q_q(\phi)$

Unless otherwise mentioned, we assume throughout this section that the function ϕ is given by the series

$$\phi(z) = 1 + A_1 z + A_2 z^2 + A_3 z^3 + \dots, \quad A_1 > 0. \tag{2.1}$$

The following two lemmas for the class $\mathcal{P}$ will be used to prove our results.

Lemma 2.1. *([7]) Let the function $p \in \mathcal{P}$ be given by*

$$p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \dots, \quad z \in \mathbb{U}. \tag{2.2}$$

Then the sharp estimate

$$|c_n| \leq 2, \quad n \in \mathbb{N} \tag{2.3}$$

holds.

Lemma 2.2. *([15], [16]) If the function $p \in \mathcal{P}$ is given by (2.2), then*

$$2c_2 = c_1^2 + x(4 - c_1^2) \tag{2.4}$$

$$4c_3 = c_1^3 + 2(4 - c_1^2)c_1 x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)z \tag{2.5}$$

for some x, z with $|x| \leq 1$ and $|z| \leq 1$.

In the next theorem, we shall determine the upper bound for the Hankel determinant $H_2(2)$ for the class $\mathcal{S}Q_q(\phi)$.

Theorem 2.1. *Let $q \in (0,1)$ and let $\delta = (1 + 1/q)^2(1 + q^2)$. Suppose that $f \in \mathcal{S}Q_q(\phi)$ is given by (1.1).*

1. *If A_1, A_2 and A_3 satisfy the inequalities*

$$|A_1^2 + 2A_2| \leq (\delta - 1)A_1 \text{ and } |A_1^2 A_2 + (\delta + 1)A_1 A_3 - \frac{\delta}{4}A_1^4 - \delta A_2^2| \leq \delta A_1^2$$

then

$$|a_2 a_4 - a_3^2| \leq \frac{4q^2}{\delta + 1} A_1^2.$$

2. If A_1, A_2 and A_3 satisfy the inequalities

$$|A_1^2 + 2A_2| \geq (\delta - 1)A_1 \quad \textit{and}$$
$$2|A_1^2A_2 + (\delta+1)A_1A_3 - \frac{\delta}{4}A_1^4 - \delta A_2^2| \geq A_1|A_1^2 + 2A_2| + (\delta+1)A_1^2$$

or the inequalities

$$|A_1^2 + 2A_2| \leq (\delta - 1)A_1 \ \textit{and} \ |A_1^2A_2 + (\delta+1)A_1A_3 - \frac{\delta}{4}A_1^4 - \delta A_2^2| \geq \delta A_1^2$$

then

$$|a_2a_4 - a_3^2| \leq \frac{4}{q^2\delta(\delta+1)}|A_1^2A_2 + (\delta+1)A_1A_3 - \frac{\delta}{4}A_1^4 - \delta A_2^2|.$$

3. If A_1, A_2 and A_3 satisfy the inequalities

$$|A_1^2 + 2A_2| > (\delta - 1)A_1 \ \textit{and}$$
$$2|A_1^2A_2 + (\delta+1)A_1A_3 - \frac{\delta}{4}A_1^4 - \delta A_2^2| \leq A_1|A_1^2 + 2A_2| + (\delta+1)A_1^2$$

then

$$|a_2a_4 - a_3^2| \leq \frac{A_1^2}{q\delta(\delta+1)} \times \left(\frac{\begin{array}{c} 4\delta|A_1^2A_2 + (\delta+1)A_1A_3 - \frac{\delta}{4}A_1^4 - \delta A_2^2| \\ -2(\delta+1)A_1|A_1^2 + 2A_2| - |A_1^2 + 2A_2|^2 - (\delta+1)^2A_1^2 \end{array}}{|A_1^2A_2 + (\delta+1)A_1A_3 - \frac{\delta}{4}A_1^4 - \delta A_2^2| - A_1|A_1^2 + 2A_2| - A_1^2} \right).$$

Proof. Assume that $f \in \mathcal{S}Q(\phi)$. Then there exists an analytic self map $w(z)$ of $\mathbb{U}$ with $w(0) = 0$ such that

$$\sqrt{\mathcal{D}_q f(z)} = \phi(w(z)), \quad z \in \mathbb{U}. \tag{2.6}$$

Define the function $p \in \mathcal{P}$ by

$$p(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + c_1z + c_2z^2 + c_3z^3 + \dots, \quad z \in \mathbb{U}$$

or equivalently

$$w(z) = \frac{p(z)-1}{p(z)+1} = \frac{1}{2}\left[c_1 z + \left(c_2 - \frac{c_1^2}{2}\right) z^2 + \left(c_3 - c_1 c_2 + \frac{c_1^3}{4}\right) z^3 + \dots\right]. \tag{2.7}$$

Making use of (2.7) together with (2.1) we have

$$\phi\left(\frac{p(z)-1}{p(z)+1}\right) = 1 + \frac{1}{2}A_1 c_1 z + \frac{1}{2}\left[A_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{A_2 c_1^2}{2}\right] z^2$$
$$+\frac{1}{2}\left[A_1\left(c_3 - c_1 c_2 + \frac{c_1^3}{4}\right) + A_2 c_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{A_3 c_1^3}{4}\right] z^3 + \dots. \tag{2.8}$$

In view of (1.5), we have

$$\sqrt{\mathcal{D}_q f(z)} = 1 + \frac{1}{2}[2]_q a_2 z + \frac{1}{2}\left([3]_q a_3 - \frac{[2]_q^2}{4} a_2^2\right) z^2$$
$$+\frac{1}{2}\left([4]_q a_4 - \frac{[2]_q[3]_q}{2} a_2 a_3 + \frac{[2]_q^3}{8} a_2^3\right) z^3 + \dots$$

or, by using (1.2)

$$\sqrt{\mathcal{D}_q f(z)} = 1 + \frac{1+q}{2} a_2 z + \frac{1}{2}\left[(1+q+q^2)a_3 - \frac{(1+q)^2}{4} a_2^2\right] z^2$$
$$+\frac{1}{2}\left[(1+q)(1+q^2)a_4 - \frac{(1+q)(1+q+q^2)}{2} a_2 a_3 + \frac{(1+q)^3}{8} a_2^3\right] z^3 + \dots$$

Equating the coefficients of z, z^2 and z^3, from (2.6) and (2.8), we obtain

$$a_2 = \frac{A_1 c_1}{1+q} \tag{2.9}$$

$$a_3 = \frac{1}{1+q+q^2}\left(A_1 c_2 + \frac{A_1^2 c_1^2}{4} - \frac{A_1 c_1^2}{2} + \frac{A_2 c_1^2}{2}\right) \tag{2.10}$$

$$a_4 = \frac{1}{(1+q)(1+q^2)}\left(A_1 c_3 - A_1 c_1 c_2 + A_2 c_1 c_2 + \frac{A_1^2 c_1 c_2}{2}\right.$$
$$\left. + \frac{A_1 c_1^3}{4} - \frac{A_1^2 c_1^3}{4} - \frac{A_2 c_1^3}{2} + \frac{A_1 A_2 c_1^3}{4} + \frac{A_3 c_1^3}{4}\right). \tag{2.11}$$

A lengthy computation leads to

$$a_2a_4 - a_3^2 = \frac{A_1}{4q^2\delta(\delta+1)}$$
$$\times[(-A_1^2 + A_1 - 2A_2 + A_1A_2 + (\delta+1)A_3 - \delta/4A_1^3 - \delta A_2^2/A_1)c_1^4$$
$$+(2A_1^2 - 4A_1 + 4A_2)c_1^2c_2 + 4(\delta+1)A_1c_1c_3 - 4\delta A_1c_2^2]$$

where $\delta = (1+1/q)^2(1+q^2)$.

In order to simplify computation, let

$$\Lambda = \frac{A_1}{4q^2\delta(\delta+1)}$$

$$\lambda_1 = -A_1^2 + A_1 - 2A_2 + A_1A_2 + (\delta+1)A_3 - \frac{\delta}{4}A_1^3 - \delta\frac{A_2^2}{A_1} \tag{2.12}$$

$$\lambda_2 = 2A_1^2 - 4A_1 + 4A_2 \quad \lambda_3 = 4(\delta+1)A_1 \quad \lambda_4 = -4\delta A_1. \tag{2.13}$$

It follows that

$$|a_2a_4 - a_3^2| = \Lambda|\lambda_1c_1^4 + \lambda_2c_1^2c_2 + \lambda_3c_1c_3 + \lambda_4c_2^2|. \tag{2.14}$$

Since $p \in \mathcal{P}$, the function $p(e^{i\theta}z)$ $(\theta \in \mathbb{R})$ is also in the class $\mathcal{P}$ and therefore we can assume without loss of generality that $c_1 = c \in [0,2]$.

Substituting in (2.14) the values of c_2 and c_3 from (2.4) and (2.5) respectively , we get

$$|a_2a_4 - a_3^2| = \Lambda\left|c^4\left(\lambda_1 + \frac{1}{2}\lambda_2 + \frac{1}{4}\lambda_3 + \frac{1}{4}\lambda_4\right) + \frac{1}{2}c^2x(4-c^2)(\lambda_2+\lambda_3+\lambda_4)\right.$$
$$\left.+\frac{1}{4}x^2(4-c^2)[-\lambda_3c^2 + \lambda_4(4-c^2)] + \frac{1}{2}\lambda_3c(4-c^2)(1-|x|^2)z\right|.$$

Furthermore, substituting the values of $\lambda_1, \lambda_2, \lambda_3$ and λ_4 from (2.12) and (2.13), in view of triangle inequality, we obtain

$$|a_2a_4 - a_3^2| \le \Lambda\left[|A_1A_2 + (\delta+1)A_3 - \frac{\delta}{4}A_1^3 - \delta\frac{A_2^2}{A_1}|c^4 + c^2\mu(4-c^2)|A_1^2 + 2A_2|\right.$$
$$\left.+\mu^2(4-c^2)A_1(c-2)(c-2\delta) + 2(\delta+1)A_1c(4-c^2)\right] := F(c,\mu), \tag{2.15}$$

where $\mu = |x| \in [0,1]$.

Now, we maximize the function $F(c,\mu)$, given by (2.15), on the closed rectangle $[0,2]\times[0,1]$. Since

$$\frac{\partial F(c,\mu)}{\partial\mu} = \Lambda[c^2(4-c^2)|A_1^2 + 2A_2| + 2\mu(4-c^2)A_1(c-2)(c-2\delta)] > 0$$

it follows that $F(c,\mu)$ is an increasing function of μ. Hence

$$\max_{0\leq\mu\leq 1} F(c,\mu) = F(c,1) := G(c), \tag{2.16}$$

where

$$G(c) = \Lambda\left[\left(|A_1A_2 + (\delta+1)A_3 - \frac{\delta}{4}A_1^3 - \delta\frac{A_2^2}{A_1}| - |A_1^2 + 2A_2| - A_1\right)c^4 \right.$$
$$\left. +4c^2\left(|A_1^2+2A_2| + (1-\delta)A_1\right) + 16\delta A_1\right].$$

Define

$$A = |A_1A_2 + (\delta+1)A_3 - \delta/4A_1^3 - \delta A_2^2/A_1| - |A_1^2+2A_2| - A_1$$
$$B = 4|A_1^2+2A_2| + 4(1-\delta)A_1$$
$$C = 16\delta A_1.$$

and let $c^2 = t \in [0,4]$. Then, in view of (2.15) and (2.16), we have

$$|a_2a_4 - a_3^2| \leq \max_{0\leq c\leq 2} G(c) = \Lambda \max_{0\leq t\leq 4}(At^2 + Bt + C).$$

Since

$$\max_{0\leq t\leq 4}(At^2+Bt+C) = \begin{cases} C & , B\leq 0,\ \ A\leq -B/4 \\ 16A+4B+C & , B\geq 0,\ \ A\geq -B/8 \ \text{ or } \ B\leq 0,\ A\geq -B/4 \\ \dfrac{4AC-B^2}{4A} & , B>0,\ \ A\leq -B/8. \end{cases}$$

a routine calculation yields the desired result. □

If in Theorem (2.1) we let $q\to 1^-$ and set $\phi(z) = 1/(1-z)$, we obtain the upper bound for the second Hankel determinant $H_2(2)$ for the class of functions which satisfy inequality (1.7).

Corollary 2.1. *Suppose that $f\in\mathcal{A}$, given by (1.1), satisfies inequality (1.7). Then*

$$|a_2a_4 - a_3^2| \leq \frac{4}{9}.$$

Setting $\phi(z) = (1+(1-2\alpha)z)/(1-z)$ in Theorem 2.1, we get the estimate for $H_2(2)$ for the class $\mathcal{S}Q_q(\alpha)$ $(0\leq\alpha<1)$.

Corollary 2.2. *Let $q\in(0,1)$ and let $\delta = (1+1/q)^2(1+q^2)$. If $f\in\mathcal{S}Q_q(\alpha)$ is given by (1.1), then*

$$|a_2a_4 - a_3^2| \leq \frac{16q^2}{\delta+1}(1-\alpha)^2.$$

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