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Some characteristic properties of analytic functions

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Abstract

In this paper, we consider a class $\mathcal{L}\,(\lambda,\mu;\phi)$ of analytic functions f defined in the open unit disk $\mathbb{U}$ satisfying the subordination condition that

$$q(z)\frac{\mathcal{D}^{\lambda+1}f(z)}{\mathcal{D}^{\lambda}f(z)}\prec\phi(z)\ (\lambda\in\mathbb{N}_0,\mu\geq 0;z\in\mathbb{U}),$$

where $q(z)=\left(\frac{z}{\mathcal{D}^{\lambda}f(z)}\right)^{\mu-2}$, $\mathcal{D}^{\lambda}$ is the Sălăgean operator and $\phi(z)$ is a convex function with positive real part in $\mathbb{U}$. We obtain some characteristic properties giving the coefficient inequality, radius and subordination results, and an inclusion result for the above class when the function $\phi(z)$ is a bilinear mapping in the open unit disk. For these functions $f\,(z)$, sharp bounds for the initial coefficient and for the Fekete-Szegö functional are determined, and also some integral representations are given.

1 Introduction

Let $\mathcal{A}$ denote a class of functions f analytic in the open unit disk $\mathbb{U}=\{z:|z|<1\}$ with the normalization that $f(0)=0=f'(0)-1$, that is the function f has the series expansion

$$f(z)=z+\sum_{k=1}^{\infty}a_{k+1}\ z^{k+1},\ z\in\mathbb{U}. \tag{1.1}$$

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For $f \in \mathcal{A}$ of the form (1.1), we define the operator denoted $\mathcal{D}^\lambda$, $\lambda \in \mathbb{Z} = \mathbb{N} \cup \{0\} \cup -\mathbb{N} = \{\cdots, -2, -1, 0, 1, 2, \cdots\}$ by

$$\mathcal{D}^\lambda f(z) = z + \sum_{k=1}^{\infty} (k+1)^\lambda a_{k+1} \, z^{k+1}, \; z \in \mathbb{U}.$$

The operator $\mathcal{D}^\lambda$ was considered in [16] and for $\lambda \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ it is known as Sălăgean operator of order λ. In this case, it can be defined equivalently by

$$\mathcal{D}^0 f(z) = f(z), \; \mathcal{D}^1 f(z) = \mathcal{D} f(z) = z f'(z), \; \mathcal{D}^\lambda f(z) = \mathcal{D}\left(\mathcal{D}^{\lambda-1} f(z)\right), \; \lambda \in \mathbb{N}.$$

We note that $\mathcal{D}^\lambda \mathcal{D}^{-\lambda} f(z) = f(z)$, for all $\lambda \in \mathbb{Z}$.

Classes of analytic functions $f \in \mathcal{A}$ involving the quotient $\frac{\frac{zf'(z)}{f(z)}}{\frac{f(z)}{z}} = \frac{z^2 f'(z)}{f^2(z)}$ have been studied in [2, 10, 11, 17, 19]. Also, the classes involving the quotient $\frac{\frac{zf'(z)}{f(z)}}{\left(\frac{f(z)}{z}\right)^\mu} = f'(z)\left(\frac{z}{f(z)}\right)^{1+\mu}$ have been studied for $\mu > -1$ in [13] (for $-1 < \mu < 0$ in [9] and for $0 < \mu < 1$ in [22]). Moreover, for $\mu \neq 0$, a class involving a certain linear operator under a subordination condition is investigated in [4]. Interestingly, a combination of both $f'(z)\left(\frac{z}{f(z)}\right)^{1+\mu}$ and $\left(\frac{z}{f(z)}\right)^{\mu}$ for $0 < \mu < 1$ was studied in [23] (see also [18, Definition 1.1, p. 5]).

It may be observed that the operator $\mathcal{D}^\lambda$ preserves the class $\mathcal{A}$ and hence $\mathcal{D}^\lambda f(z) = 0$ at $z = 0$. Let $\lambda \in \mathbb{N}_0$ and let $f \in \mathcal{A}$ be such that $\mathcal{D}^\lambda f(z) \neq 0$ for $z \in \mathbb{U} \setminus \{0\}$. We define a function $q(z)$ by

$$q(z) = \left(\frac{z}{\mathcal{D}^\lambda f(z)}\right)^{\mu-2} \quad (\mu \geq 0, \;\; \mu \neq 2, \;\; z \in \mathbb{U} \setminus \{0\}) \;\; \text{and} \;\; q(0) = 1, \quad (1.2)$$

where we assume that only principal values of $\left(\frac{z}{\mathcal{D}^\lambda f(z)}\right)^{\mu-2}$ are taken into consideration. Clearly, the function $q(z)$ is analytic in the open unit disk $\mathbb{U}$.

Recently, by considering the expression $q(z)\frac{\mathcal{D}^{\lambda+1} f(z)}{\mathcal{D}^\lambda f(z)}, \mu \geq 0$, Prajapat and Raina [14] investigated a class $\mathcal{B}(\lambda, \mu; \alpha)$ of functions $f \in \mathcal{A}$ satisfying the condition that

$$\left| q(z)\frac{\mathcal{D}^{\lambda+1} f(z)}{\mathcal{D}^\lambda f(z)} - 1 \right| < 1 - \alpha, \; \mu \geq 0, 0 \leq \alpha < 1, z \in \mathbb{U}.$$

It may be noted that for $\lambda = 0, \alpha = 0, \mu = 3$, the class $\mathcal{B}(0, 3; 0) = \mathcal{U}$ was earlier studied by Ozaki and Nunukawa in [11] (see also Obradovic et al. [10] and Singh [19]), where it is proved that the functions $f \in \mathcal{U}$ are univalent.

For two analytic functions p, q such that $p(0) = 1 = q(0)$, we say that p is subordinate to q in $\mathbb{U}$ and write $p(z) \prec q(z), \; z \in \mathbb{U}$, if there exists a Schwarz function w, analytic in $\mathbb{U}$ with $w(0) = 0$, and $|w(z)| < 1, z \in \mathbb{U}$ such that $p(z) = q(w(z)), z \in \mathbb{U}$. Furthermore, if the function q is univalent in $\mathbb{U}$, then we have the following equivalence:

$$p(z) \prec q(z) \Leftrightarrow p(0) = q(0) \text{ and } p(\mathbb{U}) \subset q(\mathbb{U}).$$

Janowski [5] defined a class $\mathcal{P}(A, B)$ of analytic functions $p(z), z \in \mathbb{U}$, with $p(0) = 1$, if $p(z) \prec \frac{1+Az}{1+Bz}, -1 \leq B < A \leq 1, z \in \mathbb{U}$. If $p \in \mathcal{P}(A, B)$, then it follows that

$$\left| p(z) - \frac{1 - AB}{1 - B^2} \right| < \frac{A - B}{1 - B^2} \text{ for } -1 < B < A \leq 1, z \in \mathbb{U} \tag{1.3}$$

and for $B = -1$,

$$\Re\left(p(z)\right) > \frac{1 - A}{2}, -1 < A \leq 1, z \in \mathbb{U}. \tag{1.4}$$

The class $\mathcal{P}(1, -1) = \mathcal{P}$ is a Carathéodory class of functions which are analytic with positive real part in $\mathbb{U}$.

In this paper, we consider a new class $\mathcal{L}(\lambda, \mu; \phi)$ of analytic functions (which evidently generalizes the class $\mathcal{B}(\lambda, \mu; \alpha)$) comprising of functions $f \in \mathcal{A}$ if and only if (for $\frac{z}{\mathcal{D}^\lambda f(z)} \neq 0$ in $\mathbb{U}$) :

$$q(z)\frac{\mathcal{D}^{\lambda+1} f(z)}{\mathcal{D}^\lambda f(z)} \prec \phi(z) \; (\lambda \in \mathbb{N}_0, \mu \geq 0; z \in \mathbb{U}),$$

where $q(z)$ is given by (1.2), $\mathcal{D}^\lambda$ is the Sălăgean operator and $\phi \in \mathcal{P}$ is a convex function in $\mathbb{U}$; see also the works in [20] and [21].

We note that $\mathcal{L}(0, 2; \phi) = S^*[\phi]$ and $\mathcal{L}(1, 2; \phi) = K[\phi]$ are the classes introduced by Ma and Minda [7] which include several well-known starlike and convex mappings as special cases.

For the bilinear transformation $\phi(z) = \frac{1+Az}{1+Bz}$ $(-1 \leq B < A \leq 1, z \in \mathbb{U})$, we denote $\mathcal{L}\left(\lambda, \mu; \frac{1+Az}{1+Bz}\right)$ by $\mathcal{T}(\lambda, \mu; A, B)$.

We observe that the class $\mathcal{T}(\lambda, 2; A, B) = \mathcal{P}_\lambda^{\lambda+1}(A, B)$ was earlier considered by Kuroki and Owa [6, Remark 2, p. 4] for any integer λ, and for complex parameters A and B, the class $\mathcal{T}(0, 3; A, B) = \mathcal{T}(A, B)$ was studied by Shanmugam and Gangadharan [17]. The class $\mathcal{T}(0, 2; A, B) = \mathcal{S}(A, B)$ is the class of Janowski starlike functions [5]. Further, the classes $\mathcal{T}(\lambda, \mu; 1 - \alpha, 0) = \mathcal{B}(\lambda, \mu; \alpha)$ and $\mathcal{T}(0, 3; 1 - \alpha, 0) = \mathcal{B}(\alpha)$ $(0 \leq \alpha < 1)$ were studied in [2] and various subordination properties and sufficient conditions were investigated in these classes of functions.

For the purpose of this paper, we consider the functions $f \in \mathcal{A}$ of the form (1.1) such that the coefficients b_k $(k \in \mathbb{N})$ defined by

$$q(z) = \left(\frac{z}{\mathcal{D}^\lambda f(z)} \right)^{\mu - 2} = 1 + \sum_{k=1}^{\infty} b_k \; z^k, z \in \mathbb{U}, \tag{1.5}$$

to be non-negative.

Example 1. *Let $\mu \geq 0$, $\mu \neq 2$ and let $\lambda \in \mathbb{N}_0$; if we consider $f \in \mathcal{A}$ of the form $f(z) = \mathcal{D}^{-\lambda} \left(z e^{z/(2-\mu)} \right)$, then $q(z) = e^z$ has the form (1.5).*

Example 2. *Let $0 < \mu < 2$, $\mu = \frac{p}{r}$, $p, r \in \mathbb{N}$ and let $\lambda = 1$; if*

$$f(z) = \frac{1}{r+1} \left[(1+z)^{r+1} - 1 \right], \text{ then } q(z) = (1+z)^{2r-p}.$$

Example 3. *Let $\mu > 2$ and let $\lambda \in \mathbb{N}_0$; if we consider $f \in \mathcal{A}$, $f(z) = z - a_n z^n$, where $a_n > 0$ and $n \geq 2$, then*

$$q(z) = \left[1 - n^\lambda a_n z^{n-1} \right]^{2-\mu} =$$

$$1 + \sum_{k=1}^{\infty} \frac{(-\mu+2)(-\mu+1)(-\mu)\cdots(-\mu-(k-3))}{k!} \left(-n^\lambda a_n z^{n-1} \right)^k =$$

$$1 + \sum_{k=1}^{\infty} \frac{(\mu-2)(\mu-1)(\mu)\cdots(\mu+k-3)}{k!} \, n^{k\lambda} \left(a_n \right)^k z^{k(n-1)}.$$

Example 4. *Let $\mu \geq 0$, $\mu \neq 2$ and let $\lambda \in \mathbb{N}_0$; if we consider $f \in \mathcal{A}$ of the form $f(z) = \mathcal{D}^{-\lambda} \left(z(1+z)^{1/(2-\mu)} \right)$, then $q(z) = 1 + z$.*

Example 5. *Evidently, for f of the form (1.1) with $a_{k+1} \geq 0$ and for $\mu = 1$ and $\lambda \in \mathbb{N}_0$, the coefficients b_k are given by $b_k = (k+1)^\lambda a_{k+1}, k \in \mathbb{N}$.*

In this paper, we concentrate ourselves in investigating some basic characteristic properties such as the coefficient inequality, the radius result, subordination and inclusion properties for the functions $f \in \mathcal{T}(\lambda, \mu; A, B)$. Sharp bounds for the initial coefficient, the Fekete-Szegö functional of functions $f(z)$ and integral representations belonging to this class are also determined.

2 A Coefficient Inequality

We begin to investigate the coefficient inequality of functions $f \in \mathcal{T}(\lambda, \mu; A, B)$, which is contained in the following:

Theorem 1. *Let $-1 \leq B < A \leq 1, \mu \in [1,3] \setminus \{2\}$, let $f \in \mathcal{A}$ of the form (1.1) and let $b_k, k \in \mathbb{N}$ defined by (1.5) be non-negative. If*

$$\sum_{k=1}^{\infty} \frac{k-\mu+2}{|\mu-2|} b_k \leq \frac{A-B}{1+|B|}, \tag{2.1}$$

then $f \in \mathcal{T}(\lambda, \mu; A, B)$. The condition (2.1) is necessary for $f \in \mathcal{T}(\lambda, \mu; A, B)$ provided that $-1 \leq B \leq 0 < A \leq 1, \mu \in (2,3]$.

Proof. Let

$$p(z) = q(z) \frac{\mathcal{D}^{\lambda+1} f(z)}{\mathcal{D}^{\lambda} f(z)}, z \in \mathbb{U}, \tag{2.2}$$

where $q(z)$ is given by (1.2) then, we get

$$p(z) = q(z) - \frac{zq'(z)}{\mu-2}. \tag{2.3}$$

Since $f \in \mathcal{T}(\lambda, \mu; A, B)$, if and only if

$$\left| \frac{p(z)-1}{A-Bp(z)} \right| < 1, z \in \mathbb{U}, \tag{2.4}$$

therefore, if we consider

$$P = |p(z)-1| - |A-Bp(z)|,$$

then in view of (1.5) and (2.3), we get

$$\begin{aligned} P &= \left| -\sum_{k=1}^{\infty} \frac{k-\mu+2}{\mu-2} b_k \, z^k \right| - \left| A-B+\sum_{k=1}^{\infty} \frac{k-\mu+2}{\mu-2} B b_k \, z^k \right| \\ &< \sum_{k=1}^{\infty} \frac{k-\mu+2}{|\mu-2|} b_k - \left[A-B-\sum_{k=1}^{\infty} \frac{k-\mu+2}{|\mu-2|} |B| \, b_k \right] \\ &= \sum_{k=1}^{\infty} \frac{k-\mu+2}{|\mu-2|} (1+|B|) \, b_k - (A-B) \leq 0, \end{aligned}$$

on using (2.1). For the necessary part, we consider for $-1 \leq B \leq 0 < A \leq 1$, $\mu \in (2,3]$ that $f \in \mathcal{T}(\lambda, \mu; A, B)$, then from (2.4), in view of (1.5) and (2.3), we have

$$\left| \frac{-\sum\limits_{k=1}^{\infty} \frac{k-\mu+2}{\mu-2} b_k \, z^k}{A-B-\sum\limits_{k=1}^{\infty} \frac{k-\mu+2}{\mu-2} |B| \, b_k \, z^k} \right| < 1, z \in \mathbb{U}. \tag{2.5}$$

Since $p(z)$ in (2.3) is real for real z, letting $z \to 1^-$ along real axis, we get from the condition that

$$\Re\left(p(z)\right) > \frac{1-A}{1-B}$$

which ensures that the denominator under the mod sign in the inequality (2.5) remains positive and then we have

$$\frac{\sum\limits_{k=1}^{\infty} \frac{k-\mu+2}{\mu-2} b_k}{A - B - \sum\limits_{k=1}^{\infty} \frac{k-\mu+2}{\mu-2} \left|B\right| b_k} \leq 1,$$

which proves (2.1). This completes the proof of Theorem 1. ∎

From Theorem 1, for the cases when $B = 0$ and $B = -1$ ($\mu \in (2,3]$), respectively, and applying the well-known assertions (1.3) and (1.4), we get the following results.

Corollary 1. *Let* $0 < A \leq 1, \mu \in (2,3]$ *and let* $f \in \mathcal{A}$ *of the form (1.1) and let* $b_k, k \in \mathbb{N}$ *defined by (1.5) be non-negative. Then*

$$\left|q(z)\frac{\mathcal{D}^{\lambda+1}f(z)}{\mathcal{D}^{\lambda}f(z)} - 1\right| < A, \ z \in \mathbb{U},$$

if and only if

$$\sum_{k=1}^{\infty} \frac{k-\mu+2}{\mu-2} b_k \ \leq A.$$

Corollary 2. *Let* $-1 < A \leq 1, \mu \in (2,3]$ *and let* $f \in \mathcal{A}$ *of the form (1.1) and let* $b_k, k \in \mathbb{N}$ *defined by (1.5) be non-negative. Then*

$$\Re\left(q(z)\frac{\mathcal{D}^{\lambda+1}f(z)}{\mathcal{D}^{\lambda}f(z)}\right) > \frac{1-A}{2}, \ z \in \mathbb{U},$$

if and only if

$$\sum_{k=1}^{\infty} \frac{k-\mu+2}{\mu-2} b_k \ \leq \frac{1+A}{2}.$$

Remark 1. *For* $\lambda = 0, \mu = 3, A = 1$, *Corollary 1 corresponds to the known result of Ponnusamy and Sahoo [13, Theorem 7, p. 400].*

3 Radius Result

Theorem 2. *Let $-1 \leq B < A \leq 1, \mu \in [1,3] \setminus \{2\}$ and let $f \in \mathcal{A}$ of the form (1.1) and let $b_k, k \in \mathbb{N}$ defined by (1.5) be non-negative and satisfy the condition that*

$$\sum_{k=1}^{\infty} \frac{k-\mu+2}{|\mu-2|} (b_k)^2 \leq 1. \tag{3.1}$$

Then

$$\frac{1}{r} f(rz) \in \mathcal{T}(\lambda, \mu; A, B)$$

for $0 < r \leq r_0$, where $r_0 = r_0(\mu, A, B)$ is given by

$$r_0 = \frac{\eta \sqrt{2|\mu-2|}}{[3-\mu+2\eta^2|\mu-2|+E]^{1/2}} \tag{3.2}$$

where $E = \sqrt{\{3-\mu+2\eta^2|\mu-2|\}^2 + 4\eta^2|\mu-2|(\mu-2-\eta^2|\mu-2|)}$ and $\eta = \dfrac{A-B}{1+|B|}$.

Proof. Let $f \in \mathcal{A}$ be of the form (1.1) with $\mu \in [1,3] \setminus \{2\}$. Then for $0 < r \leq 1$, we have

$$q(rz) = \left(\frac{z}{\frac{1}{r}\mathcal{D}^\lambda f(rz)} \right)^{\mu-2} = 1 + \sum_{k=1}^{\infty} b_k \, r^k z^k, b_k \geq 0, z \in \mathbb{U},$$

where $q(z)$ is given by (1.2). Thus, by Theorem 1, $\frac{1}{r} f(rz) \in \mathcal{T}(\lambda, \mu; A, B)$ if

$$R := \sum_{k=1}^{\infty} \frac{k-\mu+2}{|\mu-2|} b_k \, r^k \leq \frac{A-B}{1+|B|}.$$

By Cauchy-Schwarz inequality and the condition (3.1), we obtain that

$$\begin{aligned} R &\leq \left(\sum_{k=1}^{\infty} \frac{k-\mu+2}{|\mu-2|} (b_k)^2 \right)^{1/2} \left(\sum_{k=1}^{\infty} \frac{k-\mu+2}{|\mu-2|} \, r^{2k} \right)^{1/2} \\ &\leq \frac{1}{\sqrt{|\mu-2|}} \left(\sum_{k=1}^{\infty} (k-\mu+2) \, r^{2k} \right)^{1/2} \\ &= \frac{1}{\sqrt{|\mu-2|}} \left(\frac{r^4}{(1-r^2)^2} + (3-\mu) \frac{r^2}{1-r^2} \right)^{1/2} \\ &= \frac{1}{\sqrt{|\mu-2|}} \frac{r}{1-r^2} \left\{ 3-\mu+(\mu-2) r^2 \right\}^{1/2} \leq \frac{A-B}{1+|B|}, \end{aligned}$$

provided that the inequality

$$\frac{r}{1-r^2}\left\{3-\mu+(\mu-2)\,r^2\right\}^{1/2} \le \eta\sqrt{|\mu-2|}$$

holds, where $\frac{A-B}{1+|B|}=\eta$, or equivalently

$$\frac{1}{r^4}\eta^2\,|\mu-2| - \frac{1}{r^2}\left\{3-\mu+2\eta^2\,|\mu-2|\right\} - \left\{\mu-2-\eta^2\,|\mu-2|\right\} \ge 0$$

holds, which provides the value of r_0 given by (3.2). This proves Theorem 2. ∎

Remark 2. *By setting $\lambda=0, \mu=3-\alpha$ $(0\le\alpha<1)$ and $B=0, A=\eta$, Theorem 2 coincides with the result of Ponnusamy and Sahoo [13, Theorem 5, p. 398] for univalent functions $f(z)$.*

4 Subordination Result

Theorem 3. *Let $-1\le B<A\le 1, \mu\in[1,2)$ and let $f\in\mathcal{A}$ of the form (1.1) and let $b_k, k\in\mathbb{N}$ defined by (1.5) be non-negative. If $f\in\mathcal{T}(\lambda,\mu;A,B)$, then*

$$\left(\frac{z}{\mathcal{D}^{\lambda}f(z)}\right)^{\mu-2} \prec \frac{1+Az}{1+Bz}, z\in\mathbb{U} \tag{4.1}$$

and hence,

$$b_k \le A-B, k\in\mathbb{N}. \tag{4.2}$$

Proof. Let $q(z)$ be defined by (1.2), which is analytic in $\mathbb{U}$ with $q(0)=1$, then from (2.3), we have

$$q(z)+\frac{zq'(z)}{2-\mu} \prec \frac{1+Az}{1+Bz}, z\in\mathbb{U},$$

which by the result of Hallenbeck and Ruscheweyh [3] proves (4.1). Further, on using a well-known result of Rogosinski [15] on subordination, and in view of (1.5), the subordination (4.1) gives the coefficient inequality (4.2). ∎

5 Inclusion Result

Theorem 4. *Let $\lambda\in\mathbb{N}_0, -1\le B\le 0<A\le 1, \mu\in(2,3]$ and let $f\in\mathcal{A}$ of the form (1.1) and let $b_k, k\in\mathbb{N}$ defined by (1.5) be non-negative. If*

$$\frac{\mathcal{D}^{\lambda+1}f(z)}{\mathcal{D}^{\lambda}f(z)} \prec \frac{1+Az}{1+Bz}, z\in\mathbb{U}, \tag{5.1}$$

then

$$\sum_{k=1}^{\infty}\left\{\frac{k-\frac{A-B}{1+|B|}(\mu-2)}{\mu-2}\right\}b_k\leq\frac{A-B}{1+|B|}. \tag{5.2}$$

Hence, $\mathcal{P}_{\lambda}^{\lambda+1}(A,B)\subset\mathcal{T}(\lambda,\mu;A,B)$.

Proof. From (5.1), we have

$$f\in\mathcal{P}_{\lambda}^{\lambda+1}(A,B)\Leftrightarrow\frac{\mathcal{D}^{\lambda+1}f(z)}{\mathcal{D}^{\lambda}f(z)}\prec\frac{1+Az}{1+Bz}\Leftrightarrow\left|\frac{\frac{\mathcal{D}^{\lambda+1}f(z)}{\mathcal{D}^{\lambda}f(z)}-1}{A-B\frac{\mathcal{D}^{\lambda+1}f(z)}{\mathcal{D}^{\lambda}f(z)}}\right|<1, z\in\mathbb{U}. \tag{5.3}$$

Let $q(z)$ be defined by (1.2), then on using (2.2) and (2.3), we get

$$\frac{zq'(z)}{q(z)}=(\mu-2)\left(1-\frac{\mathcal{D}^{\lambda+1}f(z)}{\mathcal{D}^{\lambda}f(z)}\right).$$

Hence, by (1.5), the condition (5.3) can equivalently be expressed as

$$\left|\frac{-\sum\limits_{k=1}^{\infty}kb_k\ z^k}{(A-B)(\mu-2)\left(1+\sum\limits_{k=1}^{\infty}b_k\ z^k\right)+B\sum\limits_{k=1}^{\infty}kb_k\ z^k}\right|<1, z\in\mathbb{U}. \tag{5.4}$$

Since $\frac{\mathcal{D}^{\lambda+1}f(z)}{\mathcal{D}^{\lambda}f(z)}$ is real for real z, letting $z\to1^-$ along the real axis, we get from (5.1) that

$$\Re\left(\frac{\mathcal{D}^{\lambda+1}f(z)}{\mathcal{D}^{\lambda}f(z)}\right)>\frac{1-A}{1-B}\Leftrightarrow\frac{\mathcal{D}^{\lambda+1}f(z)}{\mathcal{D}^{\lambda}f(z)}>\frac{1-A}{1-B}$$

and hence, for being $B\leq0$,

$$A-B\frac{\mathcal{D}^{\lambda+1}f(z)}{\mathcal{D}^{\lambda}f(z)}>A+|B|\frac{1-A}{1-B}>0,$$

which ensures that the denominator under the mod sign in the inequality (5.4) is positive. Thus, we have

$$\frac{\sum\limits_{k=1}^{\infty}kb_k}{(A-B)(\mu-2)-\sum\limits_{k=1}^{\infty}\{|B|k-(A-B)(\mu-2)\}b_k}\leq1,$$

which yields the desired inequality (5.2). Further, since $\frac{A-B}{1+|B|} \leq 1$, if $f \in \mathcal{P}_{\lambda}^{\lambda+1}(A,B)$, we have by (5.2) that

$$\sum_{k=1}^{\infty} \frac{k-\mu+2}{\mu-2} b_k \leq \sum_{k=1}^{\infty} \left\{ \frac{k - \frac{A-B}{1+|B|}(\mu-2)}{\mu-2} \right\} b_k \leq \frac{A-B}{1+|B|},$$

and consequently by Theorem 1, we conclude that $f \in \mathcal{T}(\lambda,\mu;A,B)$. This proves the inclusion result. ∎

6 Fekete-Szegö Problem

Let $f(z)$ of the form (1.1) be in the class $\mathcal{T}(\lambda,\mu;A,B)$, then for some Schwarz function $w(z)$, we get

$$q(z)\frac{\mathcal{D}^{\lambda+1}f(z)}{\mathcal{D}^{\lambda}f(z)} = \frac{1+Aw(z)}{1+Bw(z)}, z \in \mathbb{U}, \tag{6.1}$$

where $q(z)$ is given by (1.2) and upon using the series:

$$\mathcal{D}^{\lambda}f(z) = z + \sum_{k=1}^{\infty} (k+1)^{\lambda} a_{k+1}\ z^{k+1}, z \in \mathbb{U}$$

and performing elementary calculations, we can write the series expansion

$$\begin{aligned} & q(z)\frac{\mathcal{D}^{\lambda+1}f(z)}{\mathcal{D}^{\lambda}f(z)} \\ = \ & \frac{\mathcal{D}^{\lambda+1}f(z)}{z}\left(\frac{z}{\mathcal{D}^{\lambda}f(z)}\right)^{\mu-1} \\ = \ & 1+(3-\mu)\,2^{\lambda}a_2 z + (4-\mu)\left\{3^{\lambda}a_3 - (\mu-1)\,2^{2\lambda-1}a_2^2\right\} z^2 + ... \end{aligned} \tag{6.2}$$

For the Schwarz function $w(z)$, let $\phi \in \mathcal{P}$ be defined by

$$\phi(z) = \frac{1+w(z)}{1-w(z)} = 1 + c_1 z + c_2 z^2 + ... \tag{6.3}$$

Then

$$\frac{1+Aw(z)}{1+Bw(z)} = 1 + \frac{A-B}{2}c_1 z + \frac{A-B}{2}\left\{c_2 - \frac{B+1}{2}c_1^2\right\} z^2 + ..., \tag{6.4}$$

and from (6.2) and (6.4), we get

$$(3-\mu)\,2^{\lambda}a_2 = \frac{A-B}{2}c_1, \tag{6.5}$$

$$(4-\mu)\left\{3^{\lambda}a_3 - (\mu-1)\,2^{2\lambda-1}a_2^2\right\} = \frac{A-B}{2}\left\{c_2 - \frac{B+1}{2}c_1^2\right\}. \tag{6.6}$$

In order to find in this section sharp upper bound for $|a_2|$ and for the Fekete-Szegö functional $\left|a_3-\rho a_2^2\right|$ $(\rho\in\mathbb{C})$, we use the following result from [12, p. 166] (see also [1, p. 41]).

Lemma 1. *Let $\phi\in\mathcal{P}$ be of the form $\phi(z)=1+c_1z+c_2z^2+...$, then*

$$\left|c_2-c_1^2/2\right|\le 2-\left|c_1\right|^2/2$$

and $|c_k|\le 2$ for all $k\in\mathbb{N}$.

Theorem 5. *Let $-1\le B<A\le 1,\mu\in(2,3)$, and $f\in\mathcal{A}$ be of the form (1.1) belong to the class $\mathcal{T}(\lambda,\mu;A,B)$, then*

$$|a_2|\le\frac{A-B}{(3-\mu)\,2^{\lambda}},$$

and for all $\rho\in\mathbb{C}$:

$$\left|a_3-\rho a_2^2\right|\le\frac{A-B}{(4-\mu)\,3^{\lambda}}\max\left\{1,\left|\left(\frac{\mu-1}{3^{\lambda}}2^{2\lambda-1}-\rho\right)\frac{(A-B)\,(4-\mu)\,3^{\lambda}}{(3-\mu)^2\,2^{2\lambda}}-B\right|\right\}.$$

The result is sharp if $B=-1$ or if $B=0$.

Proof. Let the function $f(z)$ of the form (1.1) belong to the class $\mathcal{T}(\lambda,\mu;A,B)$, then using the Carathéodory condition: $|c_1|\le 2$ in (6.5), for the functions $\phi\in\mathcal{P}$ of the form (6.3), we get

$$|a_2|\le\frac{A-B}{(3-\mu)\,2^{\lambda}},$$

which by virtue of (6.5) and (6.6) gives

$$\begin{aligned}a_3-\rho a_2^2 \quad = \quad & \left(\frac{(\mu-1)\,2^{2\lambda-1}}{3^{\lambda}}-\rho\right)\frac{(A-B)^2}{4\,(3-\mu)^2\,2^{2\lambda}}c_1^2+\\ & +\frac{A-B}{2\,(4-\mu)\,3^{\lambda}}\left\{c_2-\frac{B+1}{2}c_1^2\right\}=\frac{A-B}{2\,(4-\mu)\,3^{\lambda}}\left(c_2-\frac{1}{2}c_1^2\right)+\\ & +\left(\left(\frac{\mu-1}{3^{\lambda}}2^{2\lambda-1}-\rho\right)\frac{(A-B)^2}{(3-\mu)^2\,2^{2\lambda+2}}-\frac{(A-B)\,B}{4\,(4-\mu)\,3^{\lambda}}\right)c_1^2.\end{aligned}$$

By Lemma 1, it follows that

$$\left|a_3-\rho a_2^2\right|\le F\left(|c_1|\right)=C+CD\frac{|c_1|^2}{4},$$

where

$$C=\frac{(A-B)}{(4-\mu)\,3^{\lambda}}>0, D=|E|-1, E=\left(\frac{\mu-1}{3^{\lambda}}2^{2\lambda-1}-\rho\right)\frac{(A-B)\,(4-\mu)\,3^{\lambda}}{(3-\mu)^{2}\,2^{2\lambda}}-B.$$

As $|c_1|\leq 2$, we infer that

$$\left|a_3-\rho a_2^2\right|\leq\left\{\begin{array}{ll} F\,(0)=C, & |E|\leq 1\\ F\,(2)=C\,|E| & |E|\geq 1\end{array}\right. .$$

In the case when $B=-1$, the sharpness can be verified for the functions given by

$$q(z)\frac{\mathcal{D}^{\lambda+1}f(z)}{\mathcal{D}^{\lambda}f(z)}=\frac{1+Az^2}{1-z^2}\left(\text{or }\frac{1+Az}{1-z}\right), z\in\mathbb{U}$$

and, in case when $B=0$, the sharpness can be verified for functions given by

$$q(z)\frac{\mathcal{D}^{\lambda+1}f(z)}{\mathcal{D}^{\lambda}f(z)}=1+Az^2\ \ (\text{or }1+Az)\,, z\in\mathbb{U},$$

where $q(z)$ is given by (1.2). This completes the proof of Theorem 5. ∎

Remark 3. *For $\lambda=0,\mu=2+\nu\ \ (0<\nu<1)$, Theorem 5 corresponds (for $A=1-2\alpha\ (0\leq\alpha<1)\,, B=-1)$ to Theorem 1, and (for $A=k\ (0<k\leq 1)\,,$ $B=0$) to Theorem 2 of Tuneski and Darus [22, pp. 64-65].*

7 Integral Representations

Theorem 6. *Let $-1\leq B<A\leq 1, 2<\mu\leq 3$ and $f\in\mathcal{A}$ be of the form (1.1). If $f\in\mathcal{T}(\lambda,\mu;A,B)$, then for some Schwarz functions $w_1(z)$ and $w_2(z)$, $w_1(0)=0=w_1'(0)-1$ (in case $2<\mu<3$):*

$$\left(\frac{z}{\mathcal{D}^{\lambda}f(z)}\right)^{\mu-2}=1-(\mu-2)\,(A-B)\,z^{\mu-2}\int_0^z\frac{w_1(t)}{t^{\mu-1}\,(1+Bw_1(t))}dt, z\in\mathbb{U}, \tag{7.1}$$

and $w_2(0)=0=w_2'(0)$ *(in case $\mu=3$):*

$$\frac{z}{\mathcal{D}^{\lambda}f(z)}=1-2^{\lambda}a_2z-(A-B)\,z\int_0^z\frac{w_2(t)}{t^2\,(1+Bw_2(t))}dt, z\in\mathbb{U}. \tag{7.2}$$

Proof. Let $f \in \mathcal{A}$ be of the form (1.1), then from (6.2), we have

$$q(z)\frac{\mathcal{D}^{\lambda+1} f(z)}{\mathcal{D}^{\lambda} f(z)} = 1 + (3-\mu)\, 2^{\lambda} a_2 z + (4-\mu)\left\{3^{\lambda} a_3 - (\mu-1)\, 2^{2\lambda-1} a_2^2\right\} z^2 + \ldots,$$

where $q(z)$ is given by (1.2). Hence, if $f \in \mathcal{T}(\lambda, \mu; A, B)$, the Schwarz function $w(z)$ in (6.1) is given by

$$w(z) = \begin{cases} w_1(z): \quad w_1(0) = 0 = w_1'(0) - 1, & \text{if} \quad 2 < \mu < 3, \\ w_2(z): \quad w_2(0) = 0 = w_2'(0), & \text{if} \quad \mu = 3. \end{cases}$$

It is easy to verify that

$$\frac{d}{dz}\left(\frac{1}{\left(\mathcal{D}^{\lambda} f(z)\right)^{\mu-2}} - \frac{1}{z^{\mu-2}}\right) = -\frac{\mu-2}{z^{\mu-1}}\left(q(z)\frac{\mathcal{D}^{\lambda+1} f(z)}{\mathcal{D}^{\lambda} f(z)} - 1\right),$$

where $q(z)$ is given by (1.2) and therefore by (6.1), we get

$$\frac{d}{dz}\left(\frac{1}{\left(\mathcal{D}^{\lambda} f(z)\right)^{\mu-2}} - \frac{1}{z^{\mu-2}}\right) = -\frac{(\mu-2)(A-B)\, w(z)}{z^{\mu-1}\left(1 + Bw(z)\right)}, z \in \mathbb{U}. \tag{7.3}$$

From (1.5), we also have

$$\frac{1}{\left(\mathcal{D}^{\lambda} f(z)\right)^{\mu-2}} - \frac{1}{z^{\mu-2}} = \frac{1}{z^{\mu-2}}\left[q(z) - 1\right] = \sum_{k=1}^{\infty} b_k\, z^{k-\mu+2},$$

which yields that

$$\left(\frac{1}{\left(\mathcal{D}^{\lambda} f(z)\right)^{\mu-2}} - \frac{1}{z^{\mu-2}}\right)_{z=0} = \begin{cases} b_1, & \mu = 3 \\ 0, & 2 < \mu < 3 \end{cases}.$$

By using (1.5) and (6.2), and equating the coefficient of z on both the sides of (2.3), we find that $b_1 = -(\mu-2)\, 2^{\lambda} a_2$. Now, upon integrating (7.3), we obtain the desired representations given by (7.1) and (7.2). ∎

Remark 4. *For $\mu = 3$ and $\lambda = 0$, the above representation (7.2) corresponds to the representation due to Shanmugam and Gangadharan in [17, Theorem 2.1, pp. 2-3] and corresponds to Theorem 1 of Obradovic et al. [10] if $A = 1$ and $B = 0$.*

Corollary 3. *Let $-1 \leq B \leq 0 < A \leq 1$ and $f \in \mathcal{A}$ be of the form (1.1). If $f \in \mathcal{T}(\lambda, \mu; A, B)$, then for $\mu \in (2,3)$:*

$$\left|\left(\frac{z}{\mathcal{D}^{\lambda} f(z)}\right)^{\mu-2} - 1\right| \tag{7.4}$$
$$\leq \begin{cases} \frac{(\mu-2)A}{3-\mu} |z|, B = 0, z \in \mathbb{U}, \\ (\mu-2)(A-B) \frac{|z|}{3-\mu} \; {}_2F_1(1, 3-\mu, 4-\mu; |B| |z|), -1 \leq B < 0, z \in \mathbb{U}, \end{cases}$$

and for $\mu = 3$:

$$\left|\frac{z}{\mathcal{D}^{\lambda} f(z)} - 1\right| \leq \begin{cases} 2^{\lambda} a_2 |z| + A|z|^2, B = 0, z \in \mathbb{U}, \\ 2^{\lambda} a_2 |z| + \frac{(A-B)|z|}{2\sqrt{|B|}} \log\left(\frac{1+|z|\sqrt{|B|}}{1-|z|\sqrt{|B|}}\right), -1 \leq B < 0, z \in \mathbb{U}. \end{cases} \tag{7.5}$$

Proof. From (7.1) when $\mu \in (2,3)$, and on substituting $t = zu$ and noting that
$|w_1(zu)| \leq |z| u$, we get

$$\left|\left(\frac{z}{\mathcal{D}^{\lambda} f(z)}\right)^{\mu-2} - 1\right| \leq (\mu-2)(A-B) \int_0^1 \frac{|z|}{u^{\mu-2}(1-|B||z|u)} du, z \in \mathbb{U}.$$

Now if $B = 0$, the above integral gives simply

$$\int_0^1 \frac{|z|}{u^{\mu-2}(1-|B||z|u)} du = \frac{|z|}{3-\mu}, z \in \mathbb{U},$$

and if $-1 \leq B < 0$, making use of the known integral representation of the Gaussian hypergeometric function mentioned, for instance, see [8, p. 7], we get

$$\int_0^1 \frac{|z|}{u^{\mu-2}(1-|B||z|u)} du = \frac{|z|}{3-\mu} \; {}_2F_1(1, 3-\mu, 4-\mu; |B| |z|)$$

and hence, we have the inequality (7.4). Also, from (7.2), by substituting $t = zu$ and noting that $|w_2(zu)| \leq |z|^2 u^2$, we get

$$\left|\frac{z}{\mathcal{D}^{\lambda} f(z)} - 1\right| \leq 2^{\lambda} a_2 |z| + (A-B) \int_0^1 \frac{|z|^2}{1-|B||z|^2 u^2} du, z \in \mathbb{U}.$$

Using now the following integral (for $-1 \le B \le 0$):

$$\int_0^1 \frac{|z|^2}{1-|B|\,|z|^2\,u^2}\,du = \begin{cases} |z|^2 & B=0, z \in \mathbb{U}, \\ \frac{|z|}{2\sqrt{|B|}} \log\left(\frac{1+|z|\sqrt{|B|}}{1-|z|\sqrt{|B|}}\right), & -1 \le B < 0, z \in \mathbb{U}, \end{cases}$$

we are lead to the second inequality (7.5) of Corollary 3. ∎

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