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An Inequality on Quaternionic CR-Submanifolds

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Abstract

We establish an inequality for an intrinsic invariant of Chen-type defined on quaternionic CR-submanifolds in quaternionic space forms, in terms of the squared mean curvature, the main extrinsic invariant, by using the method of constrained extrema.

1 Introduction

To find simple relationships between main extrinsic invariants and the main intrinsic invariants of a submanifold represents one of the most fundamental problems in the theory of submanifolds. Among intrinsic invariants, the δ-invariants are very important because of the different nature from the classical Ricci and scalar curvature. The non-trivial δ-invariants are obtained from scalar curvature by substracting a certain amount of sectional curvatures.

In 1978, A. Bejancu [3] introduced the notion of *CR-submanifolds*, which is a generalization of holomorphic and totally real submanifolds in an almost Hermitian manifold (see also [6]).

Let $\tilde{M}$ be a Kaehler manifold with complex structure J and let M be a Riemannian manifold isometrically immersed in $\tilde{M}$. One denotes by $\mathcal{D}_x,\ x \in M$ the maximal complex subspace $T_xM \cap J(T_xM)$ of the tangent space T_xM of M. If the dimension of $\mathcal{D}_x$ is constant for all $x \in M$, then $\mathcal{D} : x \to \mathcal{D}_x$ defines a *holomorphic distribution* $\mathcal{D}$ on M. A subspace ν of $T_xM,\ x \in M$ is

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called *totally real* if $J(\nu)$ is a subspace of the normal space $T_x^\perp M$ at x. If each tangent space of M is totally real, then M is called *totally real submanifold* of the Kaehler manifold $\tilde{M}$.

If there exists a totally real distribution $\mathcal{D}^\perp$ on M whose orthogonal complement is the holomorphic distribution $\mathcal{D}$, i.e., $TM = \mathcal{D} \oplus \mathcal{D}^\perp,\ J\mathcal{D}_x^\perp \subset T_x^\perp M$, $x \in M$ then the submanifold M is called *CR-submanifold*.

The totally real distribution $\mathcal{D}^\perp$ of every CR-submanifold of a Kaehler manifold is an integrable distribution (see [4]).

According to the embedding theorem of J.F. Nash [7], every Riemannian manifold can be isometrically embedded in some Euclidean space with sufficiently high codimension. If a Riemannian manifold is regarded as a Riemannian submanifold, then one can use the extrinsic help.

In order to give answers to an open question concerning minimal immersions proposed by S.S. Chern in the 1960's and to provide applications of the Nash embedding theorem, B.-Y. Chen introduced the notion of δ-invariants. In the case of a CR-submanifold M of a Kaehler manifold, Chen introduced a δ-invariant $\delta(\mathcal{D})$, called *CR δ-invariant*, defined by

$$\delta(\mathcal{D})(x) = \tau(x) - \tau(\mathcal{D}_x),$$

where τ is the scalar curvature of M and $\tau(\mathcal{D})$ is the scalar curvature of the holomorphic distribution $\mathcal{D}$ of M.

In [1], Al-Solamy, Chen and Deshmukh proved an inequality involving the δ-invariant $\delta(\mathcal{D})$, in the case of an anti-holomorphic submanifold in a complex space form, in terms of squared mean curvature. In this paper, we consider a quaternionic CR-submanifold in a quaternionic space form with minimal codimension.

2 Basics on Quaternionic Manifolds and Their Submanifolds

Let $\tilde{M}$ be a Riemannian manifold and $M \subset \tilde{M}$ a Riemannian submanifold of $\tilde{M}$ with the Riemannian metric induced by the metric of $\tilde{M}$. We denote by TM and $T^\perp M$ the tangent bundle, respectively the normal bundle of M, with ∇ and $\tilde{\nabla}$ the Levi-Civita connections of M, respectively $\tilde{M}$.

The Gauss and Weingarten formulae are given by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X,Y)\ ,$$

$$\tilde{\nabla}_X V = -A_V X + \nabla_X^\perp V,$$

$\forall X, Y \in \Gamma(TM),\ V \in \Gamma(T^\perp M)$, where $\nabla^\perp$ is the normal connection on $T^\perp M$.

We also have the relation

$$g(h(X,Y),V) = g(A_V X, Y).$$

If $\tilde{M}$ is a $4m$-dimensional manifold with the Riemannian metric $\tilde{g}$, then $\tilde{M}$ is called a *quaternionic Kaehler manifold* if there exist a 3-dimensional vector bundle σ of type (1,1) with local basis of almost Hermitian structures J_1, J_2, J_3 such that

$$J_\alpha \circ J_{\alpha+1} = -J_{\alpha+1} \circ J_\alpha = J_{\alpha+2},\ J_\alpha^2 = -\operatorname{Id},$$

where $\alpha, \alpha+1, \alpha+2$ are taken modulo 3.

In this case, σ is called the *almost quaternionic structures on* $\tilde{M}$, $\{J_1, J_2, J_3\}$ is the *canonical local basis of* σ. So, $(\tilde{M}, \sigma)$ is called an *almost quaternionic manifold*, with $\dim \tilde{M} = 4m,\ m \geq 1$.

A Riemannian metric $\tilde{g}$ on $\tilde{M}$ is said to be *adapted to the almost quaternionic structure* σ if it satisfies

$$\tilde{g}(J_\alpha X, J_\alpha Y) = \tilde{g}(X,Y),\ \forall \alpha = \overline{1,3}.$$

If σ is parallel with respect to $\tilde{\nabla}$ of $\tilde{g}$, then $(\tilde{M}, \sigma, \tilde{g})$ is called *quaternionic Kaehler manifold*. Equivalent, locally defined 1-forms $\omega_1, \omega_2, \omega_3$ exists such that $\forall \alpha = \overline{1,3},\ \tilde{\nabla}_X J_\alpha = \omega_{\alpha+2}(X) J_{\alpha+1} - \omega_{\alpha+1}(X) J_{\alpha+2}$, where $\alpha, \alpha+1, \alpha+2$ are taken modulo 3.

Remark. Any quaternionic Kaehler manifold $\tilde{M}$ ($\dim \tilde{M} \geq 4$) is an Einstein manifold.

Let $(\tilde{M}, \sigma, \tilde{g})$ be a quaternionic Kaehler manifold and X be a non-null vector on $\tilde{M}$. Then the 4-plane spanned by $\{X, J_1X, J_2X, J_3X\}$ denoted by $Q(X)$ is called a *quaternionic* 4*-plane*. Any 2-plane in $Q(X)$ is called a *quaternionic plane*. The sectional curvature of a quaternionic plane is called a *quaternionic sectional curvature*.

A quaternionic Kaehler manifold is called a *quaternionic space form* if its quaternionic sectional curvature is constant, say c. $(\tilde{M}, \sigma, \tilde{g})$ is a quaternionic space form if and only if

$$\tilde{R}(X,Y)Z = \frac{c}{4}\left\{\tilde{g}(Y,Z)X - \tilde{g}(X,Z)Y + \sum_{\alpha=1}^{3} [\tilde{g}(Z, J_\alpha Y) J_\alpha X - \right.$$

$$\left. - \tilde{g}(Z, J_\alpha X) J_\alpha Y + 2\tilde{g}(X, J_\alpha Y) J_\alpha Z]\right\},$$

$\forall X, Y, Z \in \Gamma(T\tilde{M})$.

For a submanifold M of $\tilde{M}$, if $\{e_1, \ldots, e_n\}$ is an orthonormal basis of T_pM and $\{e_{n+1}, \ldots, e_{4m}\}$ is an orthonormal basis of $T_p^{\perp}M,\ p \in M$,

$$H(p) = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i)$$

represents the *mean curvature vector*.

One denotes by

$$h_{ij}^r = g(h(e_i, e_j), e_r),\ i, j = \overline{1, n},\ r = \overline{n+1, 4m},$$

$$\|h\|^2(p) = \sum_{i,j=1}^{n} g(h(e_i, e_j), h(e_i, e_j)).$$

For a quaternionic Kaehler manifold, we have

$$\tilde{\nabla}_X J_\alpha = \sum_{\beta=1}^{3} Q_{\alpha\beta}(X) J_\beta,\ \alpha = \overline{1, 3},\ \forall X \in \Gamma(T\tilde{M}),$$

where $Q_{\alpha\beta}$ are certain 1-forms locally defined on $\tilde{M}$ such that $Q_{\alpha\beta} + Q_{\beta\alpha} = 0$.

Let $\tilde{M}$ be a quaternionic Kaehler manifold and M be a real submanifold of $\tilde{M}$. A distribution $\mathcal{D} : x \to \mathcal{D}_x \subset T_xM$ is called a *quaternionic distribution* if $J_\alpha(\mathcal{D}) \subset \mathcal{D},\ \forall \alpha = 1, 2, 3$, so $\mathcal{D}$ is carried into itself by the quaternionic structure.

M is called a *quaternionic CR-submanifold* if it admits a differential quaternionic distribution $\mathcal{D}$ such that its orthogonal complementary distribution $\mathcal{D}^{\perp}$ is totally real, i.e., $J_\alpha(\mathcal{D}_x^{\perp}) \subset T_x^{\perp}M,\ \alpha = 1, 2, 3,\ \forall x \in M$.

A submanifold M in a quaternionic manifold $\tilde{M}$ is called *quaternionic submanifold* (respectively, a *totally real submanifold*) if $\dim \mathcal{D}_x^{\perp} = 0$ (respectively, $\dim \mathcal{D}_x = 0$). A quaternionic CR-submanifold is called *proper* if it is neither totally real nor quaternionic.

Let $\mathcal{D}_{\alpha x} = J_\alpha(\mathcal{D}_x^{\perp}),\ \nu_x^{\perp} = \mathcal{D}_{1x} \oplus \mathcal{D}_{2x} \oplus \mathcal{D}_{3x}$ a $3q$-dimensional distribution $\nu^{\perp} : x \to \nu_x^{\perp}$ globally defined on M, where $q = \dim \mathcal{D}_x^{\perp}$ and ν the orthogonal complementary distribution of $\nu^{\perp}$.

Then

$$T\tilde{M} = TM \oplus T^{\perp}M,\ TM = \mathcal{D} \oplus \mathcal{D}^{\perp},$$

$$T^{\perp}M = \nu \oplus \nu^{\perp},\ \nu, \nu^{\perp} \subset T^{\perp}M,\ \nu_x^{\perp} = \mathcal{D}_{1x} \oplus \mathcal{D}_{2x} \oplus \mathcal{D}_{3x}.$$

M is called *mixed geodesic* if $h(X, Y) = 0,\ \forall X \in \Gamma(\mathcal{D}),\ Y \in \Gamma(\mathcal{D}^{\perp})$.

M is called $\mathcal{D}$*-geodesic* if $h(X, Y) = 0,\ \forall X, Y \in \Gamma(\mathcal{D})$.

Let $\pi = \mathrm{Sp}\{X, Y\}$ be a tangent plane to $\tilde{M}$ at a point $p \in \tilde{M}$. The sectional curvature of π is

$$K(\pi) = \frac{\tilde{R}(X,Y,X,Y)}{\tilde{g}(X,X)\tilde{g}(Y,Y) - \tilde{g}^2(X,Y)}.$$

Thus, we obtain

$$\tilde{K}(X \wedge Y) = \frac{c}{4}\left[1 + 3\sum_{\alpha=1}^{3} \tilde{g}^2(J_\alpha X, Y)\right],$$

$\forall X, Y \in \Gamma(T\tilde{M})$ unit vector fields; moreover, from the Gauss equation, we have

$$K(X \wedge Y) = \tilde{K}(X \wedge Y) + \tilde{g}(h(X,X), h(Y,Y)) - \tilde{g}(h(X,Y), h(X,Y)).$$

In this article we will use the convention $R(X,Y,Z,W) = g(R(X,Y)W,Z)$ and similar for $\tilde{R}$.

3 A Chen Invariant and a Chen-type Inequality

The CR δ-invariant $\delta(\mathcal{D})$ ([5], [1]) is given by

$$\delta(\mathcal{D})(x) = \tau(x) - \tau(\mathcal{D}_x),\ x \in \tilde{M},$$

where τ and $\tau(\mathcal{D})$ denote the scalar curvature of M and the scalar curvature of the quaternionic distribution $\mathcal{D} \subset TM$, respectively.

If M is a quaternionic CR-submanifold of minimal codimension, i.e., $\dim \nu_x = 0$ for $x \in M$, we choose the following orthonormal basis:

$$\mathcal{D}_x = Sp\{\ e_1, \ldots, e_n\},$$

$$\mathcal{D}_x^\perp = Sp\{e_{n+1}, \ldots, e_{n+q}\},$$

and then

$$TM = Sp\{e_1, \ldots, e_n;\ e_{n+1}, \ldots, e_{n+q}\},$$

$$T^\perp M = Sp\{J_1 e_{n+1}, \ldots, J_1 e_{n+q};\ J_2 e_{n+1}, \ldots, J_2 e_{n+q};\ J_3 e_{n+1}, \ldots, J_3 e_{n+q}\},$$

which correspond to the definition of a quaternionic CR-submanifold given in [2].

For $x \in M$, we have

$$\dim \mathcal{D}_x = n;\ \dim \mathcal{D}_x^{\perp} = q;\ \dim T_x M = n + q;$$

$$\dim \nu_x = 0,\ \dim T_x^{\perp} M = 3q = \dim \nu_x^{\perp}.$$

We will use the following convention on range of indices, unless mentioned otherwise:

$$i, j, k = \overline{1, n}\ ;\ \alpha, \beta, \gamma = \overline{1, 3}\ ;\ r, s, t = \overline{n+1, n+q}\ ;\ A, B, C = \overline{1, n+q}.$$

In [1], the authors proved an inequality for $\delta(\mathcal{D})$ in case of an anti-holomorphic submanifold of a complex space form:

Theorem 3.1. [1] *Let M be an anti-holomorphic submanifold of a complex space form $\tilde{M}^{h+p}(c)$ with $h = \operatorname{rank}_{\mathbb{C}} \mathcal{D} \geq 1$ and $p = \operatorname{rank} \mathcal{D}^{\perp} \geq 2$. Then we have*

$$\delta(\mathcal{D}) \leq \frac{(2h+p)^2}{2} \cdot \frac{p-1}{p+2} \|H\|^2 + \frac{p(4h+p-1)}{2} \cdot \frac{c}{4}.$$

The equality sign holds identically if and only if the following three conditions are satisfied:
(a) M is $\mathcal{D}$-minimal,
(b) M is mixed totally geodesic, and
(c) there exists an orthonormal frame $\{e_{2h+1}, \ldots, e_n\}$ of $\mathcal{D}^{\perp}$ such that the second fundamental form σ of M satisfies
$\sigma_{rr}^r = 3\sigma_{ss}^r$, for $2h+1 \leq r \neq s \leq 2h+p$, and
$\sigma_{rs}^t = 0$ for distinct $r, s, t \in \{2h+1, \ldots, 2h+p\}$.

In this paper we prove a corresponding inequality for a quaternionic CR-submanifold with minimal codimension of a quaternionic space form, by using a different method, more precisely the method of constrained extrema.

Theorem 3.2. *If M is a quaternionic CR-submanifold of a quaternionic space form $\tilde{M}$, of minimal codimension, i.e. $\dim \nu_x = 0$, for $x \in M$, $\dim \mathcal{D}_x = n$, $\dim \mathcal{D}_x^{\perp} = q$ and $\dim \nu_x^{\perp} = 3q = \dim T_x^{\perp} M$ then*

$$(*) \qquad \delta(\mathcal{D}) \leq \frac{(n+q)^2}{2} \cdot \frac{q+2}{q+5} \|H\|^2 + \frac{q(2q+n-1)}{2} \cdot \frac{c}{4}.$$

The equality sign holds at a point $x \in M$ if and only if the following conditions are satisfied:
a) M is mixed totally geodesic;

b) there is an orthonormal basis $\{e_1, e_2, \dots, e_{n+q}\}$ *at* x *such that with respect to this basis the second fundamental form* h *satisfies the following conditions:*

(i)

$$\sum_{i=1}^{n} \tilde{g}(h(e_i, e_i), J_\alpha e_r) = \tilde{g}(h(e_r, e_r), J_\alpha e_r) = 3\tilde{g}(h(e_s, e_s), J_\alpha e_r),$$

$\forall \alpha = \overline{1, 3},\ \forall r \neq s \in \{n+1, \dots, n+q\}$,

(ii)

$$\tilde{g}(h(e_r, e_s), J_\alpha e_t) = 0,$$

$\forall \alpha = \overline{1, 3},\ r, s, t = \overline{n+1, n+q},\ r \neq s \neq t \neq r.$

Proof.

With the above notations and orthonormal basis, we have

$$\tau(x) = \sum_{1 \le i < j \le n} K(e_i \wedge e_j) + \sum_{n+1 \le r < s \le n+q} K(e_r \wedge e_s) + \sum_{i=1}^{n} \sum_{r=n+1}^{n+q} K(e_i \wedge e_r),$$

$$\tau(\mathcal{D}_x) = \sum_{1 \le i < j \le n} K(e_i \wedge e_j).$$

From these two relations, we obtain

$$\delta(\mathcal{D})(x) = \sum_{i=1}^{n} \sum_{r=n+1}^{n+q} K(e_i \wedge e_r) + \sum_{n+1 \le r < s \le n+q} K(e_r \wedge e_s). \tag{3.1}$$

From the Gauss equation (see also section 2), one obtains:

$$K(X \wedge Y) = \frac{c}{4}\left[1 + 3\sum_{\alpha=1}^{3} \tilde{g}^2(J_\alpha X, Y)\right] + \tag{3.2}$$

$$+\tilde{g}(h(X, X), h(Y, Y)) - \tilde{g}(h(X, Y), h(X, Y)).$$

Applying (3.2) for $X = e_i,\ Y = e_r,\ i = \overline{1, n},\ r = \overline{n+1, n+q}$ we obtain

$$K(e_i \wedge e_r) = \frac{c}{4}\left[1 + 3\sum_{\alpha=1}^{3} \tilde{g}^2(J_\alpha e_i, e_r)\right] + \tag{3.3}$$

$$+\tilde{g}(h(e_i,e_i),h(e_r,e_r))-\tilde{g}(h(e_i,e_r),h(e_i,e_r)).$$

Because $J_\alpha e_i \in \mathcal{D}$ and $e_r \in \mathcal{D}^\perp$, we have

$$\tilde{g}(J_\alpha e_i, e_r)=0$$

and it follows that

$$K(e_i\wedge e_r)=\frac{c}{4}+\tilde{g}(h(e_i,e_i),h(e_r,e_r))-\tilde{g}(e_i,e_r),h(e_i,e_r)). \tag{3.4}$$

By summation in (3.4) over $i=\overline{1,n}$, and $r=\overline{n+1,n+q}$ we find

$$\sum_{i=1}^{n}\sum_{r=n+1}^{n+q}K(e_i\wedge e_r)=nq\frac{c}{4}+ \tag{3.5}$$

$$+\sum_{i=1}^{n}\sum_{r=n+1}^{n+q}\left[\tilde{g}(h(e_i,e_i),h(e_r,e_r))-\tilde{g}(h(e_i,e_r),h(e_i,e_r))\right].$$

Applying Gauss equation for $X=e_r,\ Y=e_s,\ r,s=\overline{n+1,n+q},\ r\neq s$, we get

$$K(e_r\wedge e_s)=\frac{c}{4}\left[1+3\sum_{\alpha=1}^{3}\tilde{g}^2(J_\alpha e_r,e_s)\right]+ \tag{3.6}$$

$$+\tilde{g}(h(e_r,e_r),h(e_s,e_s))-\tilde{g}(h(e_r,e_s),h(e_r,e_s)).$$

Because $J_\alpha e_r \in T^\perp M$ and $e_s\in TM,\ \alpha=1,3,\ r,s=\overline{n+1,n+q},\ r\neq s$, it follows that $\tilde{g}(J_\alpha e_r,e_s)=0$.

By summation in (3.6) over $r,s=\overline{n+1,n+q},\ r<s$ we obtain

$$\sum_{n+1\le r<s\le n+q}K(e_r\wedge e_s)=\frac{q(q-1)}{2}\frac{c}{4}+ \tag{3.7}$$

$$+\sum_{n+1\le r<s\le n+q}\left[\tilde{g}(h(e_r,e_r),h(e_s,e_s))-\tilde{g}(h(e_r,e_s),h(e_r,e_s))\right].$$

Using the relations (3.5) and (3.7) in (3.1), it follows that

$$\delta(\mathcal{D})(x)=\frac{q(q-1)}{2}\frac{c}{4}+nq\cdot\frac{c}{4}+\sum_{n+1\le r<s\le n+q}\tilde{g}(h(e_r,e_r),h(e_s,e_s))+ \tag{3.8}$$

$$+\sum_{i=1}^{n}\sum_{r=n+1}^{n+q}\tilde{g}(h(e_i,e_i),h(e_r,e_r))-\sum_{n+1\leq r<s\leq n+q}\|h(e_r,e_s)\|^2-$$

$$-\sum_{i=1}^{n}\sum_{r=n+1}^{n+q}\|h(e_i,e_r)\|^2.$$

Thus, we get

$$\delta(\mathcal{D})(x)=\sum_{n+1\leq r<s\leq n+q}\tilde{g}(h(e_r,e_r),h(e_s,e_s))+\sum_{i=1}^{n}\sum_{r=n+1}^{n+q}\tilde{g}(h(e_i,e_i),h(e_r,e_r))-$$

$$-\sum_{n+1\leq r<s\leq n+q}\|h(e_r,e_s)\|^2-\sum_{i=1}^{n}\sum_{r=n+1}^{n+q}\|h(e_i,e_r)\|^2+\frac{q(2n+q-1)}{2}\cdot\frac{c}{4},$$

which implies

$$\delta(\mathcal{D})(x)\leq\frac{q(2n+q-1)}{2}\cdot\frac{c}{4}+\sum_{n+1\leq r<s\leq n+q}\tilde{g}(h(e_r,e_r),h(e_s,e_s))+ \tag{3.9}$$

$$+\sum_{i=1}^{n}\sum_{r=n+1}^{n+q}\tilde{g}(h(e_i,e_i),h(e_r,e_r))-\sum_{n+1\leq r<s\leq n+q}\|h(e_r,e_s)\|^2.$$

We denote

$$h^t_{AB}=\tilde{g}(h(e_A,e_B),J_1e_t),$$
$$\tilde{h}^t_{AB}=\tilde{g}(h(e_A,e_B),J_2e_t),$$
$$\tilde{\tilde{h}}^t_{AB}=\tilde{g}(h(e_A,e_B),J_3e_t),$$

where $A,B=\overline{1,n+q},\ t=\overline{n+1,n+q}$.

From (3.9), we get

$$\delta(\mathcal{D})(x)\leq\frac{q(2n+q-1)}{2}\cdot\frac{c}{4}+ \tag{3.10}$$

$$+\sum_{n+1\leq r<s\leq n+q}\sum_{t=n+1}^{n+q}(h^t_{rr}h^t_{ss}+\tilde{h}^t_{rr}\tilde{h}^t_{ss}+\tilde{\tilde{h}}^t_{rr}\tilde{\tilde{h}}^t_{ss})+$$

$$+\sum_{i=1}^{n}\sum_{r=n+1}^{n+q}\sum_{t=n+1}^{n+q}(h_{ii}^t h_{rr}^t+\tilde{h}_{ii}^t\tilde{h}_{rr}^t+\tilde{\tilde{h}}_{ii}^t\tilde{\tilde{h}}_{rr}^t)-$$

$$-\sum_{n+1\le r<s\le n+q}\sum_{t=n+1}^{n+q}\left[(h_{rs}^t)^2+(\tilde{h}_{rs}^t)^2+(\tilde{\tilde{h}}_{rs}^t)^2\right].$$

We consider the following sums (3.11.1)-(3.11.3):

$$S=\sum_{n+1\le r<s\le n+q}\sum_{t=n+1}^{n+q}h_{rr}^t h_{ss}^t+\sum_{i=1}^{n}\sum_{r=n+1}^{n+q}\sum_{t=n+1}^{n+q}h_{ii}^t h_{rr}^t-$$
$$-\sum_{n+1\le r<s\le n+q}\sum_{t=n+1}^{n+q}(h_{rs}^t)^2, \tag{3.11.1}$$

$$\tilde{S}=\sum_{n+1\le r<s\le n+q}\sum_{t=n+1}^{n+q}\tilde{h}_{rr}^t \tilde{h}_{ss}^t+\sum_{i=1}^{n}\sum_{r=n+1}^{n+q}\sum_{t=n+1}^{n+q}\tilde{h}_{ii}^t \tilde{h}_{rr}^t-$$
$$-\sum_{n+1\le r<s\le n+q}\sum_{t=n+1}^{n+q}(\tilde{h}_{rs}^t)^2, \tag{3.11.2}$$

$$\tilde{\tilde{S}}=\sum_{n+1\le r<s\le n+q}\sum_{t=n+1}^{n+q}\tilde{\tilde{h}}_{rr}^t \tilde{\tilde{h}}_{ss}^t+\sum_{i=1}^{n}\sum_{r=n+1}^{n+q}\sum_{t=n+1}^{n+q}\tilde{\tilde{h}}_{ii}^t \tilde{\tilde{h}}_{rr}^t-$$
$$-\sum_{n+1\le r<s\le n+q}\sum_{t=n+1}^{n+q}(\tilde{\tilde{h}}_{rs}^t)^2. \tag{3.11.3}$$

From the relations (3.10) and (3.11) we obtain

$$\delta(\mathcal{D})(x)\le\frac{q(2n+q-1)}{2}\cdot\frac{c}{4}+S+\tilde{S}+\tilde{\tilde{S}}. \tag{3.12}$$

For each of the sums S, $\tilde{S}$ and $\tilde{\tilde{S}}$ we must find the maximum. Let's consider first the sum S (we'll proceed in the same manner for the other two sums $\tilde{S}$ and $\tilde{\tilde{S}}$).

$$S=\sum_{n+1\le r<s\le n+q}\sum_{t=n+1}^{n+q}h_{rr}^t h_{ss}^t+\sum_{i=1}^{n}\sum_{r=n+1}^{n+q}\sum_{t=n+1}^{n+q}h_{ii}^t h_{rr}^t-$$

$$-\sum_{n+1\le r<s\le n+q}\sum_{t=n+1}^{n+q}(h^t_{rs})^2=$$

$$=\sum_{n+1\le r<s\le n+q}(h^r_{rr}h^r_{ss}+h^s_{rr}h^s_{ss})+\sum_{n+1\le r<s\le n+q}\sum_{n+1\le t\le n+q}^{t\notin\{r,s\}}h^t_{rr}h^t_{ss}+$$

$$+\sum_{i=1}^{n}\sum_{r=n+1}^{n+q}h^r_{ii}h^r_{rr}+\sum_{i=1}^{n}\sum_{n+1\le r<t\le n+q}h^t_{ii}h^t_{rr}-$$

$$-\sum_{n+1\le r<s\le n+q}\left[(h^r_{rs})^2+(h^s_{rs})^2\right]-\sum_{n+1\le r<s\le n+q}\sum_{n+1\le t\le n+q}^{t\notin\{r,s\}}(h^t_{rs})^2,$$

which implies

$$S\le\sum_{n+1\le r<s\le n+q}(h^r_{rr}h^r_{ss}+h^s_{rr}h^s_{ss})+\sum_{n+1\le r<s\le n+q}\sum_{n+1\le t\le n+q}^{t\notin\{r,s\}}h^t_{rr}h^t_{ss}+ \tag{3.13}$$

$$+\sum_{i=1}^{n}\sum_{r=n+1}^{n+q}h^r_{ii}h^r_{rr}+\sum_{i=1}^{n}\sum_{n+1\le r<t\le n+q}h^t_{ii}h^t_{rr}-\sum_{n+1\le r<s\le n+q}\left[(h^s_{rr})^2+(h^r_{ss})^2\right].$$

We recall the following result.

Let (M,g) be a Riemannian submanifold of a Riemannian manifold $(\tilde{M},\tilde{g})$ and $f\in\mathcal{C}^\infty(\tilde{M})$, with the attached optimum problem:

$$\min_{x\in M} f(x). \tag{3.14}$$

Theorem. [8] *If $x_0\in M$ is a solution of the problem* (3.14)*, then*
(a) $(\operatorname{grad} f)(x_0)\in T^\perp_{x_0}M$;
(b) The bilinear form $\alpha : T_{x_0}M\times T_{x_0}M\to\mathbb{R}$,

$$\alpha(X,Y)=\operatorname{Hess}_f(X,Y)+\tilde{g}(h(X,Y),(\operatorname{grad} f)(x_0))$$

is semipositive definite, where h is the second fundamental form of the submanifold M in $\tilde{M}$.

For $t=\overline{n+1,n+q}$, we consider the quadratic forms $f_t:\mathbb{R}^{n+q}\to\mathbb{R}$ defined by

$$f_t(h^t_{11},h^t_{22},\ldots,h^t_{nn},h^t_{n+1n+1},\ldots,h^t_{n+qn+q})= \tag{3.15}$$

$$= \sum_{n+1\le r<s\le n+q} h^t_{rr}h^t_{ss} + \sum_{i=1}^{n}\sum_{r=n+1}^{n+q} h^t_{ii}h^t_{rr} - \sum_{n+1\le r\le n+q}^{r\neq t} \left(h^t_{rr}\right)^2.$$

Now, we consider f_{n+1} as

$$f_{n+1}(h^{n+1}_{11}, h^{n+1}_{22}, \ldots, h^{n+1}_{nn}, h^{n+1}_{n+1n+1}, \ldots, h^{n+1}_{n+qn+q}) =$$

$$= \sum_{n+1\le r<s\le n+q} h^{n+1}_{rr}h^{n+1}_{ss} + \sum_{i=1}^{n}\sum_{r=n+1}^{n+q} h^{n+1}_{ii}h^{n+1}_{rr} - \sum_{n+2\le r\le n+q} \left(h^{n+1}_{rr}\right)^2.$$

We must find an upper bound for f_{n+1}, subject to

$$P : h^{n+1}_{11} + h^{n+1}_{22} + \ldots + h^{n+1}_{nn} + h^{n+1}_{n+1n+1} + \ldots + h^{n+1}_{n+qn+q} = c^{n+1}, \tag{3.16}$$

where c^{n+1} is a real number.

The bilinear form $\alpha : T_xP \times T_xP \to \mathbb{R}$ has the expression

$$\alpha(X, Y) = \mathrm{Hess}(f_r)(X, Y) + \langle h'(X, Y), \operatorname{grad} f_r(q)\rangle,$$

where h' is the second fundamental form of P in $\mathbb{R}^{n+q}$ and $\langle\ ,\ \rangle$ is the standard inner-product on $\mathbb{R}^{n+q}$.

Searching for the partial derivatives of the function f_{n+1}, we get

$$\frac{\partial f_{n+1}}{\partial h^{n+1}_{ii}} = \sum_{r=n+1}^{n+q} h^{n+1}_{rr},\ i = \overline{1, n},$$

$$\frac{\partial f_{n+1}}{\partial h^{n+1}_{n+1n+1}} = \sum_{s=n+2}^{n+q} h^{n+1}_{ss} + \sum_{i=1}^{n} h^{n+1}_{ii},$$

$$\frac{\partial f_{n+1}}{\partial h^{n+1}_{rr}} = \sum_{n+1\le s\le n+q}^{s\neq n+2} h^{n+1}_{ss} + \sum_{i=1}^{n} h^{n+1}_{ii} - 2h^{n+1}_{rr},\ r = \overline{n+2, n+q}.$$

In the standard frame of $\mathbb{R}^{n+q}$, the Hessian of f_{n+1} has the matrix

$$\begin{pmatrix} O_n & A & B \\ A^t & 0 & C \\ B^t & C^t & D \end{pmatrix},$$

where $O_n \in \mathcal{M}_n(\mathbb{R})$, with all the elements equals to 0, $A \in \mathcal{M}_{n,1}(\mathbb{R})$, with all the elements equals to 1, $B \in \mathcal{M}_{n,q-1}(\mathbb{R})$, with all the elements equals to 1,

$C \in \mathcal{M}_{1,q-1}(\mathbb{R})$, with all the elements equals to 1 and D is the matrix

$$D = \begin{pmatrix} -2 & 1 & 1 & \dots & 1 \\ 1 & -2 & 1 & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 1 & 1 & \dots & -2 \end{pmatrix}, \quad D \in \mathcal{M}_{q-1,q-1}(\mathbb{R}) .$$

As P is totally geodesic in $\mathbb{R}^{n+q}$ (see [8]), we obtain

$$\alpha(X,X) = 2\sum_{r=n+1}^{n+q} X_1X_r + 2\sum_{r=n+1}^{n+q} X_2X_r + \dots + 2\sum_{r=n+1}^{n+q} X_nX_r +$$

$$+2\sum_{r=n+1}^{n+q} X_{n+1}X_r + 2\sum_{n+2\le r<s\le n+q} X_rX_s - 2\sum_{n+2}^{n+q}(X_r)^2 =$$

$$= \left(\sum_{A=1}^{n+q} X_A\right)^2 - 2\sum_{1\le i<j\le n} X_iX_j - \sum_{i=1}^{n}(X_i)^2 - (X_{n+1})^2 - 3\sum_{r=n+2}^{n+q}(X_r)^2 =$$

$$= \left(\sum_{A=1}^{n+q} X_A\right)^2 - \left(\sum_{i=1}^{n} X_i\right)^2 - (X_{n+1})^2 - 3\sum_{r=n+2}^{n+q}(X_r)^2 < 0$$

and then the Hessian of f_{n+1} is negative semidefinite.

Searching for the critical point $(h_{11}^{n+1}, h_{22}^{n+1}, \dots, h_{n+qn+q}^{n+1})$ of f_{n+1}, we find

$$\frac{\partial f_{n+1}}{\partial h_{11}^{n+1}} = \frac{\partial f_{n+1}}{\partial h_{n+1n+1}^{n+1}} \implies$$

$$h_{n+1n+1}^{n+1} = \sum_{i+1}^{n} h_{ii}^{n+1} = 3\lambda. \tag{3.17}$$

$$\frac{\partial f_{n+1}}{\partial h_{n+2n+2}^{n+1}} = \frac{\partial f_{n+1}}{\partial h_{n+3n+3}^{n+1}} \implies$$

$$h_{n+2n+2}^{n+1} = h_{n+3n+3}^{n+1} = \dots = h_{n+qn+q}^{n+1} = \lambda. \tag{3.18}$$

$$\frac{\partial f_{n+1}}{\partial h_{n+2n+2}^{n+1}} = \frac{\partial f_{n+1}}{\partial h_{n+1n+1}^{n+1}} \implies$$

$$h_{n+1n+1}^{n+1} = 3h_{n+2n+2}^{n+1} = 3\lambda. \tag{3.19}$$

From (3.16), (3.17), (3.18) and (3.19) we obtain

$$3\lambda + 3\lambda + (q-1)\lambda = c^{n+1} \implies \lambda = \frac{c^{n+1}}{q+5},$$

which gives

$$\sum_{i+1}^{n} h_{ii}^{n+1} = h_{n+1n+1}^{n+1} = \frac{3c^{n+1}}{q+5} \tag{3.20}$$

and

$$h_{n+2n+2}^{n+1} = h_{n+3n+3}^{n+1} = \ldots = h_{n+qn+q}^{n+1} = \frac{c^{n+1}}{q+5}. \tag{3.21}$$

Using the relations (3.20) and (3.21) in the expression of f_{n+1} we have

$$f_{n+1}(h_{11}^{n+1}, h_{22}^{n+1}, \ldots, h_{nn}^{n+1}, h_{n+1n+1}^{n+1}, \ldots, h_{n+qn+q}^{n+1}) \leq \tag{3.22}$$

$$\leq \frac{3c^{n+1}}{q+5} \cdot (q-1) \cdot \frac{c^{n+1}}{q+5} + \frac{(q-1)(q-2)}{2} \cdot \left(\frac{c^{n+1}}{q+5}\right)^2 + \left(\frac{3c^{n+1}}{q+5}\right)^2 +$$

$$+(q-1) \cdot \frac{3c^{n+1}}{q+5} \cdot \frac{c^{n+1}}{q+5} - (q-1) \cdot \left(\frac{c^{n+1}}{q+5}\right)^2 =$$

$$= \left(\frac{c^{n+1}}{q+5}\right)^2 \cdot \frac{6q-6+q^2-3q+2+18+6q-6-2q+2}{2} =$$

$$= \left(\frac{c^{n+1}}{q+5}\right)^2 \cdot \frac{q^2+7q+10}{2} = \left(\frac{c^{n+1}}{q+5}\right)^2 \cdot \frac{(q+2)(q+5)}{2} = (c^{n+1})^2 \cdot \frac{q+2}{2(q+5)}.$$

Then

$$f_{n+1} \leq \frac{q+2}{2(q+5)} \cdot (n+q)^2 (H^{n+1})^2, \tag{3.23}$$

where $H^{n+1} = \dfrac{1}{n+q} \displaystyle\sum_{A=1}^{n+q} h_{AA}^{n+1}$.

In a similar manner one can prove that

$$f_r \leq \frac{q+2}{2(q+5)} \cdot (n+q)^2 (H^r)^2, \ \forall r = \overline{n+1, n+q}, \tag{3.24}$$

where $H^r = \dfrac{1}{n+q} \displaystyle\sum_{A=1}^{n+q} h_{AA}^r$.

For $\tilde{S}$ we have the functions $\tilde{f}_r$ and the relations

$$\tilde{f}_r \leq \frac{q+2}{2(q+5)} \cdot (n+q)^2 (\tilde{H}^r)^2, \ \forall r = \overline{n+1, n+q}, \tag{3.25}$$

where $\tilde{H}^r = \frac{1}{n+q} \sum_{A=1}^{n+q} \tilde{h}^r_{AA}$ and for $\tilde{\tilde{S}}$ we have the functions $\tilde{\tilde{f}}_r$ and the relations

$$\tilde{\tilde{f}}_r \leq \frac{q+2}{2(q+5)} \cdot (n+q)^2 (\tilde{\tilde{H}}^r)^2, \ \forall r = \overline{n+1, n+q}, \tag{3.26}$$

where $\tilde{\tilde{H}}^r = \frac{1}{n+q} \sum_{A=1}^{n+q} \tilde{\tilde{h}}^r_{AA}$.

Using the relations (3.24), (3.25) and (3.26) in (3.12), we get the relation (*); the conditions for the equality case are obtained from (3.9), (3.13), (3.20) and (3.21).

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