DOI: 10.1515/auom-2015-0038
An. Şt. Univ. Ovidius Constanţa

VERSITA
Vol. 23(2),2015, 213–222

A multi-dimensional FBSDE with quadratic generator and its applications

Eduard Rotenstein

Abstract

We consider, in the Markovian framework, a multi-dimensional forward - backward stochastic differential equation with quadratic growth for the generator function of the backward system. We prove an existence result of the solution and we use this result for pricing and hedging of contingent claims that depend on non-tradeable indexes by portfolios consisting in correlated risky assets.

1 Introduction. The financial background

In 1990, Pardoux and Peng [11] introduced the notion of nonlinear backward stochastic differential equation (for short, BSDE) and obtained existence and uniqueness result for this kind of equation. Since then, the interest in BSDEs has kept growing and there have been a lot of works on that subject. The main reason is that BSDEs are encountered in many fields of mathematics such as finance, stochastic games, optimal control, partial differential equations. This kind of stochastic systems were first studied under Lipschitz or monotonicity conditions imposed on the generator function.

A decade after, in her PhD thesis, Kobylanski [7] considered a new framework for such equations, which is better adapted for the study of financial markets. She considered real-valued BSDEs with quadratic growth for the coefficient. The adopted technique for proving the existence of a solution consists in some suitably exponential transformation of the equation, followed by

Key Words: Forward-backward SDEs, Quadratic growth, Financial derivatives
2010 Mathematics Subject Classification: Primary 60H10, 46L05, 91G20, 91G80
Received: December, 2013.
Revised: February, 2014.
Accepted: February, 2014.

approximations, in a monotone manner, by Lipschitz functions, of the new generator. This approach permits the use of exponential utility functions, which role is to capture the sensitiveness with respect to risk of the investors in the market.

In the present exposition, a multidimensional forward-backward stochastic differential equation with quadratic growth for the generator function of the backward system is considered, in the Markovian framework. Namely, an existence result is established for this multidimensional equation; an application in pricing and hedging contingent claims that depend on non-tradeable indexes is presented. The hedging will be done by portfolios which consists in correlated (with respect to the non-tradable index) risky assets. This situation may appear when there exists in the financial market securities that are not the subject of transaction. For example, the choice of a company that produce kerosene (which is not tradable on a liquid market) for covering the risk caused by depreciation of the oil price is to invest in some assets that are correlated with the price of the kerosene. Therefore, to price and hedge a contingent claim (possible a multi-dimensional one) that is based on the non-tradable asset, the investor can construct portfolios that contain the tradable correlated underlyings.

The paper is organized as follows. Section 2 is dedicated to the introduction of some notations and preliminary results concerning Markovian forward backward stochastic differential equations, Section 3 contains the existence result for the solution of a multi-dimensional BSDE with quadratic coefficient, while Section 4 deals with the modelization of a financial markets with tradable and non-tradable assets. Considering given a utility function for the investor, the goal consists in hedging European contingent claims within the utility maximization paradigm (see [1], [6] or [10]).

2 Preliminaries. Notations. Hypothesis

The purpose of this section is to introduce some basic notations and results concerning the Markovian framework of forward-backward SDEs with quadratic growth for the coefficient of the backward system, which will be needed throughout this paper. In all that follows we shall consider a finite horizon $T > 0$ and a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which is defined a standard k-dimensional Brownian motion $W = (W_t)_{t \leq T}$ whose natural filtration is denoted $\mathbb{F} = \{\mathcal{F}_t,\ 0 \leq t \leq T\}$. More precisely, $\mathbb{F}$ is the filtration generated by the process W and augmented by $\mathcal{N}_{\mathbb{P}}$, the set of all $\mathbb{P}$-null sets, *i.e.* $\mathcal{F}_t = \sigma\{W_s,\, s \leq t\} \vee \mathcal{N}_{\mathbb{P}}$.

Let us consider:

(i) $\mathcal{P}$, the σ-algebra of $\mathcal{F}_t$-progressively measurable sets on $[0, T] \times \Omega$;

(ii) $\mathcal{L}^2_T(\mathbb{R}^d)$, the set of $\mathcal{P}$-measurable and $\mathbb{R}^d$-valued processes $z = (z_t)_{t\leq T}$ such that $\int_0^T |z_t|^2 dt < \infty$, $\mathbb{P}$-a.s.; $\mathcal{H}^2_T(\mathbb{R}^d)$ is the subspace of $\mathcal{L}^2_T(\mathbb{R}^d)$, such that $\mathbb{E}[\int_0^T |z_t|^2 dt] < \infty$;

(iii) $\mathcal{S}^2$, the set of $\mathcal{P}$-measurable and continuous processes $Y = (Y_t)_{t\leq T}$ such that $\mathbb{E}[\sup\limits_{t\leq T} |Y_t|^2] < \infty$ (this space will be also denoted by $\mathcal{H}^\infty_T(\mathbb{R}^d)$);

Let us now introduce the four objects which define the forward backward stochastic differential system that we will consider along this paper:

1. the measurable functions $b : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ (the drift coefficient) and $\sigma : [0,T] \times \mathbb{R}^n \to \mathbb{R}^{n\times k}$ (the diffusion coefficient) satisfying, $\mathbb{P}$-a.s., for $L, K > 0$:

(i) $|b(t,x) - b(t,y)| + |\sigma(t,x) - \sigma(t,y)| \leq L|x-y|, \quad \forall (t,x,y) \in [0,T] \times \mathbb{R}^n \times \mathbb{R}^n;$

(ii) $|b(t,x)|^2 + |\sigma(t,x)|^2 \leq K(1+|x|^2), \quad \forall (t,x) \in [0,T] \times \mathbb{R}^n;$

(iii) the *non-degeneracy condition* (see Hamadène, Lepeltier, Peng [5]): there exists a constant $C > 0$ such that, for every $(t,x) \in [0,T] \times \mathbb{R}^n$,

$$\frac{1}{C} I_n \leq \sigma(t,x)\sigma(t,x)^t \leq C I_n,$$

where I_n represents the identity matrix of dimension $n \times n$. (H1)

2. the measurable generator function $F : [0,T] \times \mathbb{R}^n \times \mathbb{R}^d \times \mathbb{R}^{d\times k} \to \mathbb{R}^d$, satisfying, for some constants $\alpha > 0$ and $C = (C_1, ..., C_d) \in (\mathbb{R}^*_+)^d$, $\mathbb{P}$-a.s.,

(i) $F(t,x,\cdot,\cdot)$ is continuous for every pair $(t,x) \in [0,T] \times \mathbb{R}^n$;

(ii) $|F_i(t,x,y,z) - C_i|z_i|^2| \leq \alpha(1+|y_i|), \ \forall i = \overline{1,d}, \quad \forall (t,x,y,z) \in [0,T] \times \mathbb{R}^n \times \mathbb{R}^d \times \mathbb{R}^{d\times k},$ by the generic h_i understanding the i^{th} component of the vector h. (H2)

3. $g : \mathbb{R}^n \to \mathbb{R}^d$ is a bounded and measurable function.

Consider the pair $(t_0, x_0) \in [0,T] \times \mathbb{R}^n$ fixed. We will denote by $(X_t^{t_0,x_0})_{t\in[0,T]}$, the solution of the forward SDE:

$$\begin{cases} dX_t^{t_0,x_0} = b(t, X_t^{t_0,x_0})dt + \sigma(t, X_t^{t_0,x_0})dW_t, \quad t \in [t_0, T], \\ X_t^{t_0,x_0} = x_0, \quad t \in [0, t_0]. \end{cases} \tag{1}$$

It is well known (see Karatzas, Shreve [8]) that, under the assumptions (H1), the SDE (1) admits a unique solution X^{t_0,x_0}. Moreover, for every $p \geq 2$, there exists $C_p > 0$ such that, for all $t \in [0,T]$, $x, x' \in \mathbb{R}^n$, we have, $\mathbb{P}$-a.s.:

$$\mathbb{E}\left[\sup_{s\in[t_0,T]} |X_s^{t_0,x_0} - X_s^{t_0,x_0'}|^p \middle| \mathcal{F}_t\right] \leq C_p|x_0 - x_0'|^p;$$

$$\mathbb{E}\left[\sup_{s\in[t_0,T]} |X_s^{t_0,x_0}|^p \middle| \mathcal{F}_t\right] \leq C_p(1 + |x_0|^p),$$

where the constant C_p depends only on the Lipschitz and the linear growth constants of b and σ.

We will study in Section 3 the existence of a solution for the multi-dimensional backward stochastic differential system characterized by the functions F and g (and denoted by QG-mFBSDE(F,g)):

$$\begin{cases} -dY_t^{t_0,x_0} = F(t, X_t^{t_0,x_0}, Y_t^{t_0,x_0}, Z_t^{t_0,x_0})dt - Z_t^{t_0,x_0}dW_t, \quad t \in [0,T], \\ Y_T^{t_0,x_0} = g(X_T^{t_0,x_0}), \end{cases} \tag{2}$$

where $X^{t,x}$ is the unique solution of Eq.(1).

Definition 1. *By a solution of Eq.(2) we understand a couple*

$$\left(Y^{t_0,x_0}, Z^{t_0,x_0}\right) \in \mathcal{H}_T^2(\mathbb{R}^d) \times \mathcal{H}_T^2(\mathbb{R}^{d\times k}),$$

satisfying, $\mathbb{P}$*-a.s.,*

$$Y_t^{t_0,x_0} = g(X_T^{t_0,x_0}) + \int_t^T F(r, X_r^{t_0,x_0}, Y_r^{t_0,x_0}, Z_r^{t_0,x_0})dr - \int_t^T Z_r^{t_0,x_0}dW_r, \forall t \in [0,T].$$

For the case of a real-valued function F, we can find a complete approach in Kobylanski [7].

3 An existence result for QG-mFBSDE

In this Section we will provide an existence result for the solution of the BSDE (2). The main idea that will be used consists in an exponential transformation of the generator function, used on its components.

Theorem 2. *Consider the functions* $F : [0,T] \times \mathbb{R}^n \times \mathbb{R}^d \times \mathbb{R}^{d\times k} \to \mathbb{R}^d$ *satisfying hypothesis (H2) and* $g \in L^\infty(\mathbb{R}^n)$ *which give the terminal condition of the BSDE (2), where the coefficients* $b : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ *and* $\sigma : [0,T] \times \mathbb{R}^n \to \mathbb{R}^{n\times k}$ *of the forward SDE (1) satisfy hypothesis (H1). Then, the QG-mFBSDE*(F,g) *admits a unique solution*

$$(Y,Z) \in \mathcal{H}_T^\infty(\mathbb{R}^d) \times \mathcal{H}_T^2(\mathbb{R}^{d\times k}).$$

Proof. For every $i \in \overline{1,d}$, we consider the following exponential transformation

$$y_i := e^{2C_iY_i} \quad (\iff Y_i = (2C_i)^{-1} \ln y_i).$$

The obtained new generator function $f = (f_i)_{i=\overline{1,d}}$ is given by

$$f_i(t,x,y,z) := \begin{cases} 2C_i y_i F_i\left(t, x, \dfrac{\ln y}{2C}, \dfrac{z}{2Cy}\right) - \dfrac{|z_i|^2}{2y_i}, & y \in (\mathbb{R}_+^*)^d, \\ 0, & \text{elsewhere}, \end{cases}$$

where $y = (y_1, ..., y_d) \in \mathbb{R}^d$, $z = (z_1, ..., z_d)^t \in \mathbb{R}^{d\times k}$ and

$$\frac{\ln y}{2C} := \left(\frac{\ln y_1}{2C_1}, ..., \frac{\ln y_d}{2C_d}\right) \quad \text{and} \quad \frac{z}{2Cy} := \left(\frac{z_1}{2C_1y_1}, ..., \frac{z_d}{2C_dy_d}\right)^t.$$

We obtain, for $y \in (\mathbb{R}_+^*)^d$,

$$\begin{aligned} f_i(t,x,y,z) &= 2C_iy_iF_i\left(t, x, \frac{\ln y}{2C}, \frac{z}{2Cy}\right) - \frac{|z_i|^2}{2y_i} \\ &\overset{(H2-ii)}{\leq} 2C_iy_i\left[\alpha\left(1 + \frac{|\ln y_i|}{2C_i}\right) + C_i\frac{|z_i|^2}{(2C_iy_i)^2}\right] - \frac{|z_i|^2}{2y_i} \\ &= \alpha y_i(2C_i + |\ln y_i|). \end{aligned}$$

In the same manner we obtain the left inequality, therefore

$$|f_i(t,x,y,z)| \leq \alpha y_i(2C_i + |\ln y_i|), \quad \forall i = \overline{1,d},$$

hence the function f_i is bounded provided that y_i has the same property.

Considering that $g \in L^\infty(\mathbb{R}^n)$, we define, for each $i = \overline{1,d}$, $M_i := e^{\alpha T}(\alpha T + \|g_i\|_{L^\infty})$. For every $K > 0$, let $\Phi_K : [0,1] \to \mathbb{R}$, $\Phi_K \in C^\infty(\mathbb{R})$, $\Phi_K = 1$ on $[-K, K]$ and $\Phi_K = 0$ on $(-K-1, K+1)^c$ and we consider, for each i, the function

$$\varnothing_i(y) := \begin{cases} \Phi_{M_i}\left(\ln \dfrac{y}{2C_i}\right), & y \geq 0, \\ 0, & y < 0, \end{cases}$$

for which $\operatorname{supp}(\varnothing_i) \subset (e^{-2C_i(M_i+1)}, e^{2C_i(M_i+1)})$.

For each $i = \overline{1,d}$, let $\hat{f}_i : \Omega \times [0,T] \times \mathbb{R}^n \times \mathbb{R}^d \times \mathbb{R}^{d\times k} \to \mathbb{R}$,

$$\hat{f}_i(\omega, t, x, y, z) := f_i(\omega, t, x, y, z) \cdot \prod\nolimits_{j=1}^d \varnothing_j(y_j),$$

with y_j being the j^{th} component of y. The functions $\hat{f}_i$ are continuous (for every fixed (t,x)) and bounded since, when y_i is bounded, then f_i is bounded and, if $|y_i|$ exceeds $e^{2C_i(M_i+1)}$, then $\hat{f}_i = 0$. Therefore, denoting by $\hat{f} := (\hat{f}_1, ..., \hat{f}_d)$, from Hamadène, Hdhiri [4] or Hamadène, Lepeltier, Peng [5], there exists the pair

$$(y,z) \in \mathcal{H}_T^2(\mathbb{R}^d) \times \mathcal{H}_T^2(\mathbb{R}^{d\times k}) \text{ solution for } BSDE(\hat{f}, e^{2Cg}).$$

We have, for every $i = \overline{1,d}$, the following estimates, $\mathbb{P}$-a.s.:

$$|\hat{f}_i(t,x,y,z)| \leq |f_i(t,x,y,z)| \cdot |\varnothing_i(y_i)| \cdot \underbrace{\left|\prod\nolimits_{j\neq i} \varnothing_j(y_j)\right|}_{\leq 1} \leq \alpha y_i (2C_i + |\ln y_i|) \cdot \varnothing_i(y_i),$$

for every (t,x,y,z) . The functions $h_i := \alpha y_i (2C_i + |\ln y_i|) \cdot \varnothing_i(y_i)$ are bounded Lipschitz functions and, since $-h_i \leq \hat{f}_i \leq h_i$, we obtain (by the comparison principle from El Karoui, Peng, Quenez [2]) that, for each $1 \leq i \leq d$, there exist and they are unique the pairs

$$(y_{i,t}^1, 0)_{t\in[0,T]} \text{ (resp., } (y_{i,t}^2, 0)_{t\in[0,T]}) \text{ from } \mathcal{H}_T^2(\mathbb{R}^d) \times \mathcal{H}_T^2(\mathbb{R}^{d\times k}),$$

solutions for the following equations with deterministic generators and terminal conditions

$$\text{BSDE}(-h_i, e^{-2C_i\|g_i\|_{L^\infty}}) \text{ (resp., BSDE}(h_i, e^{2C_i\|g_i\|_{L^\infty}})),$$

solutions that verify, for every $1 \leq i \leq d$,

$$y_{i,t}^1 \leq y_{i,t} \leq y_{i,t}^2 \ , \quad \forall t \in [0,T]$$

Writing the second equation under differential form

$$dy_{i,t}^2 = -\alpha y_{i,t}^2 (2C_i + |\ln y_{i,t}^2|) \varnothing_i(y_{i,t}^2) dt, \quad y_{i,T}^2 = e^{2C_i\|g_i\|_{L^\infty}},$$

we can compute its explicit solution via the following system

$$\begin{cases} d\tilde{y}_{i,t}^2 = -\alpha \tilde{y}_{i,t}^2 (2C_i + \ln \tilde{y}_{i,t}^2) dt, \\ \tilde{y}_{i,T}^2 = e^{2C_i\|g_i\|_{L^\infty}}. \end{cases}$$

Denoting $w_{i,t}^2 := \ln \tilde{y}_{i,t}^2$, it follows immediately that

$$\ln(2C_i + w_{i,t}^2) = \ln 2C_i + \ln(1 + \|g_i\|_{L^\infty}),$$

equality that implies

$$0 \le \ln \tilde{y}^2_{i,t} \le e^{\alpha T}(2C_i + 2C_i \|g_i\|_{L^\infty}) - 2C_i \le 2C_i e^{\alpha T}\|g_i\|_{L^\infty} + 2C_i(e^{\alpha T} - 1)$$
$$\le 2C_i e^{\alpha T}(\alpha T + \|g_i\|_{L^\infty}) = 2C_i M_i \ ,$$

from where we have that $1 \le \tilde{y}^2_{i,t} \le e^{2C_i M_i}$, $\forall t \in [0,T]$. It follows that $\varnothing_i(\tilde{y}^2_{i,t}) = 1$, $\forall t \in [0,T]$, which means that

$$y^2_{i,t} = \tilde{y}^2_{i,t} \ , \quad \forall t \in [0,T] \quad \forall i = \overline{1,d}.$$

Exactly in the same manner, for the equation with the solution $(y^1_{i,t}, 0)_{t\in[0,T]}$, we consider the attached system

$$\begin{cases} d\tilde{y}^1_{i,t} = \alpha \tilde{y}^1_{i,t}(2C_i - \ln \tilde{y}^1_{i,t})dt, \\ \tilde{y}^1_{i,T} = e^{-2C_i\|g_i\|_{L^\infty}}, \end{cases}$$

we denote once again $w^1_{i,t} := \ln \tilde{y}^1_{i,t}$ and we obtain that $e^{-2C_i M_i} \le \tilde{y}^1_{i,t} \le 1$, which implies $\varnothing_i(\tilde{y}^1_{i,t}) = 1$, $\forall t \in [0,T]$. Therefore,

$$y^1_{i,t} = \tilde{y}^1_{i,t} \ , \quad \forall t \in [0,T]\,, \quad \forall i = \overline{1,d}.$$

Putting together the above estimations, we obtain, for every index i,

$$e^{-2C_i M_i} \le y^1_{i,t} \le y_{i,t} \le y^2_{i,t} \le e^{2C_i M_i}, \quad \forall t \in [0,T]$$

and, from the definition of $\hat{f}_i$, we find that $\hat{f}_i = f_i$ and $(y,z) \in \mathcal{H}^\infty_T(\mathbb{R}^d) \times \mathcal{H}^2_T(\mathbb{R}^{d\times k})$, i.e. (y,z) is the solution of the BSDE(f, e^{2Cg}).

To finish the proof, we denote by

$$Y := \frac{\ln y}{2C} \quad \text{and} \quad Z := \frac{z}{2Cy}.$$

It follows that $(Y,Z) \in \mathcal{H}^\infty_T(\mathbb{R}^d) \times \mathcal{H}^2_T(\mathbb{R}^{d\times k})$. We have $Y_T = g(X^{t_0,x_0}_T)$ and, by Itô's formula, we find, for every $i = \overline{1,d}$,

$$dY_{i,t} = \frac{1}{2C_i y_{i,t}} dy_{i,t} = \frac{1}{2C_i e^{2C_i Y_{i,t}}} \left[-f_i(t, X^{t_0,x_0}_t, y_{i,t}, z_{i,t})dt + z_{i,t} dW_t\right]$$
$$= \frac{1}{2C_i e^{2C_i Y_{i,t}}} \left[-2C_i e^{2C_i Y_{i,t}} F_i\left(t, X^{t_0,x_0}_t, Y_{i,t}, \frac{z_{i,t}}{2C_i y_{i,t}}\right)dt + \frac{z_{i,t}}{2C_i y_{i,t} e^{2C_i Y_{i,t}}} dW_t\right]$$
$$= -F_i(t, X^{t_0,x_0}_t, Y_{i,t}, Z_{i,t})dt + Z_{i,t} dW_t \ .$$

Therefore, (Y,Z) is the solution of the QG-mFBSDE(F,g). Since the uniqueness is clear, the proof is complete. $\square$

4 Applications

In the Introduction we discuss a few words about non-tradable assets and hedging contingent claims by portfolios which depends on correlated tradable assets. We will consider a complete financial market characterized as follows (see Imkeller, Dos Reis, Zhang [6]). We assume that there exists a risky non-tradable underlying, a risk-free account and d risky assets, correlated with the non-tradable one. Consider the Brownian motion $W = (W_1, W_2)$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$ and let $\tilde{W}_1, ..., \tilde{W}_d$ some W_1-correlated new Brownian motions, given by

$$\tilde{W}_{i,t} := \rho_i W_{1,t} + \sqrt{1-\rho_i^2} W_{2,t} \ , \quad \forall t \in [0,T] \, , \ i = \overline{1,d}. \quad (\rho_i \in [-1,1])$$

The correlation between $\tilde{W}_i$ and W_1 is ρ_i, for every $1 \leq i \leq d$. We assume that the dynamics of the non-tradable index is given by

$$dX_t^{0,x_0} = b(t, X_t^{0,x_0})dt + \sigma(t, X_t^{0,x_0})dW_{1,t}, \quad X_0^{0,x_0} = x_0, \tag{3}$$

where the real-valued coefficients b and σ satisfy hypothesis (H1) and the d risky assets are given by

$$dS_{i,t} = \alpha_i(t, X_t^{0,x_0})S_{i,t}dt + \beta_i(t, X_t^{0,x_0})S_{i,t}d\tilde{W}_{i,t}, S_{i,0} = s_{i,0}, t \in [0,T] \, , \ \forall i = \overline{1,d}$$

We suppose that α_i, $\beta_i : [0,T] \times \mathbb{R} \to \mathbb{R}$ are bounded, measurable and strictly positive functions and we denote, for every $(t,x) \in [0,T] \times \mathbb{R}$, $\theta_i(t,x) := \alpha_i(t,x) / \beta_i(t,x)$. The square integrable measurable processes λ_i, $1 \leq i \leq d$, represents the strategy of investment in the risky asset i and the sets of these admissible strategies on $[t,T]$ will be denoted by $\mathcal{A}_{i,t}$. Let ν_t the $\mathcal{F}_t$-measurable bounded r.v. - the initial investment (at time t) in the risk free account. The gain (or loss) of the investor at time s, obtained by investing in the risky asset i is given by

$$dG_{i,s} = \lambda_{i,s} \frac{dS_{i,s}}{S_{i,s}}, \quad G_{i,t} = 0, \quad s \in [t,T] \, , \quad \forall i = \overline{1,d}.$$

We will hedge European contingent claims of the form $g : \mathbb{R} \to \mathbb{R}^d$, $g = g(X_T^{0,x_0})$, with g a bounded and measurable function and, given a nonzero constant risk attitude parameter $\eta \in \mathbb{R}^d$, the investor intends to optimize the utility function $U : \mathbb{R}^d \to \mathbb{R}^d$, $U(x) := -(e^{-\eta_1 x_1}, ..., e^{-\eta_d x_d})$, where $x = (x_1, ..., x_d)$ (see El Karoui, Rouge [3] or Mania, Schweizer [9]). Therefore, we have

$$V_t := \sup_{(\lambda_1, ... \lambda_d) \in \mathcal{A}_{1,t} \times \mathcal{A}_{d,t}} \mathbb{E}\left[U\left(\nu_t \cdot \mathbf{1}_d + G_T + g(X_T^{0,x_0})\right) \Big| \mathcal{F}_t \right], \tag{4}$$

where $\mathbf{1}_d := (1, ..., 1) \in \mathbb{R}^d$ and $G_T = (G_{1,T}, ..., G_{d,T})$.

A characterization of the utility V given by (4) can be obtained via a QG-mFBSDE. For this, consider $(\mathcal{G}_s)_{s\in[0,T]}$ the natural filtration generated by the Brownian motion W_1, augmented with the set of all $\mathbb{P}$-null sets. In the perspective of Theorem 2 and of the results from [6], we have:

Proposition 3. *Consider the generator function* $F : [0,T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$, *given by components:*

$$F_i(t,x,z) := \frac{\theta_i^2\,(t,x)}{2\eta_i} - z_i\rho_i\theta_i\,(t,x) - \frac{\eta_i}{2}\left(1-\rho_i^2\right)z_i^2, \quad \forall i = \overline{1,d}. \tag{5}$$

Then, the following QG-mFBSDE(F,g)*:*

$$Y_s^{0,x_0} = g(X_T^{0,x_0}) + \int_s^T F(r, X_r^{0,x_0}, Z_r^{0,x_0})dr - \int_s^T Z_r^{0,x_0} d(\mathbf{1}_d \cdot W_{1,r}), \quad s \in [0,T], \tag{6}$$

with $(X_s^{0,x_0})_{s\in[0,T]}$ *the solution of (3), admits a unique solution* $\left(Y^{0,x_0}, Z^{0,x_0}\right) \in \mathcal{H}_T^\infty(\mathbb{R}^d) \times \mathcal{H}_T^2(\mathbb{R}^d)$ *such that* $V_t = (V_{i,t})_{i=\overline{1,d}} = \left(-e^{-\eta_i(\nu_t + Y_{i,t})}\right)_{i=\overline{1,d}}$, $\mathbb{P}$*-a.s. Moreover, the investment strategy is given by*

$$\lambda_{i,s} = -\frac{\rho_i}{\beta_i(s, X_s^{0,x_0})} Z_i^{0,x_0} + \frac{\theta_i(s, X_s^{0,x_0})}{\eta_i\beta_i(s, X_s^{0,x_0})}, \quad \forall s \le T, \quad \forall i = \overline{1,d},$$

where Z^{0,x_0} *is the control component from Eq.(6). This investment strategy is also the optimal point from (4).*

Remark 4. *It is enough to characterize the utility function* V *defined in (4) by the solution of the QG-mFBSDE*(F,g) *constructed in (6) because pricing the European contingent claim* $g(X_T^{0,x_0})$ *is made within the utility maximization paradigm based on the identity*

$$V_t(0,\nu_t) =: \bar{V}_t = \tilde{V}_t := V_t(g, \nu_t - p_t).$$

According to this identity, the investor is indifferent about a portfolio with initial endowment ν_t *without receiving one quantity of the contingent claim* $g(X_T^{0,x_0})$ *and a portfolio with initial endowment* $\nu_t - p_t$*, but receiving one quantity of the European contingent claim;* p_t *is interpreted as the time-t indifference price of the contingent claim* $g(X_T^{0,x_0})$*. The indifference price does not depend on the initial endowment* ν_t*.*

Acknowledgements. 1. The work for this paper was supported by funds from the Project POSDRU/159/1.5/S/137750, *Programe doctorale şi postdoctorale - suport pentru creşterea competitivităţii cercetării în domeniul Stiinţelor exacte.*

2. The author would like to express his gratitude to the anonymous referees for their comments and suggestions, which have resulted in the improvement of the presentation of this article.

References

[1] Ankirchner, S; Imkeller, P; Dos Reis, G. - *Pricing and hedging of derivatives based on non-tradable underlyings*, Mathematical Finance, Vol. 20, No. 2, pp. 289–312 (2010)

[2] El Karoui, N; Peng, S; Quenez, M.C. - *Backward stochastic differential equations in finance*, Math. Finance, 7(1), pp. 1–71 (1997)

[3] El Karoui, N.; Rouge, R. - *Pricing via utility maximization and entropy*, Math. Finance, 10(2), pp. 259–276 (2000)

[4] Hamadène, S.; Hdhiri I. - *Backward stochastic differential equations with two distinct reflecting barriers and quadratic growth generator*, Journal of Applied Mathematics and Stochastic Analysis, Vol. 2006 (Issue 1), pp. 1–28 (2006)

[5] Hamadène, S.; Lepeltier, J.-P.; Peng, S. - *BSDEs with continuous coefficients and stochastic differential games*, Backward Stochastic Differential Equations, editors N. El. Karoui and L. Mazliak, Longman, Harlow, pp. 115–128 (1997)

[6] Imkeller, P; Dos Reis, G; Zhang, J - *Results on numerics for FBSDE with drivers of quadratic growth,* Contemporary Quantitative Finance (Essays in Honour of Eckhard Platen), pp. 159–182 (2010)

[7] Kobylanski, M. - *Backward Stochastic Differential Equations and Partial Differential Equations with Quadratic Growth*, The Annals of Probability, Vol. 28, No. 2, pp. 558–602 (2000)

[8] Karatzas, I.; Shreve, S.E. - *Brownian motion and Stochastic Calculus*, Springer-Verlag, N.Y. (1991)

[9] Mania, M; Schweizer, M. - *Dynamic exponential utility indifference valuation*, Ann. Appl. Probab., 15(3), pp. 2113–2143 (2005)

[10] Morlais, M.-A. - *Quadratic BSDEs driven by a continuous martingale and applications to the utility maximization problem*, Finance Stoch., 13(1), pp. 121–150 (2009)

[11] Pardoux, E.; Peng, S. - *Solution of a Backward Stochastic Differential Equation, Systems and Control Letters*, 14, pp. 55–61, (1990)

Eduard Rotenstein,
Faculty of Mathematics,
"Al. I. Cuza" University of Iaşi,
9 Carol I Blvd., 700506 Iaşi, România.
Email: eduard.rotenstein@uaic.ro