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Characterizations of Generalized Exponential Trichotomies for Linear Discrete-time Systems

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Abstract

The concept of generalized exponential trichotomy for linear time-varying systems is investigated in relationship with the classical notion of uniform exponential trichotomy. Some key properties of generalized exponential trichotomy are explored through supplementary projections. These results are also extended to the case of projection sequences, while certain applications for adjoint systems are suggested.

1 Introduction

The trichotomy concept involves splitting the state space at any moment into three subspaces (a stable, unstable and a central one) and represents the most complex description of the asymptotic behavior of linear time-varying (LTV) systems. The first notable study on the uniform exponential trichotomy (UET) for discrete-time LTV systems was done by S. Elaydi and K. Janglajew in [5]. Numerous extensions have followed. For example in [7], [12], [13] (and the references therein), various nonuniform exponential concepts are presented, where some exponential loss of hyperbolicity along the trajectories is allowed. Some nonuniform concepts of polynomial type are presented in [15].

Motivated by results obtained by A. Castaneda and G. Robledo in [2] for generalized exponential dichotomy (GED) for difference equations, this notion

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was extended to generalized exponential trichotomy (GET) in [10], where we have established the relation between this notion and the classical (uniform) exponential trichotomy from [5]. It should be noted that our notion of GET is not a kind of nonuniform hyperbolicity. In fact our notion represents a kind of uniform hyperbolicity. In this context, we can point out some important results obtained in this direction in [2], [6].

In this paper we give a simple and concrete example illustrating the relationship between the concepts of UET and GET. Also, motivated by the lead given in [5], we present some theorems of characterization for discrete-time LTV systems in terms of GET. More precisely, we will show in Section 2 how the mutual orthogonality property matrix projections can be replaced for the case of the GET property. Also, these characterizations are extended in Section 3 for the case of invariant projection sequences. Subsequently, in the last section of the paper a necessary and sufficient condition for GET property for the dual system is developed. This paper is a companion of our earlier work [11] where some preliminary results have been presented.

Notations. The notations used in this paper are generally standard. For the readers' convenience we recall some of them: $\mathbb{Z}$ denotes the set of real integers, $\mathbb{Z}_+$ is the set of all $n \in \mathbb{Z}$, $n \geq 0$, $\mathbb{Z}_-$ is the set of all $n \in \mathbb{Z}$, $n \leq 0$, while $\mathbb{R}$ denotes the set of real numbers and $\|.\|$ represents a matrix norm.

2 Generalized exponential trichotomy

Let us consider the LTV system

$$x_{n+1} = A_n x_n, \quad n \in \mathbb{Z}, \tag{$\mathfrak{A}$}$$

where $(A_n)_{n\in\mathbb{Z}}$ is a sequence of $d \times d$ invertible matrices. By W_n we denote the fundamental matrix of ($\mathfrak{A}$), i.e., $W_{n+1} = A_n W_n$ and $W_0 = I$, where I represents the identity matrix. Further on, we shall consider a strictly positive sequence $(a_n)_{n\in\mathbb{Z}}$ satisfying the properties

$$\sum_{j=p}^{q} a_j \to +\infty \text{ as } q \to +\infty \text{ for fixed } p \in \mathbb{Z}, \tag{1}$$

$$\sum_{j=p}^{q} a_j \to +\infty \text{ as } p \to -\infty \text{ for fixed } q \in \mathbb{Z}. \tag{2}$$

Definition 2.1. ([10]) The LTV system ($\mathfrak{A}$) admits a *generalized exponential trichotomy* (GET) on $\mathbb{Z}$ if there exist projections (P^i), $i \in \{1, 2, 3\}$, satisfying

$$P^1 + P^2 + P^3 = I \text{ and } P^i P^j = P^j P^i = 0, \tag{3}$$

for all $i, j \in \{1, 2, 3\}$ with $i \neq j$, together with the constants $K \geq 1$, $p \in (0, 1)$, and a strictly positive sequence $(a_n)_{n\in\mathbb{Z}}$ satisfying (1) and (2) such that

$$\|W_n P^1 W_m^{-1}\| \leq K p^{\sum_{j=m}^{n} a_j}, \ \ n \geq m, \tag{4}$$

$$\|W_n P^2 W_m^{-1}\| \leq K p^{\sum_{j=n}^{m} a_j}, \ \ m \geq n, \tag{5}$$

$$\|W_n P^3 W_m^{-1}\| \leq K p^{\sum_{j=n}^{m} a_j}, \ \ 0 \geq m \geq n, \tag{6}$$

$$\|W_n P^3 W_m^{-1}\| \leq K p^{\sum_{j=m}^{n} a_j}, \ \ n \geq m \geq 0. \tag{7}$$

In order to simplify the notations further we will denote by $t_{mn} = \sum\limits_{k=m}^{n} a_k$, for all $m, n \in \mathbb{Z}$, with $n \geq m$. Also, we point out that the projections P^i, $i \in \{1, 2, 3\}$, satisfying relation (3) are called *supplementary.*

One may notice that the LTV system $(\mathfrak{A})$ admits a *generalized exponential dichotomy* (GED) if it admits a GET with $P^3 = 0$, for all $n \in \mathbb{Z}$. The notion of GED has been introduced by A. Castaneda and G. Robledo in [2]. For the particular case when $a_j = \alpha > 0$, for any $j \in \mathbb{Z}$, we obtain the notion of $\alpha-$exponential dichotomy from [8]. For a deeper discussion about discrete dichotomies we refer the reader to [1], [2], [4], [9] and the references therein.

There are examples of GETs for which a_j cannot be replaced by a constant α, as shown by the example below. For more details one may consult [10]. Consider a sequence $(b_n)_{n\in\mathbb{Z}}$ satisfying the following properties

- If $n \in \mathbb{N}$ then $0 < b_0 \leq b_1 \leq \ldots \leq b_n \leq b_{n+1} \leq \ldots < 1$ and $(b_n)_{n\in\mathbb{Z}}$ monotonically increasing to 1 as $n \to \infty$.
- If $n \in \mathbb{Z} \setminus \mathbb{N}$ then $b_n = b_{-n}$.

We consider $(c_n)_{n\in\mathbb{Z}}$ defined by $c_n = \begin{cases} b_n & n \in \mathbb{Z} \\ 1/b_n & n \in \mathbb{Z} \setminus \mathbb{N}. \end{cases}$

It is shown in particular that on $\mathbb{R}^3$ endowed with the Euclidean norm that the LTV system generated by

$$A_n = \begin{pmatrix} b_n & 0 & 0 \\ 0 & 1/b_n & 0 \\ 0 & 0 & c_n \end{pmatrix},$$

has a GET and not an $\alpha-$exponential one.

Proposition 2.1. *The LTV system* $(\mathfrak{A})$ *has a GET if and only if there exist two projections* T^1 *and* T^2 *satisfying* $T^1T^2 = T^2T^1$ *and* $T^1 + T^2 - T^1T^2 = I = T^1 + T^2 - T^2T^1$*, some constants* $D \geq 1$, $p \in (0, 1)$ *and a strictly positive sequence* $(a_n)_{n\in\mathbb{Z}}$ *satisfying (1) and (2), such that*

$$\|W_n(I - T^2)W_m^{-1}\| \leq Dp^{\,t_{mn}}, \;\; n \geq m, \tag{8}$$

$$\|W_n(I - T^1)W_m^{-1}\| \leq Dp^{\,t_{nm}}, \;\; m \geq n, \tag{9}$$

$$\|W_nT^2W_m^{-1}\| \leq Dp^{\,t_{nm}}, \;\; 0 \geq m \geq n, \tag{10}$$

$$\|W_nT^1W_m^{-1}\| \leq Dp^{\,t_{mn}}, \;\; n \geq m \geq 0, \tag{11}$$

Proof. Necessity. We consider $T^1 = P^1 + P^3$ and $T^2 = P^2 + P^3$. It can be easily seen that T^1 and T^2 are projections and $T^1T^2 = T^2T^1 = P^3$. From $P^1 + P^2 + P^3 = I$ we have that

$$T^1 + T^2 - T^1T^2 = I = T^1 + T^2 - T^2T^1.$$

Also, we have that

$$I - T^2 = T^1 - T^1T^2 = P^1 \text{ and } I - T^1 = T^2 - T^2T^1 = P^2.$$

By direct calculation from equation (4) we obtain (8) and by (5) we get (9). Finally, by equations (5) and (6) one obtains (10), and similarly by (4) and (7) it follows that (11) holds.

Sufficiency. Let $P^1 = I - T^2$, $P^2 = I - T^1$ and $P^3 = T^2T^1 = T^1T^2$. Then P^i, $i \in \{1, 2, 3\}$ are supplementary projections.

First note that (8) implies (4) and (9) implies (5). Now, from (9) for $m = n$ we obtain that

$$\begin{aligned}\|W_nT^1W_n^{-1}\| &= \|W_n(-T^1)W_n^{-1}\| \leq \|W_n(I - T^1)W_n^{-1}\| + \|I\| \\ &\leq Dp^{a_n} + 1 \leq 2D.\end{aligned}$$

Analogously, from (8) for $m = n$ we get

$$\begin{aligned}\|W_nT^2W_n^{-1}\| &= \|W_n(-T^2)W_n^{-1}\| \leq \|W_n(I - T^2)W_n^{-1}\| + \|I\| \\ &\leq Dp^{a_n} + 1 \leq 2D.\end{aligned}$$

Further, for $0 \geq m \geq n$, by (10) and the two previous inequalities we obtain

$$\begin{aligned}\|W_nP^3W_m^{-1}\| &= \|W_nT^1T^2W_m^{-1}\| = \|W_nT^1W_n^{-1}W_nT^2W_m^{-1}\| \\ &\leq 2D^2p^{t_{nm}}.\end{aligned}$$

Finally, for $n \geq m \geq 0$ taking into account (11) we have that

$$\|W_nP^3W_m^{-1}\| \leq 2D^2p^{t_{mn}}.$$

which ends the proof. □

Proposition 2.2. *The LTV system* ($\mathfrak{A}$) *has a GET if and only if there exist two projections* G^1 *and* G^2 *satisfying* $G^1G^2 = G^2G^1 = G^1$*, some constants* $D \geq 1$, $p \in (0,1)$ *and a strictly positive sequence* $(a_n)_{n\in\mathbb{Z}}$ *satisfying (1) and (2), such that*

$$\|W_n G^1 W_m^{-1}\| \leq D p^{t_{mn}}, \;\; n \geq m, \tag{12}$$

$$\|W_n (I - G^2) W_m^{-1}\| \leq D p^{t_{nm}}, \;\; m \geq n, \tag{13}$$

$$\|W_n (I - G^1) W_m^{-1}\| \leq D p^{t_{nm}}, \;\; 0 \geq m \geq n, \tag{14}$$

$$\|W_n G^2 W_m^{-1}\| \leq D p^{t_{mn}}, \;\; n \geq m \geq 0. \tag{15}$$

Proof. Let T^1 and T^2 be the projections considered in Proposition 2.1. Define the projections G^1 and G^2 by $G^1 = I - T^2$ and $G^2 = T^1$. Clearly, these projections satisfy $G^1G^2 = G^2G^1 = G^1$.

Conversely, suppose that G^1 and G^2 are two projections satisfying the condition $G^1G^2 = G^2G^1 = G^1$. By letting $T^1 = G^2$ and $T^2 = I - G^1$, it follows that $T^1T^2 = T^2T^1 = G^2 - G^1$ is a projection because

$$(T^1T^2)^2 = (G^2 - G^1)^2 = G^2 - G^1 = T^1T^2.$$

Moreover,

$$T^1 + T^2 - T^1T^2 = T^1 + T^2 - T^2T^1 = I.$$

Therefore, the equivalence between the equations (8)-(11) and (12)-(15) can be directly obtained. □

Proposition 2.3. *For every* $m \in \mathbb{Z}^*$ *we have that*

$$W_m = \begin{cases} A_{m-1}A_{m-2}\cdots A_1A_0, & \text{if } m > 0 \\ (A_{-1}A_{-2}\cdots A_m)^{-1}, & \text{if } m < 0. \end{cases}$$

Proof. If $m > 0$, then

$$\begin{aligned} W_m &= A_{m-1}W_{m-1} = A_{m-1}A_{m-2}W_{m-1} = \ldots = A_{m-1}A_{m-2}\cdots A_1A_0W_0 \\ &= A_{m-1}A_{m-2}\cdots A_1A_0, \end{aligned}$$

while for $m < 0$ we have that

$$\begin{aligned} W_m &= A_m^{-1}W_{m+1} = A_m^{-1}A_{m+1}^{-1}W_{m+2} = \ldots = A_m^{-1}A_{m+1}^{-1}\cdots A_{-1}^{-1}W_0 \\ &= A_m^{-1}A_{m+1}^{-1}\cdots A_{-1}^{-1} = (A_{-1}A_{-2}\cdots A_m)^{-1}. \end{aligned}$$

□

3 Generalized exponential trichotomy with projection sequences

The principal aim of this section is to give a characterization of GET in terms of two invariant projection sequences. We begin with some definitions.

A sequence $(P_n)_{n\in\mathbb{Z}}$ is called a *projection sequence* if $(P_n)^2 = P_n$, for $n \in \mathbb{Z}$. A projection sequence $(P_n)_{n\in\mathbb{Z}}$ with the property $P_{n+1}A_n = A_nP_n$, for all $n \in \mathbb{Z}$ is called *invariant* for the LTV system $(\mathfrak{A})$. Three projection sequences $(P_n^i)_{n\in\mathbb{Z}}$, $i \in \{1, 2, 3\}$, are called *supplementary* if

$$P_n^1 + P_n^2 + P_n^3 = I, \text{ for } n \in \mathbb{Z}, \tag{16}$$

$$P_n^iP_n^j = 0, \text{ for } n \in \mathbb{Z} \text{ and } i, j \in \{1, 2, 3\},\ i \neq j. \tag{17}$$

Proposition 3.1. *Let P be a projection. If $S_m = W_mPW_m^{-1}$, for every $m \in \mathbb{Z}$, then $(S_m)_{m\in\mathbb{Z}}$ is a projection sequence such that*

$$S_{m+1}A_m = A_mS_m,$$

for all $m \in \mathbb{Z}$.

Proof. Let $m \in \mathbb{Z}$. It is easily seen that $S_m^2 = S_m$ and

$$\begin{aligned} S_{m+1}A_m &= W_{m+1}PW_{m+1}^{-1}A_m = (A_mW_m)P(W_m^{-1}A_m^{-1})A_m \\ &= A_m(W_mPW_m^{-1})(A_m^{-1}A_m) = A_mS_m. \end{aligned}$$

□

A property of the invariant projection sequences was reported in [14].

Proposition 3.2. *If $(P_n)_{n\in\mathbb{Z}}$ is an invariant projection sequence for the LTV system $(\mathfrak{A})$, then $P_m = W_mP_0W_m^{-1}$ for every $m \in \mathbb{Z}$.*

Proof. If $m > 0$, then

$$\begin{aligned} P_m &= A_{m-1}P_{m-1}A_{m-1}^{-1} = A_{m-1}A_{m-2}P_{m-2}A_{m-2}^{-1}A_{m-1}^{-1} \\ &= \ldots = A_{m-1}A_{m-2}\cdots A_1A_0P_0A_0^{-1}A_1^{-1}\cdots A_{m-2}^{-1}A_{m-1}^{-1} \\ &= W_mP_0W_m^{-1}, \end{aligned}$$

while for $m < 0$ we have

$$\begin{aligned} P_m &= A_m^{-1}P_{m+1}A_m = A_m^{-1}A_{m+1}^{-1}P_{m+2}A_{m+1}A_m \\ &= \ldots = A_m^{-1}A_{m+1}^{-1}A_{m+2}^{-1}\cdots A_{-1}^{-1}P_0A_{-1}A_{-2}\cdots A_{m+2}A_{m+1}A_m \\ &= W_mP_0W_m^{-1}. \end{aligned}$$

□

The following theorems are the main results of the paper.

Theorem 3.1. *The LTV system* ($\mathfrak{A}$) *has a GET if and only if there exist invariant and supplementary projection sequences* $(S^k_m)_{m\in\mathbb{Z}}$*,* $k \in \{1,2,3\}$*, some constants* $D \geq 1$*,* $p \in (0,1)$ *and a sequence* $(a_n)_{n\in\mathbb{Z}}$ *satisfying (1) and (2), such that*

$$\|W_n W_m^{-1} S_m^1\| \leq Dp^{\,t_{mn}}, \;\; n \geq m, \tag{18}$$

$$\|W_n W_m^{-1} S_m^2\| \leq Dp^{\,t_{nm}}, \;\; m \geq n, \tag{19}$$

$$\|W_n W_m^{-1} S_m^3\| \leq Dp^{\,t_{nm}}, \;\; 0 \geq m \geq n, \tag{20}$$

$$\|W_n W_m^{-1} S_m^3\| \leq Dp^{\,t_{mn}}, \;\; n \geq m \geq 0. \tag{21}$$

Proof. Necessity. Let P^k, $k \in \{1,2,3\}$ be projections given in Definition 2.1. For every $m \in \mathbb{Z}$ we consider $S^k_m = W_m P^k W_m^{-1}$, $k \in \{1,2,3\}$. According to Proposition 3.1, $(S^k_m)_{m\in\mathbb{Z}}$ are projection sequences which satisfy the invariant property $S^k_{m+1}A_m = A_m S^k_m$, $m \in \mathbb{Z}$. For all $m \in \mathbb{Z}$ and $i,j \in \{1,2,3\}$ with $i \neq j$, one has

$$S_m^1 + S_m^2 + S_m^3 = W_m(P^1 + P^2 + P^3)W_m^{-1} = I,$$

$$S_m^j S_m^i = W_m P^j W_m^{-1} W_m P^i W_m^{-1} = 0,$$
$$S_m^i S_m^j = W_m P^i W_m^{-1} W_m P^j W_m^{-1} = 0,$$

hence the projections are also supplementary.

Let $m,n \in \mathbb{Z}$. Taking into account that $P^k W_m^{-1} = W_m^{-1} S^k_m$ we obtain that

$$W_n P^k W_m^{-1} = W_n W_m^{-1} S_m^k,$$

for all $k \in \{1,2,3\}$. Finally, (18)-(21) follow immediately by (4)-(7).

Sufficiency. Let $k \in \{1,2,3\}$. Based on Proposition 3.2 we have that $S^k_m = W_m S^k_0 W_m^{-1}$, or equivalently, $W_m^{-1}S^k_m = S^k_0 W_m^{-1}$. This leads to $W_n W_m^{-1} S^k_m = W_n S^k_0 W_m^{-1}$, for all $n,m \in \mathbb{Z}$. Setting $P^k = S^k_0$ we have that projections P^k, $k \in \{1,2,3\}$, verifies the conditions from Definition 2.1. Thus, we obtain the equivalence between the equations (18)-(21) and (4)-(7).

□

Remark. (a) By Definition 2.1, for $m = n = 0$ we have $\|P^k\| \leq Kp^{a_0} \leq K$, for all $k \in \{1,2,3\}$.

(b) By Theorem 3.1, for $m = n$ we have $\|S^k_m\| \leq Dp^{a_m} \leq D$, for all $k \in \{1,2,3\}$.

Theorem 3.2. *The LTV system* $(\mathfrak{A})$ *has a GET if and only if there exist two invariant projection sequences* $(Q_n^i)_{n\in\mathbb{Z}}$, $i \in \{1,2\}$, *some constants* $K \geq 1$, $p \in (0,1)$ *and a sequence* $(a_n)_{n\in\mathbb{Z}}$ *satisfying (1) and (2), such that*

$$Q_n^1 + Q_n^2 - Q_n^1 Q_n^2 = I, \;\; Q_n^1 Q_n^2 = Q_n^2 Q_n^1, \;\; \textit{for all } n \in \mathbb{Z}, \tag{22}$$

$$\|W_n W_m^{-1} Q_m^1\| \leq M p^{t_{mn}}, \;\; n \geq m \geq 0, \tag{23}$$

$$\|W_n W_m^{-1} Q_m^2\| \leq M p^{t_{nm}}, \;\; 0 \geq m \geq n, \tag{24}$$

$$\|W_n W_m^{-1}(I - Q_m^1)\| \leq M p^{t_{nm}}, \;\; m \geq n, \tag{25}$$

$$\|W_n W_m^{-1}(I - Q_m^2)\| \leq M p^{t_{mn}}, \;\; n \geq m. \tag{26}$$

Proof. Necessity. Let $n \in \mathbb{Z}$. We consider $Q_n^1 = S_n^1 + S_n^3$ and $Q_n^2 = S_n^2 + S_n^3$. One can easily see that $Q_n^1 Q_n^2 = Q_n^2 Q_n^1 = S_n^3$ and Q_n^1 and Q_n^2 are invariant projection sequences for the LTV system $(\mathfrak{A})$. Also, using the supplementary property we have that

$$Q_n^1 + Q_n^2 - Q_n^1 Q_n^2 = Q_n^1 + Q_n^2 - Q_n^2 Q_n^1 = I.$$

Further, one can easily observe that $I - Q_n^1 = S_n^2$ and $I - Q_n^2 = S_n^1$, hence $\|Q_n^1\| \leq \|S_n^1\| + \|S_n^3\| \leq 2D$, respectively $\|Q_n^2\| \leq \|S_n^2\| + \|S_n^3\| \leq 2D$.

We have to consider the following cases.

(1) For $n \geq m \geq 0$, using (18) and (21) we have that

$$\|W_n W_m^{-1} Q_m^1\| = \|W_n W_m^{-1}(S_m^1 + S_m^3)\| \leq 2Dp^{t_{mn}}.$$

(2) For $0 \geq m \geq n$ using (19) and (20) we get

$$\|W_n W_m^{-1} Q_m^2\| = \|W_n W_m^{-1}(S_m^2 + S_m^3)\| \leq 2Dp^{t_{nm}}.$$

(3) If $m \geq n$, then using (19) we obtain

$$\|W_n W_m^{-1}(I - Q_m^1)\| = \|W_n W_m^{-1} S_m^2\| \leq Dp^{t_{nm}}.$$

(4) Finally, for $n \geq m$ using (18) we deduce that

$$\|W_n W_m^{-1}(I - Q_m^2)\| = \|W_n W_m^{-1} S_m^1\| \leq Dp^{t_{mn}}.$$

Sufficiency. For each $n \in \mathbb{Z}$ we consider $S_n^1 = I - Q_n^2$, $S_n^2 = I - Q_n^1$ and $S_n^3 = Q_n^1 Q_n^2 = Q_n^2 Q_n^1$. We firstly observe that

$$S_n^1 + S_n^2 + S_n^3 = I - Q_n^2 + I - Q_n^1 + Q_n^1 Q_n^2 = I - Q_n^1 - Q_n^2 + Q_n^1 Q_n^2 + I = I,$$

and $S_n^i S_n^j = S_n^j S_n^i = 0$, for all $i, j \in \{1,2,3\}$ with $i \neq j$. Next, we have that

$$S_{n+1}^1 A_n = (I - Q_{n+1}^2) A_n = A_n - Q_{n+1}^2 A_n = A_n (I - Q_n^2) = A_n S_n^1.$$

Similarly, we have that $S_{n+1}^i A_n = A_n S_n^i$, with $i \in \{2,3\}$. Therefore, S_n^i, $i \in \{1,2,3\}$ are invariant projection sequences for the system $(\mathfrak{A})$. Also observe that for $m = n$ condition (25) implies that $\|I - Q_m^1\| \leq M p^{a_m} \leq M$, i.e., $\|Q_m^1\| \leq 1+M \leq 2M$. Similarly, from (26) we have that $\|Q_m^2\| \leq 1+M \leq 2M$.

In order to establish equivalence, first note that (26) and (25) clearly imply (18) and (19). On the other hand, setting $0 \geq m \geq n$, (24) becomes

$$\|W_n W_m^{-1} S_m^3\| \leq \|W_n W_m^{-1} Q_m^2\| \cdot \|Q_m^1\| \leq 2M^2 p^{t_{nm}}.$$

Similarly, setting $n \geq m \geq 0$, (23) becomes

$$\|W_n W_m^{-1} S_m^3\| \leq \|W_n W_m^{-1} Q_m^1\| \cdot \|Q_m^2\| \leq 2M^2 p^{t_{mn}}.$$

Finally, applying Theorem 3.1 we obtain that system $(\mathfrak{A})$ admits a GET, which ends the proof. □

Theorem 3.3. *The LTV system $(\mathfrak{A})$ has a GET if and only if there exist two invariant projection sequences $(R_n^i)_{n\in\mathbb{Z}}$, $i \in \{1,2\}$, some constants $K \geq 1$, $p \in (0,1)$ and a sequence $(a_n)_{n\in\mathbb{Z}}$ satisfying (1) and (2), such that*

$$R_n^1 R_n^2 = R_n^2 R_n^1 = R_n^2, \quad \text{for all } n \in \mathbb{Z}, \tag{27}$$

$$\|W_n W_m^{-1} R_m^1\| \leq K p^{t_{mn}}, \quad n \geq m \geq 0, \tag{28}$$

$$\|W_n W_m^{-1} R_m^2\| \leq K p^{t_{mn}}, \quad 0 \geq m \geq n, \tag{29}$$

$$\|W_n W_m^{-1} (I - R_m^1)\| \leq K p^{t_{nm}}, \quad m \geq n, \tag{30}$$

$$\|W_n W_m^{-1} (I - R_m^2)\| \leq K p^{t_{mn}}, \quad n \geq m. \tag{31}$$

Proof. Necessity. Let $n \in \mathbb{Z}$. We set $R_n^1 = Q_n^1$ and $R_n^2 = I - Q_n^2$. If follows from (22), that

$$R_n^1 R_n^2 = Q_n^1 - Q_n^1 Q_n^2 = I - Q_n^2 = R_n^2,$$

and

$$R_n^2 R_n^1 = Q_n^1 - Q_n^2 Q_n^1 = I - Q_n^2 = R_n^2.$$

It can easily be checked the invariant property for R_n^i, $i \in \{1,2\}$. Also, we have that $\|R_n^1\| \leq M$ and $\|R_n^2\| \leq 1 + M$. Therefore, the equivalence between the equations (23)-(26) and (28)-(31) can be directly obtained.

Sufficiency. Let $n \in \mathbb{Z}$. Setting $Q_n^1 = R_n^1$ and $Q_n^2 = I - R_n^2$ one can show that

$$Q_n^1 Q_n^2 = R_n^1 - R_n^1 R_n^2 = R_n^1 - R_n^2,$$

and

$$Q_n^2 Q_n^1 = R_n^1 - R_n^2 R_n^1 = R_n^1 - R_n^2.$$

Furthermore, $Q_n^1 Q_n^2$ is a projection sequence satisfying $Q_n^1 + Q_n^2 - Q_n^1 Q_n^2 = I$. Hence, Q_n^1, Q_n^2 and $Q_n^1 Q_n^2$ are projection sequences satisfying (22). We also note that for $m = n$, by (29) we have that $\|R_n^2\| \leq Kp^{a_m} \leq K$, while from (30) we obtain $\|R_n^1\| \leq 2K$.

Finally, (28)-(31) implies (23)-(26), which completes the proof. □

4 Generalized exponential trichotomy for adjoint system

If $(A_n)_{n\in\mathbb{Z}}$ is a sequence of $d \times d$ invertible matrices with complex elements, then the adjoint system associated to $(\mathfrak{A})$ is given by

$$y_n = A_n^* y_{n+1}, \quad n \in \mathbb{Z}, \tag{$\mathfrak{Y}$}$$

or, in equivalent form

$$y_{n+1} = (A_n^*)^{-1} y_n, \quad n \in \mathbb{Z}.$$

If V_n is the fundamental matrix of $(\mathfrak{Y})$, then $V_{n+1} = (A_n^*)^{-1} V_n$, and $V_0 = I$. Inductively, for $m > 0$ we have

$$\begin{aligned} V_m &= (A_{m-1}^*)^{-1} (A_{m-2}^*)^{-1} \cdots (A_1^*)^{-1} (A_0^*)^{-1} = (A_0^* A_1^* \cdots A_{m-1}^*)^{-1} \\ &= ((A_{m-1} A_{m-2} \cdots A_1 A_0)^*)^{-1} = (W_m^*)^{-1}. \end{aligned}$$

In the same way, for $m < 0$ we have that $V_m = (W_m^*)^{-1}$.

Proposition 4.1. *Let P be a projection. For every $m, n \in \mathbb{Z}$ we have*

$$\|W_n P W_m^{-1}\| = \|V_m P^* V_n^{-1}\|.$$

Proof. We have that

$$\|W_n P W_m^{-1}\| = \|(W_n P W_m^{-1})^*\| = \|(W_m^{-1})^* P^* W_n^*\| = \|V_m P^* V_n^{-1}\|.$$

□

Remark. In what follows, we will describe characterizations of GET property for the dual system $(\mathfrak{Y})$ with adjoint projections $(P^j)^*$, $j \in \{1, 2, 3\}$. The arguments of the proof are similar to the arguments used for Proposition 2.1 and thus omitted.

Proposition 4.2. *Let* ($\mathfrak{A}$) *be a system admitting a GET with constants* $K \geq 1$, $p \in (0,1)$, *strictly positive sequence* $(a_n)_{n\in\mathbb{Z}}$, *and projections* P^k, $k \in \{1,2,3\}$ *considered in Definition 2.1. Then the adjoint system* ($\mathfrak{Y}$) *also has a GET with the same constants and projections* $(P^j)^*$, $j \in \{1,2,3\}$. *More precisely, we have*

$$\|V_m(P^1)^*V_n^{-1}\| \leq Kp^{t_{mn}}, \ \ n \geq m,$$
$$\|V_m(P^2)^*V_n^{-1}\| \leq Kp^{t_{nm}}, \ \ m \geq n,$$
$$\|V_m(P^3)^*V_n^{-1}\| \leq Kp^{t_{nm}}, \ \ 0 \geq m \geq n,$$
$$\|V_m(P^3)^*V_n^{-1}\| \leq Kp^{t_{mn}}, \ \ n \geq m \geq 0.$$

Proposition 4.3. *Let* ($\mathfrak{A}$) *be a system admitting a GET with constants* $D \geq 1$, $p \in (0,1)$, *strictly positive sequence* $(a_n)_{n\in\mathbb{Z}}$, *and projections* T^k, $k \in \{1,2\}$ *considered in Proposition 2.1. Then the adjoint system* ($\mathfrak{Y}$) *also has a GET with the same constants and projections* $((T^j)^*)$, $j \in \{1,2\}$. *More precisely, we have*

$$\|V_m(I-(T^2)^*)V_n^{-1}\| \leq Dp^{t_{mn}}, \ \ n \geq m$$
$$\|V_m(I-(T^1)^*)V_n^{-1}\| \leq Dp^{t_{nm}}, \ \ m \geq n$$
$$\|V_m(T^2)^*V_n^{-1}\| \leq Dp^{t_{nm}}, \ \ 0 \geq m \geq n,$$
$$\|V_m(T^1)^*V_n^{-1}\| \leq Dp^{t_{mn}}, \ \ n \geq m \geq 0.$$

Let $m, n \in \mathbb{Z}$, with $m \geq n$ and consider

$$A_m^n = \begin{cases} A_{m-1}A_{m-2}\cdots A_n, & \text{if } m > n, \\ I, & \text{if } m = n. \end{cases}$$

For $m, n \in \mathbb{Z}$ one obtains

$$W_nW_m^{-1} = \begin{cases} A_n^m, & \text{if } n \geq m, \\ (A_m^n)^{-1}, & \text{if } m \geq n. \end{cases}$$

Proposition 4.4. *If* $(P_n)_{n\in\mathbb{Z}}$ *is an invariant projection sequence for the LTV system* ($\mathfrak{A}$), *then for every* $n \geq m$ *we have*

$$A_n^m P_m = P_n A_n^m,$$

which is equivalent to

$$P_m(A_n^m)^{-1} = (A_n^m)^{-1}P_n.$$

Proposition 4.5. *If* $(P_n)_{n\in\mathbb{Z}}$ *is a invariant projection sequences for the LTV system* ($\mathfrak{A}$) *then for every* $n, m \in \mathbb{Z}$ *we have that*

$$\|W_nW_m^{-1}P_m\| = \|V_mV_n^{-1}P_n^*\|.$$

Proof. If $n \geq m$, then

$$\begin{aligned}\|W_n W_m^{-1} P_m\| &= \|(W_n W_m^{-1} P_m)^*\| = \|(A_n^m)^* P_n^*\| \\ &= \|(W_n W_m^{-1})^* P_n^*\| = \|(W_m^{-1})^* W_n^* P_n^*\| = \|V_m V_n^{-1} P_n^*\|,\end{aligned}$$

and if $m \geq n$, then we have

$$\begin{aligned}\|W_n W_m^{-1} P_m\| &= \|(W_n W_m^{-1} P_m)^*\| = \|((A_m^n)^{-1})^* P_n^*\| \\ &= \|(W_n W_m^{-1})^* P_n^*\| = \|(W_m^{-1})^* W_n^* P_n^*\| = \|V_m V_n^{-1} P_n^*\|.\end{aligned}$$

□

Proposition 4.6. *If $(P_n)_{n\in\mathbb{Z}}$ is an invariant projection sequence for the LTV system ($\mathfrak{A}$), then $(P_n^*)_{n\in\mathbb{Z}}$ is an invariant projection sequence for the LTV system ($\mathfrak{Y}$).*

Proof. Let $n \in \mathbb{Z}$. Using the invariant property $P_{n+1}A_n = A_n P_n$ we have that $A_n^* P_{n+1}^* = P_n^* A_n^*$, which implies that $P_{n+1}^*(A_n^*)^{-1} = (A_n^*)^{-1} P_n^*$. □

Remark. The following theorem represents the extension of Theorem 3.1 for the case of the adjoint system ($\mathfrak{Y}$). This result is a natural extension of Proposition 6.1.1 from [3] for the case of LVT systems with GET. The solution is very similar to that used for Theorem 3.1, and is therefore omitted. This property can be easily checked also for Theorems 3.2 and 3.3.

Theorem 4.1. *Assume that LTV system ($\mathfrak{A}$) has a GET with supplementary projection sequences $(S_m^k)_{m\in\mathbb{Z}}$, $k \in \{1, 2, 3\}$, constants $D \geq 1$, $p \in (0, 1)$ and strictly positive sequence $(a_n)_{n\in\mathbb{Z}}$, as in Theorem 3.1. Then the adjoint system ($\mathfrak{Y}$) also has a GET with supplementary projection sequences $((S_m^k)^*)_{m\in\mathbb{Z}}$, $k \in \{1, 2, 3\}$, constants $D \geq 1$, $p \in (0, 1)$ and strictly positive sequence $(a_n)_{n\in\mathbb{Z}}$ satisfying (1) and (2). More precisely, we have*

$$\|V_m V_n^{-1}(S_n^1)^*\| \leq D p^{t_{mn}}, \quad n \geq m,$$

$$\|V_m V_n^{-1}(S_n^2)^*\| \leq D p^{t_{nm}}, \quad m \geq n,$$

$$\|V_m V_n^{-1}(S_n^3)^*\| \leq D p^{t_{nm}}, \quad 0 \geq m \geq n,$$

$$\|V_m V_n^{-1}(S_n^3)^*\| \leq D p^{t_{mn}}, \quad n \geq m \geq 0.$$

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