



Some classes of statistical distributions. Properties and Applications

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Abstract

We propose a new method of constructing statistical models which can be interpreted as the lifetime distributions of series-parallel/parallel-series systems used in characterizing coherent systems. An open problem regarding coherent systems is comparing the expected system lifetimes. Using these models, we discuss and establish conditions for ordering of expected system lifetimes of complex series-parallel/parallel-series systems. Also, we consider parameter estimation and the analysis of two real data sets. We give formulae for the reliability, hazard rate and mean hazard rate functions.

1 Introduction

Recently, several methods for deriving new parametric families of probability distributions attracted a special interest, due to their use for the development of wider statistical applications. Transmutation maps [18] represent techniques for introducing skewness or kurtosis into a symmetric or other distribution. They are based on the functional composition of the cumulative distribution function of one distribution with the inverse cumulative distribution function of another. This method can be used in Monte Carlo simulations and

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copula applications. Parametric distributions which can be obtained by transmutation include the skew-uniform [15], skew-exponential, skew-normal and skew-kurtosis-normal [18]. Many statisticians have used transmutation maps, but especially the quadratic rank transmutation map, applied to generalize well-known distributions. We can mention here the transmuted exponentiated exponential distribution introduced in 2013 by Merovci [11], the transmuted Pareto distribution, proposed in 2014 by Merovci and Puka [12] and many others. The rank transmutation map is used as a tool for generating new families of non-Gaussian distributions, by modulating a given base distribution with the aim of modifying its moments, in particular skewness and kurtosis.

Coherent systems have been so far characterized using signatures [14]. In this paper, by using the quadratic rank transmutation map, we construct a new class of skewed distributions, obtained by multiple application of the transmutation method. These new statistical models represent the lifetime distributions of complex series-parallel/parallel-series systems. An example of a series-parallel system is the E-17-AH bridge from Colorado, USA [16, 9]. Multi-girder bridges are characterized by these types of systems [16, 2, 9]. Other distributions that represent the lifetime distributions of series-parallel/parallel-series systems can be found in [13] and [7]. These few statistical models represent only a particular and small class of series-parallel/parallel-series systems. Using the method presented in this paper, we can represent a larger class of these type of systems. We cannot represent all the series-parallel/parallel-series systems using the method introduced in this paper, only a particular class of systems. These particular series-parallel/parallel-series systems have independent and identically distributed components.

Choosing the optimal structure design system can be difficult. One tool that helps us in this matter is the expected system lifetime, the simpler and most commonly used metric for system performance [14]. The problem of comparing system performance using the expected system lifetime is still an open problem. Boland and Samaniego (2004) [8] discussed this problem providing conditions for ordering the expected system lifetimes for a particular group of small systems. "Eventhough this comparison ignores the variability in system lifetime it is possible that a system whose expected lifetime exceeds that of a second system will be less reliable than the second system at the systems planned mission time" [14]. This paper discusses the stochastic ordering, namely the likelihood ratio ordering, of some particular series-parallel/parallel-series systems and provides an example.

The paper is organized as follows. In Section 2, the multiple transmuted models of order n are introduced. Section 3 is dedicated to characteristics of these new models. We give formulae for the reliability, hazard rate and mean residual life functions. We discuss the order statistics and their asymptotic

behaviour. Stochastic ordering is discussed in Section 4, while in Section 5 we discuss the maximum likelihood estimators of the parameters of the new classes of statistical models introduced. We perform data analysis using two real data sets in Section 6.

2 Method of construction

Using the transmutation map developed by Shaw and Buckley (2007) [18], we define the class of transmuted distributions of order n . The rank transmutation map for a baseline continuous distribution function G is defined as

$$F_T(x) = (1 + \lambda)G(x) - \lambda G(x)^2 \tag{1}$$

where $|\lambda| \leq 1$ is the transmutation parameter.

The corresponding density function of F_T is

$$f_T(x) = g(x)[1 + \lambda - 2\lambda G(x)] \tag{2}$$

where g is the density function corresponding to G .

The class of transmuted distributions of order n , denoted by T_n , is defined as follows. Let F be an arbitrary continuous cumulative distribution function and

$$\begin{aligned} T_1 = T_1(F, \lambda_0) : \quad & F_1(x) = (1 + \lambda_0)F(x) - \lambda_0 F(x)^2 \\ T_2 = T_2(F, \lambda_0, \lambda_1) : \quad & F_2(x) = (1 + \lambda_1)F_1(x) - \lambda_1 F_1(x)^2 \\ T_3 = T_3(F, \lambda_0, \lambda_1, \lambda_2) : \quad & F_3(x) = (1 + \lambda_2)F_2(x) - \lambda_2 F_2(x)^2 \\ \dots & \\ T_n = T_n(F, \lambda_0, \lambda_1, \dots, \lambda_{n-1}) : \quad & F_n(x) = (1 + \lambda_{n-1})F_{n-1}(x) - \lambda_{n-1} F_{n-1}(x)^2 \end{aligned} \tag{3}$$

where $|\lambda_i| \leq 1, \forall i \in \overline{0, n-1}$ are the transmutation parameters.

We denote by $T_1(F, \lambda_0)$ the random variable with F_1 as its cumulative distribution function (cdf) and parameter λ_0 , by $T_2(F, \lambda_0, \lambda_1)$ the random variable with F_2 as its cdf and parameters λ_0, λ_1 , and so on, denoting by $T_n(F, \lambda_0, \lambda_1, \dots, \lambda_{n-1})$ the random variable with F_n as its cdf and parameters $\lambda_0, \lambda_1, \dots, \lambda_{n-1}$. F_n is obtained by applying the transmutation map (1) to the previous cdf obtained at step $n - 1$, F_{n-1} . Therefore, the construction of the T_n models is recursive.

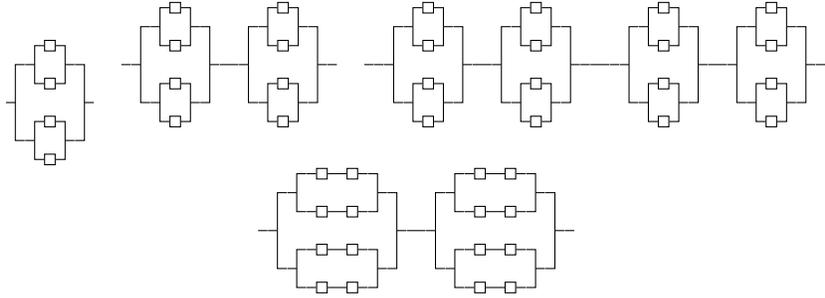


Figure 1: $T_2(F, 1, 1)$, $T_3(F, 1, 1, -1)$, $T_4(F, 1, 1, -1, -1)$ and $T_4(F, -1, 1, 1, -1)$

2.1 Motivation and interpretation

The T_n models can be interpreted as the lifetime distributions of some particular series-parallel/parallel-series system with independent and identically distributed components. A parallel system with m components is a system that fails if all the components fail, while a series system with m components is a system that fails if one of the components fails.

The cumulative distribution function F_n for $\lambda_i = -1, \forall i = \overline{0, n-1}$ can be interpreted as the lifetime distribution function of a series system with 2^n independent identically distributed components which follow a common cdf F . For $\lambda_i = 1, \forall i = \overline{0, n-1}$, F_n can be interpreted as the lifetime distribution function of a parallel system with 2^n independent identically distributed components which follow a common cdf F . When the transmutation parameters take different values, either -1 or 1 , F_n can be interpreted as a more complex system.

The series-parallel/parallel-series systems that can be represented by the T_n class of distributions are quite complex for large values of n . They build up starting from either a parallel system with two components ($\lambda_0 = 1$) or either a series system with two components ($\lambda_0 = -1$). At step n , the resulted system from the step $n - 1$ is either put into a parallel system with two components ($\lambda_{n-1} = 1$), each component representing the system from the previous step, or into a series system with also two components ($\lambda_{n-1} = -1$), each component representing the system from the previous step $n - 1$. At step $n = 1$ we can represent two types of systems, the parallel and series systems with two components, at step $n = 2$ four series-parallel/parallel-series systems, while at step n we can represent using the T_n class of distributions, 2^n series-parallel/parallel-series systems. As examples, in Figures 1 and 2, we have displayed some possible types of systems that can be represented by the transmuted distributions of order n . The empty square represents a system's

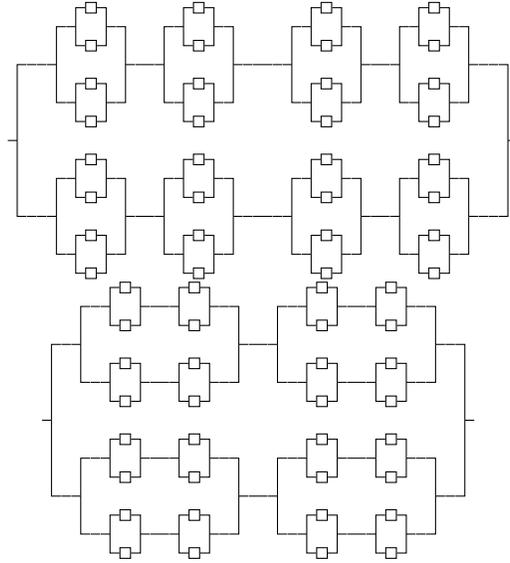


Figure 2: $T_5(F, 1, 1, -1, -1, 1)$ and $T_5(F, 1, -1, 1, -1, 1)$

component that has F as its cumulative distribution function.

2.2 Quantile function

For generating values that follow a transmuted distribution of order n , we use the quantile function. Because the construction of the T_n models is recursive, the generating of data algorithm is also recursive, but easy to implementate in software as *R* or *Matlab*. We use this algorithm in section 4 for comparing expected system lifetimes of two series-parallel/parallel-series systems.

For $\lambda_i \neq 0$, the quantile function of F_n , denoted by Q_{F_n} , is defined as

$$Q_{F_n}(u) = Q_{F_{n-1}}(u_{n-1}) = Q_{F_{n-2}}(u_{n-2}) = \dots = Q_F(u_0), \quad u \in (0, 1) \quad (4)$$

where

$$u_{n-1} = \frac{1 + \lambda_{n-1} - \sqrt{(1 + \lambda_{n-1})^2 - 4\lambda_{n-1}u}}{2\lambda_{n-1}}, \quad (5)$$

$$u_{n-2} = \frac{1 + \lambda_{n-2} - \sqrt{(1 + \lambda_{n-2})^2 - 4\lambda_{n-2}u_{n-1}}}{2\lambda_{n-2}}, \quad (6)$$

$$u_0 = \frac{1 + \lambda_0 - \sqrt{(1 + \lambda_0)^2 - 4\lambda_0 u_1}}{2\lambda_0} \quad (7)$$

and Q_F is the quantile function of F .

For generating m values for the T_n class of distributions we have the following simple algorithm.

Generating algorithm

For every $j = \overline{1, m}$ we have the following steps

1. Generate u from a Uniform(0,1) distribution
2. Calculate $u_{n-1}, u_{n-2}, \dots, u_0$
3. Set $x_j = Q_F(u_0)$.

3 Characteristics of the T_n class of distributions

The aging process of a system is represented by its hazard and mean residual life functions. Using these two notions, we can determine in a unique way the distribution function of a random variable. In this section, we discuss the reliability, hazard rate and mean residual life functions of the T_n models. Also, we discuss the order statistics of the T_n class of distributions, focusing on the asymptotic behaviour of the extreme order statistics.

3.1 Hazard rate, mean residual life and reliability functions

There are three important functions used in reliability. These are the hazard rate, denoted by h , mean residual life, denoted by M_F , and reliability, denoted by \overline{F} , functions. Let X be a random variable having F as its cdf. The functions h , M_F and \overline{F} corresponding to X are defined as follows

$$h(x) = \frac{f(x)}{1 - F(x)}, \quad M_F(x) = E(X - x | X \geq x) = \frac{1}{\overline{F}(x)} \int_x^\infty \overline{F}(t) dt \quad \text{and} \\ \overline{F}(x) = 1 - F(x) = P(X < x). \quad (8)$$

The hazard rate function give us the probability of immediate failure of X , the mean residual life function, the remaining time of functioning after X has survived x years, while the reliability function give us the probability of surviving of X .

3.1.1 Reliability functions

The reliability functions of the T_n models are

$$\begin{aligned} \bar{F}_1(x) &= (1 - \lambda_0)\bar{F}(x) + \lambda_0\bar{F}(x)^2 \\ \bar{F}_2(x) &= (1 - \lambda_1)\bar{F}_1(x) + \lambda_1\bar{F}_1(x)^2 \\ \bar{F}_3(x) &= (1 - \lambda_2)\bar{F}_2(x) + \lambda_2\bar{F}_2(x)^2 \\ &\dots \\ \bar{F}_n(x) &= (1 - \lambda_{n-1})\bar{F}_{n-1}(x) + \lambda_{n-1}\bar{F}_{n-1}(x)^2 \end{aligned} \tag{9}$$

where \bar{F} is the corresponding survival function of F .

3.1.2 Hazard rate functions

We denote by h_1, h_2, \dots, h_n the corresponding hazard rate functions of the transmuted distributions of order n , and by h the hazard rate function of the baseline distribution F . The hazard rate function of the T_n class of distributions can be represented as

$$h_n(x) = h_{n-1}(x) \frac{1 - \lambda_{n-1} + \lambda_{n-1}\bar{F}_{n-1}(x)}{1 - \lambda_{n-1} + 2\lambda_{n-1}\bar{F}_{n-1}(x)} \tag{10}$$

For $\lambda_{n-1} = 1$ (parallel system), we have that the hazard rate function of the T_n models is

$$h_n(x) = \frac{1}{2}h_{n-1}(x) \tag{11}$$

while for $\lambda_{n-1} = -1$ (series system), we have

$$h_n(x) = h_{n-1}(x) \left(1 + \frac{\bar{F}_{n-1}(x)}{2F_{n-1}(x)}\right) \tag{12}$$

3.1.3 Mean residual life function

We denote by $M_{F_1}, M_{F_2}, \dots, M_{F_n}$ the corresponding mean residual life functions of F_1, F_2, \dots, F_n . The mean residual life function of T_n class of distributions is defined as

$$M_{F_n}(x) = \frac{1}{\bar{F}_n(x)} \int_x^\infty \bar{F}_n(t) dt \tag{13}$$

Asadi and Bayranmoglu (2006) [5] have discussed the mean residual life function of a k -out-of- m structure. They gave the formulae for the mean residual life function of a k -out-of- m system. It is well-known that a parallel system is a 1-out-of- m system, while a series system is a m -out-of- m system.

Given a k -out-of- m system with m components independent identically distributed with common cumulative distribution function F , the mean residual life function, denoted by M_m^k of this system is defined as

$$M_m^k(t) = \sum_{s=0}^{k-1} \binom{m}{s} \int_t^\infty \left(\frac{\bar{F}(x)}{\bar{F}(t)}\right)^{m-s} \left(1 - \frac{\bar{F}(x)}{\bar{F}(t)}\right)^s dx \quad (14)$$

The mean residual life function of a parallel system with m components is

$$M_m^1(t) = \int_t^\infty \left(\frac{\bar{F}(x)}{\bar{F}(t)}\right)^m dx \quad (15)$$

while the mean residual life function of a series system with m components is

$$M_m^m(t) = \sum_{s=0}^{m-1} \binom{m}{s} \int_t^\infty \left(\frac{\bar{F}(x)}{\bar{F}(t)}\right)^{m-s} \left(1 - \frac{\bar{F}(x)}{\bar{F}(t)}\right)^s dx \quad (16)$$

The mean residual life functions of the T_n models can be derived using equations (15) and (16). We give the formulae for the mean residual life functions of the T_1 , T_2 models and a more generalized formulae for T_n models when λ_{n-1} is either 1 or -1 .

The mean residual life function of T_n models when $\lambda_{n-1} = 1$, using (15), is defined as

$$M_{F_n}(t) = \int_t^\infty \left(\frac{\bar{F}_{n-1}(x)}{\bar{F}_{n-1}(t)}\right)^2 dx \quad (17)$$

The mean residual life function of T_n models when $\lambda_{n-1} = -1$. using (16), is defined as

$$M_{F_n}(t) = 2 \int_t^\infty \frac{\bar{F}_{n-1}(x)}{\bar{F}_{n-1}(t)} dx - \int_t^\infty \left(\frac{\bar{F}_{n-1}(x)}{\bar{F}_{n-1}(t)}\right)^2 dx \quad (18)$$

For the T_1 models, we have two systems with two components, namely the parallel and series systems. Their mean residual life functions are obtained by taking m equals 2 in equations (15) and (16). Using the T_2 models, we can model four series-parallel/parallel-series systems as follows: (S_1)

$T_2(F, -1, -1)$; $(S_2) T_2(F, 1, 1)$; $(S_3) T_2(F, 1, -1)$ and $(S_4) T_2(F, -1, 1)$. We denote by M_{S_1} , M_{S_2} , M_{S_3} and M_{S_4} , respectively, the mean residual life functions of systems S_1, S_2, S_3 and S_4 .

The mean residual life function of system S_1 is

$$M_{S_1}(t) = 2 \int_t^\infty \frac{\bar{F}(x)[1+F(x)][1+F(x)^2]}{\bar{F}(t)[1+F(t)][1+F(t)^2]} dx - \int_t^\infty \left(\frac{\bar{F}(x)[1+F(x)][1+F(x)^2]}{\bar{F}(t)[1+F(t)][1+F(t)^2]} \right)^2 dx \quad (19)$$

The mean residual life function of system S_2 is

$$M_{S_2}(t) = \int_t^\infty \left(\frac{\bar{F}(x)^4}{\bar{F}(t)^4} \right)^2 dx \quad (20)$$

The mean residual life function of system S_3 is

$$M_{S_3}(t) = 2 \int_t^\infty \frac{\bar{F}(x)^2[2-\bar{F}(x)^2]}{\bar{F}(t)^2[2-\bar{F}(t)^2]} dx - \int_t^\infty \left(\frac{\bar{F}(x)^2[2-\bar{F}(x)^2]}{\bar{F}(t)^2[2-\bar{F}(t)^2]} \right)^2 dx \quad (21)$$

The mean residual life function of system S_4 is

$$M_{S_4}(t) = \int_t^\infty \left(\frac{\bar{F}(x)^2[1+F(x)]^2}{\bar{F}(t)^2[1+F(t)]^2} \right)^2 dx \quad (22)$$

3.2 Characteristics of the limiting distributions

In this subsection, we discuss the order statistics of the T_n models and their asymptotic behaviour. We show that the asymptotic distributions of the order statistics do not depend on the transmutation parameters. Because the transmutation parameters give us the structure of the complex series-parallel/parallel-series systems modeled by the T_n models, the asymptotic distributions of order statistics of these systems do not depend on the structure of them.

3.2.1 Order statistics

Let $X_{1:m} \leq X_{2:m} \leq \dots \leq X_{m:m}$ be the order statistics of X_1, X_2, \dots, X_m random variables independent identically distributed with common cdf F_n of the form (3). The density function $f_{i:m}(x)$ of the i th order statistics, $X_{i:m}$, is given by

$$f_{i:m}(x) = \frac{f_n(x)}{B(i, m-i+1)} F_n(x)^{i-1} \bar{F}_n(x)^{m-i} \quad (23)$$

where $B(a, b) = \int_0^1 t^{a-1}(1-t)^{b-1} dt$, $a > 0$, $b > 0$, is the beta function.

Because the construction of F_n starts with a baseline continuous distribution function F , the analytical formula in terms of F , of the density function $f_{i:m}$ is complicated and intractable for further developments. In this section, we give some general theorems which can be used to obtain the asymptotic distributions of the extreme order statistics given by $X_{1:m} = \min(X_1, X_2, \dots, X_m)$ and $X_{m:m} = \max(X_1, X_2, \dots, X_m)$.

Let \mathcal{C} be the class of continuous cumulative distribution functions. Let

$$\mathcal{C}_1 = \left\{ F \in \mathcal{C} \mid \lim_{t \rightarrow 0} \frac{F(tx)}{F(t)} < \infty, \forall x \right\}, \quad (24)$$

$$\mathcal{C}_2 = \left\{ F \in \mathcal{C} \mid \lim_{t \rightarrow \infty} \frac{1 - F(t+x)}{1 - F(t)} < \infty, \forall x \right\} \quad (25)$$

and

$$\mathcal{C}_3 = \left\{ F \in \mathcal{C} \mid \lim_{x \rightarrow \infty} \frac{d}{dx} \left(\frac{1}{h(x)} \right) = 0 \right\} \quad (26)$$

The following three theorems we use to give the asymptotic distributions of the extreme order statistics.

Theorem 1. *If $F \in \mathcal{C}_1$, then also $F_n \in \mathcal{C}_1$.*

Proof. For $n = 1$ we have

$$\lim_{t \rightarrow 0} \frac{F_1(tx)}{F_1(t)} = \lim_{t \rightarrow 0} \frac{F(tx)[1 + \lambda_0 - \lambda_0 F(tx)]}{F(t)[1 + \lambda_0 - \lambda_0 F(tx)]} = l(x) < \infty,$$

where $l(x) = \lim_{t \rightarrow 0} \frac{F(tx)}{F(t)}$.

Now, for $(n-1) \rightarrow n$, we have

$$\lim_{t \rightarrow 0} \frac{F_n(tx)}{F_n(t)} = \lim_{t \rightarrow 0} \frac{F_{n-1}(tx)[1 + \lambda_{n-1} - \lambda_{n-1} F_{n-1}(tx)]}{F_{n-1}(t)[1 + \lambda_{n-1} - \lambda_{n-1} F_{n-1}(t)]} = l(x) < \infty.$$

By mathematical induction, we conclude the proof. \square

Remark 1. *We notice that the limit of F_n from \mathcal{C}_1 is invariant to the transmutation parameters λ_i , $i = 0, n-1$.*

Theorem 2. *If $F \in \mathcal{C}_2$, then also $F_n \in \mathcal{C}_2$.*

Proof. For proof see Appendix A. □

Remark 2. *It is very important to remark that the limit $\lim_{t \rightarrow \infty} \frac{1 - F_n(t+x)}{1 - F_n(t)} = l(x)$, where $l(x) = \lim_{t \rightarrow \infty} \frac{1 - F(t+x)}{1 - F(t)}$, is invariant to the transmutation parameters λ_i .*

Theorem 3. *If $F \in \mathcal{C}_3$, then also $F_n \in \mathcal{C}_3$.*

Proof. For proof see Appendix A. □

3.2.2 Extreme order statistics

In this section, we give the asymptotic distributions of the order statistics of the T_n models.

Theorem 4. *Let $X_{1:m}$ and $X_{m:m}$ be the minimum and maximum of a random sample X_1, X_2, \dots, X_m from F_n defined for $x > 0$. The following statements hold*

1. *If $F \in \mathcal{C}_1$, $Q_F(0) = 0$ and $\lim_{t \rightarrow 0} \frac{F(tx)}{F(t)} = x^{\theta_1}$, for each $x > 0$, $\theta_1 > 0$, then*

$$\lim_{m \rightarrow \infty} P \left\{ \frac{X_{1:m} - a_m}{b_m} \leq x \right\} = 1 - \exp(-x^{\theta_1}), \quad x > 0.$$

2. *If $F \in \mathcal{C}_2$, $x_{F_n} = \infty$ and $\lim_{t \rightarrow \infty} \frac{1 - F(t+x)}{1 - F(t)} = \exp(-\theta_2 x)$ for each $x > 0$, $\theta_2 > 0$, then*

$$\lim_{m \rightarrow \infty} P \left\{ a_m^* (X_{m:m} - b_m^*) \leq x \right\} = \exp(-\exp(-\theta_2 x)).$$

3. *If $F \in \mathcal{C}_3$, $h_F \neq 0$, $h_{F_n} \neq 0$ and $Q_F(1) = \infty$ then*

$$\lim_{m \rightarrow \infty} P \left\{ \frac{X_{m:m} - c_m}{d_m} \leq x \right\} = \exp(-\exp(-x)), \quad x > 0$$

where $a_m, b_m, a_m^*, b_m^*, c_m, d_m$ are normalizing constants and $x_{F_n} = \sup\{x | F_n(x) < 1\}$ [4].

Proof. For (i), we apply Theorem 1 and Theorem 8.3.3 from [4] and for (ii), Theorem 2 and Theorem 1.6.2 from [10].

The last part, (iii), follows from Theorem 3 and Theorem 8.3.3 from [4]. \square

The form of the normalizing constants can be determined following the Corollary 1.6.3 from [10] and the results from [4].

Theorem 5. *Let $X_{1:m} \leq X_{2:m} \leq \dots \leq X_{m:m}$ be the order statistics of a random sample X_1, X_2, \dots, X_m from F_n defined for $x > 0$. The following statements hold*

1. *If $F \in \mathcal{C}_1$, $Q_F(0) = 0$ and $\lim_{t \rightarrow 0} \frac{F(tx)}{F(t)} = x^{\theta_1}$ for each $x > 0$, $\theta_1 > 0$, then for each $i = \overline{1, m}$*

$$\lim_{m \rightarrow \infty} P \left\{ \frac{X_{i:m} - a_m}{b_m} \leq x \right\} = 1 - \sum_{k=0}^{i-1} \frac{x^{k\theta_1}}{k!} \exp\{-x^{\theta_1}\}, \quad x > 0$$

2. *If $F \in \mathcal{C}_3$, $h_F \neq 0$, $h_{F_n} \neq 0$ and $Q_F(1) = \infty$ then for each $i = \overline{1, m}$*

$$\lim_{m \rightarrow \infty} P \left\{ \frac{X_{m-i+1:m} - c_m}{d_m} \leq x \right\} = \sum_{r=0}^{i-1} \frac{\exp(-rx)}{r!} \exp(-\exp(-x))$$

where a_m, b_m, c_m and d_m are normalizing constants [4].

Proof. The proof follows from Theorem 1, Theorem 3 and Eqs. (8.4.2) and (8.4.3) of [4]. \square

4 Stochastic ordering

In this section, we discuss stochastic ordering, namely the likelihood ratio ordering of the transmuted distributions of order n with applications in comparing the expected system lifetimes of complex series-parallel/parallel-series systems.

Definition 1. [17] *Let X_1 and X_2 be two random variables with respective cumulative distribution functions F_1 and F_2 and hazard rates h_1 and h_2 , respectively. Also, let f_1 and f_2 be the probability density functions corresponding to F_1 and F_2 , respectively.*

1. X_1 is said to be smaller than X_2 in the likelihood ratio order (denoted by $X_1 \underset{LR}{\leq} X_2$), if

$$\frac{f_2(x)}{f_1(x)} \text{ is non-decreasing over the union of the supports of } X_1 \text{ and } X_2,$$

2. X_1 is said to be stochastically smaller than X_2 , denoted by $X_1 \underset{ST}{\leq} X_2$, if $F_1(x) \geq F_2(x)$ for all x .
3. X_1 is said to be smaller than X_2 in the hazard rate order, denoted by $X_1 \underset{HR}{\leq} X_2$, if $h_1(x) \leq h_2(x)$ for all x .

Remark 3. It is well-known that the likelihood ratio is stronger than the hazard rate order and the stochastic order, $X_1 \underset{LR}{\leq} X_2 \Rightarrow X_1 \underset{HR}{\leq} X_2 \Rightarrow X_1 \underset{ST}{\leq} X_2$. Also, we have that $X_1 \underset{ST}{\leq} X_2$ implies $E(X_1) \leq E(X_2)$ [17].

4.1 Stochastic ordering of the T_n models

Let F and G be two arbitrary continuous cumulative distribution functions and let

$$\begin{aligned} F_1(x) &= (1 + \lambda_0)F(x) - \lambda_0F(x)^2 \\ F_2(x) &= (1 + \lambda_1)F_1(x) - \lambda_1F_1(x)^2 \\ F_3(x) &= (1 + \lambda_2)F_2(x) - \lambda_2F_2(x)^2 \\ &\dots \\ F_n(x) &= (1 + \lambda_{n-1})F_{n-1}(x) - \lambda_{n-1}F_{n-1}(x)^2 \end{aligned}$$

and

$$\begin{aligned} G_1(x) &= (1 + \lambda'_0)G(x) - \lambda'_0G(x)^2 \\ G_2(x) &= (1 + \lambda'_1)G_1(x) - \lambda'_1G_1(x)^2 \\ G_3(x) &= (1 + \lambda'_2)G_2(x) - \lambda'_2G_2(x)^2 \\ &\dots \\ G_n(x) &= (1 + \lambda'_{n-1})G_{n-1}(x) - \lambda'_{n-1}G_{n-1}(x)^2. \end{aligned}$$

We denote the corresponding probability density functions by f_1, f_2, \dots, f_n and g_1, g_2, \dots, g_n , respectively.

Theorem 6. Let X, X_1, X_2, \dots, X_n be random variables with cumulative distribution functions F, F_1, F_2, \dots, F_n , respectively. Let Y, Y_1, Y_2, \dots, Y_n be random variables with cumulative distribution functions G, G_1, G_2, \dots, G_n , respectively. If $-1 \leq \lambda_i \leq 0 \leq \lambda'_i \leq 1$, for all $i = \overline{0, n-1}$ and $Y \underset{LR}{\leq} X$, then $Y_i \underset{LR}{\leq} X_i$ for all $i = \overline{1, n}$.

Proof. We have

$$\begin{aligned} \frac{f_i(x)}{g_i(x)} &= \frac{f_{i-1}(x)[1 + \lambda_{i-1} - 2\lambda_{i-1}F_{i-1}(x)]}{g_{i-1}(x)[1 + \lambda'_{i-1} - 2\lambda'_{i-1}G_{i-1}(x)]} \\ &= \frac{f_{i-2}(x)[1 + \lambda_{i-2} - 2\lambda_{i-2}F_{i-2}(x)]}{g_{i-2}(x)[1 + \lambda'_{i-2} - 2\lambda'_{i-2}G_{i-2}(x)]} \frac{[1 + \lambda_{i-1} - 2\lambda_{i-1}F_{i-1}(x)]}{[1 + \lambda'_{i-1} - 2\lambda'_{i-1}G_{i-1}(x)]} = \dots = \\ &= \frac{f(x)}{g(x)} \frac{[1 + \lambda_0 - 2\lambda_0F(x)][1 + \lambda_1 - 2\lambda_1F_1(x)] \cdots [1 + \lambda_{i-1} - 2\lambda_{i-1}F_{i-1}(x)]}{[1 + \lambda'_0 - 2\lambda'_0G(x)][1 + \lambda'_1 - 2\lambda'_1G_1(x)] \cdots [1 + \lambda'_{i-1} - 2\lambda'_{i-1}G_{i-1}(x)]} \end{aligned}$$

and

$$\begin{aligned} \log \frac{f_i(x)}{g_i(x)} &= \log \frac{f(x)}{g(x)} + \log[1 + \lambda_0 - 2\lambda_0F(x)] - \log[1 + \lambda'_0 - 2\lambda'_0G(x)] \\ &\quad + \log[1 + \lambda_1 - 2\lambda_1F_1(x)] - \log[1 + \lambda'_1 - 2\lambda'_1G_1(x)] \\ &\quad + \cdots + \log[1 + \lambda_{i-1} - 2\lambda_{i-1}F_{i-1}(x)] - \log[1 + \lambda'_{i-1} - 2\lambda'_{i-1}G_{i-1}(x)] \end{aligned}$$

$\frac{f_i(x)}{g_i(x)}$ is non-decreasing if and only if $\left(\log \frac{f_i(x)}{g_i(x)}\right)' \geq 0$ for all x , where

$$\begin{aligned} \left(\log \frac{f_i(x)}{g_i(x)}\right)' &= \left(\log \frac{f(x)}{g(x)}\right)' - \frac{2\lambda_0 f(x)}{1 + \lambda_0 - 2\lambda_0 F(x)} + \frac{2\lambda'_0 g(x)}{1 + \lambda'_0 - 2\lambda'_0 G(x)} \\ &\quad - \frac{2\lambda_1 f_1(x)}{1 + \lambda_1 - 2\lambda_1 F_1(x)} + \frac{2\lambda'_1 g_1(x)}{1 + \lambda'_1 - 2\lambda'_1 G_1(x)} \\ &\quad - \cdots - \frac{2\lambda_{i-1} f_{i-1}(x)}{1 + \lambda_{i-1} - 2\lambda_{i-1} F_{i-1}(x)} + \frac{2\lambda'_{i-1} g_{i-1}(x)}{1 + \lambda'_{i-1} - 2\lambda'_{i-1} G_{i-1}(x)} \end{aligned}$$

It is easy to see that $-1 \leq \lambda_i \leq 0 \leq \lambda'_i \leq 1$ for all $i = \overline{0, n-1}$ and $Y \underset{LR}{\leq} X$ imply $\left(\log \frac{f_i(x)}{g_i(x)}\right)' \geq 0$ for all x , and the result holds. \square

The theorem above generalizes a proven statement that tells us that a parallel system is always preferable than a series system [14].

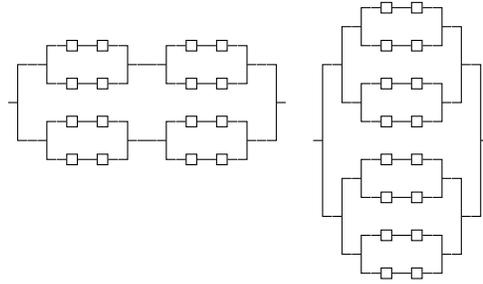


Figure 3: The serie-parallel/parallel-series systems represented by X and Y .

Remark 4. In Theorem 6 if $\lambda_{n-1} \leq \lambda'_{n-1}$ and $F_{n-1} = G_{n-1}$, then $Y_n \leq_{LR} X_n$.

4.2 Comparing series-parallel/parallel-series systems

Let $F(x) = 1 - \exp\{-(x/\rho)^\mu\}$, $x > 0$ be a Weibull cumulative distribution function of parameters $\mu > 0$ and $\rho > 0$. Let X and Y be random variables having F_4 and G_4 as their cumulative distribution functions obtained using the method described in Section 1 as follows: F_4 is the cumulative distribution function of $T_4(F, -1, 1, -1, 1)$, while G_4 is the cumulative distribution function of $T_4(F, -1, 1, 1, 1)$. In Figure 3, we have displayed the series-parallel/parallel-series systems that these random variables X and Y represent.

Using Theorem 6 and Remarks 3 and 4, we have $Y \leq_{LR} X$, and therefore, the expected system lifetime of Y is smaller than the expected system lifetime of X . Generating values from X and Y using the algorithm presented in subsection 2.2, we have calculated the expected system lifetime of them. These values are displayed in Table 1.

(μ, ρ)	$E(X)$	$E(Y)$	(μ, ρ)	$E(X)$	$E(Y)$
(2, 4)	3.701494	2.358217	(0.7, 0.5)	0.4525109	0.1389469
(0.7, 1)	0.9122977	0.2844805	(1, 0.9)	0.8163832	0.3459043
(3, 0.8)	0.7531299	0.5558009	(7, 1)	0.971466	0.8491724
(4, 7)	6.688466	5.285205	(5, 5)	4.812751	3.988665
(3, 2)	1.886318	1.389314	(10, 20)	19.60123	17.82688

Table 1: The expected system lifetimes of X and Y .

5 Parameter estimation

Let x_1, x_2, \dots, x_m be a random sample of size m from $T_n(F, \lambda_0, \lambda_1, \dots, \lambda_{n-1})$ of parameters (θ, Λ_t) , where $\theta = (\theta_1, \theta_2, \dots, \theta_k)$ is the vector parameter of length k of the underlying distribution F and $\Lambda_t = (\lambda_0, \lambda_1, \dots, \lambda_t)$ are the transmutation parameters, $t = \overline{0, n-1}$. The method used to estimate the parameters is the maximum likelihood estimation.

The log-likelihood function is given by

$$\mathcal{L} = \sum_{i=1}^m \left\{ \log[f_{n-1}(x_i; \theta, \Lambda_{n-2})] + \log[1 + \lambda_{n-1} - 2\lambda_{n-1}F_{n-1}(x_i; \theta, \Lambda_{n-2})] \right\} \quad (27)$$

In order to maximize the log-likelihood function, we solve the nonlinear likelihood system obtained by differentiating (27). This can be done using *R*, *Matlab* and *Mathcad*, among other packages. The elements of the score vector $V(\rho_t)$, $\rho_t = (\theta, \Lambda_t)$, $\Lambda_t = (\lambda_0, \lambda_1, \dots, \lambda_t)$ and $\theta = (\theta_1, \theta_2, \dots, \theta_k)$ are

$$V_{\theta_s} = \sum_{i=1}^m \left\{ \frac{\frac{\partial}{\partial \theta_s} f_{n-1}(x_i; \rho_{n-2})}{f_{n-1}(x_i; \rho_{n-2})} - \frac{2\lambda_{n-1} \frac{\partial}{\partial \theta_s} F_{n-1}(x_i; \rho_{n-2})}{1 + \lambda_{n-1} - 2\lambda_{n-1}F_{n-1}(x_i; \rho_{n-2})} \right\}, \quad s = \overline{1, k} \quad (28)$$

where

$$\begin{aligned} \frac{\partial}{\partial \theta_s} f_{n-1}(x_i; \rho_{n-2}) &= \frac{\partial}{\partial \theta_s} f_{n-2}(x_i; \rho_{n-3}) [1 + \lambda_{n-2} - 2\lambda_{n-2}F_{n-2}(x_i; \rho_{n-3})] \\ &\quad - 2\lambda_{n-2} f_{n-2}(x_i; \rho_{n-3}) \frac{\partial}{\partial \theta_s} F_{n-2}(x_i; \rho_{n-3}) \end{aligned}$$

$$\frac{\partial}{\partial \theta_s} F_{n-2}(x_i; \rho_{n-3}) = \frac{\partial}{\partial \theta_s} F_{n-3}(x_i; \rho_{n-3}) [1 + \lambda_{n-3} - 2\lambda_{n-3}F_{n-3}(x_i; \rho_{n-3})]$$

and

$$V_{\lambda_j} = \sum_{i=1}^m \left\{ \frac{\frac{\partial}{\partial \lambda_j} f_{n-1}(x_i; \rho_{n-2})}{f_{n-1}(x_i; \rho_{n-2})} - \frac{2\lambda_{n-1} \frac{\partial}{\partial \lambda_j} F_{n-1}(x_i; \rho_{n-2})}{1 + \lambda_{n-1} - 2\lambda_{n-1}F_{n-1}(x_i; \rho_{n-2})} \right\}, \quad j = \overline{0, n-2} \quad (29)$$

and

$$V_{\lambda_{n-1}} = \sum_{i=1}^m \frac{1 - 2F_{n-1}(x_i; \rho_{n-2})}{1 + \lambda_{n-1} - 2\lambda_{n-1}F_{n-1}(x_i; \rho_{n-2})} \quad (30)$$

where

$$\begin{aligned} \frac{\partial}{\partial \lambda_j} f_n(x_i; \rho_{n-1}) &= [1 + \lambda_{n-1} - 2\lambda_{n-1}F_{n-1}(x_i; \rho_{n-2})] \frac{\partial}{\partial \lambda_j} f_{n-1}(x_i; \rho_{n-2}) \\ &\quad - 2\lambda_{n-1}f_{n-1}(x_i; \rho_{n-2}) \frac{\partial}{\partial \lambda_j} F_{n-1}(x_i; \rho_{n-2}) \end{aligned}$$

$$\begin{aligned} \frac{\partial}{\partial \lambda_j} F_n(x_i; \rho_{n-1}) &= [1 + \lambda_{n-1} - 2\lambda_{n-1}F_{n-1}(x_i; \rho_{n-2})] \frac{\partial}{\partial \lambda_j} F_{n-1}(x_i; \rho_{n-2}), \\ &\quad j \neq n - 1 \end{aligned}$$

Because we can have a relatively large number of parameters this can cause problems especially when the sample size is not large. A good set of initial values is essential. The second partial derivatives of the log-likelihood function for the construction of the Fisher information matrix are also obtained. These are presented in the Appendix B.

6 Goodness-of-fit

In this section, using two data sets we compare the fits of the T_n distributions considering as baseline distribution functions the exponentiated power Lindley (EPL) [6] and Pareto [12] distributions. In each case the parameters are estimated by maximum likelihood using *fitdist()* function in *R* with Nelder-Mead options used as an iterative process for maximizing the log-likelihood function. First we give the MLEs of the parameters and the values of the Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC) statistics. The lower the values of these criteria the better the fit. Next we perform Kolmogorov-Smirnov tests. The Kolmogorov-Smirnov test is performed using the *ks.test()* function in *R*. Finally, we provide the empirical and theoretical cdf for data set 1 to show a visual comparison of the fitted cdf functions. Also, we consider the *TTT*-plot transformation which gives us the shape of the hazard rate function [1]. If the *TTT* curve is convex, then the hazard rate is increasing, if it is concave, then the hazard rate is decreasing, if it is convex then concave, then it is upside-down shaped, otherwise it is bathtub shaped [1]. The data sets are as follows.

	Log-lik	AIC	BIC	K-S	p-value
$T_5(PL, \lambda_0)$	-120.8433	247.6866	254.6788	0.078578	0.7061
$T_5(EPL, \lambda_0)$	-120.5105	249.0209	258.3438	0.080024	0.6849
$T_1(EL, \lambda_0)$	-121.2876	248.5752	255.5674	0.08935983	0.5485178
PL	-122.4001	248.8001	253.4616	0.1123079	0.2721916
EPL	-121.8663	249.7326	256.7248	0.09929166	0.4150025
$T_1(EPL, \lambda_0)$	-121.2683	250.5366	259.8596	0.09099421	0.5253884
$T_7(EPL, \lambda_0)$	-120.3936	248.7873	258.1102	0.07837921	0.7090126

Table 2: The criteria information for data set 1.

Data Set 1

The first data set [3] we consider is the times of fatigue fracture of Kevlar 373/epoxy that are subject to constant pressure at 90% stress level until all have failed: $x_1 = (0.0251, 0.0886, 0.0891, 0.2501, 0.3113, 0.3451, 0.4763, 0.5650, 0.5671, 0.6566, 0.6748, 0.6751, 0.6753, 0.7696, 0.8375, 0.8391, 0.8425, 0.8645, 0.8851, 0.9113, 0.9120, 0.9836, 1.0483, 1.0596, 1.0773, 1.1733, 1.2570, 1.2766, 1.2985, 1.3211, 1.3503, 1.3551, 1.4595, 1.4880, 1.5728, 1.5733, 1.7083, 1.7263, 1.7460, 1.7630, 1.7746, 1.8275, 1.8375, 1.8503, 1.8808, 1.8878, 1.8881, 1.9316, 1.9558, 2.0048, 2.0408, 2.0903, 2.1093, 2.1330, 2.2100, 2.2460, 2.2878, 2.3203, 2.3470, 2.3513, 2.4951, 2.5260, 2.9911, 3.0256, 3.2678, 3.4045, 3.4846, 3.7433, 3.7455, 3.9143, 4.8073, 5.4005, 5.4435, 5.5295, 6.5541, 9.0960)$. This data set has an upside-down hazard rate function, see Figures 4 and 5.

Data Set 2

Merovci and Puka (2014) [12] used two data sets for the fitting of the transmuted Pareto distribution, $X \sim Pareto(x_0, a)$, $a > 0$ and x_0 the necessarily positive minimum possible value of X and compared it to Pareto, generalized Pareto and exponentiated Weibull distributions. They showed that the transmuted Pareto distribution is a better fit model in both cases. Using one of the two data sets from [12], we extend the data analysis study, fitting the transmuted Pareto of order 2 ($T_2(Pareto, \lambda_0, \lambda_1)$) and transmuted Pareto of order 3 ($T_3(Pareto, \lambda_0, \lambda_1, \lambda_2)$) distributions to this chosen data set. The data set corresponds to the Floyd River located in James, Iowa, USA.

For data set 1 we consider the fittings of the transmuted power Lindley of order 5 distribution ($T_5(PL, \lambda_0)$) with $\lambda_0 = \lambda_1 = \lambda_2 = \lambda_3 = \lambda_4$, transmuted exponentiated power Lindley of order 5 distribution ($T_5(EPL, \lambda_0)$) with $\lambda_0 = \lambda_1 = \lambda_2 = \lambda_3 = \lambda_4$, transmuted exponentiated Lindley distribution ($T_1(EP, \lambda_0)$), power Lindley distribution (PL), exponentiated power Lindley distribution

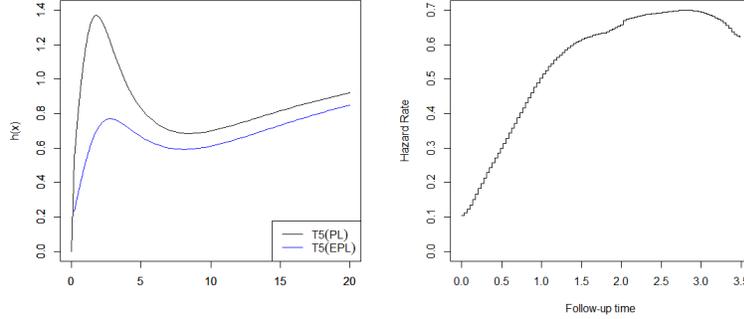


Figure 4: Hazard rate function for example 1.

	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\lambda}_0$
$T_5(PL, \lambda_0)$	1.3342506	-	0.2709258	0.6203250
$T_5(EPL, \lambda_0)$	1.4128086	0.9035090	0.1880978	0.4456854
$T_1(EL, \lambda_0)$	-	1.4145876	0.7177399	0.7058371
PL	1.1422729	-	0.7047831	-
EPL	0.9499608	1.5355626	1.0205424	-
$T_1(EPL, \lambda_0)$	1.0493532	1.3046813	0.6640713	0.6889066
$T_7(EPL, \lambda_0)$	1.5596811	0.8153664	0.1200829	0.3746730

Table 3: The MLE’s for data set 1.

(EPL), transmuted exponentiated power Lindley distribution ($T_1(EPL, \lambda_0)$) and transmuted exponentiated power Lindley of order 7 distribution ($T_7(EPL, \lambda_0)$) with $\lambda_0 = \lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = \lambda_6$. In Table 2, the values corresponding to the information criteria considered, AIC and BIC, along with the p-values of the Kolmogorov-Smirnov tests are displayed. The values of MLE estimators are presented in Table 3. As it can be noticed, the $T_5(PL, \lambda_0)$ distribution has the lowest AIC and BIC. In Figure 5, we have plotted the TTT plot transformation.

For data set 2, the transmuted Pareto of order 3 distribution has the lowest AIC and BIC. The results are displayed in Table 4. Also, for this data set, we consider hypothesis tests. We compare the $T_3(Pareto, \lambda_0, \lambda_1, \lambda_2)$ model with $T_2(Pareto, \lambda_0, \lambda_1)$ and $T_1(Pareto, \lambda_0)$ models. The results are displayed in Table 5.

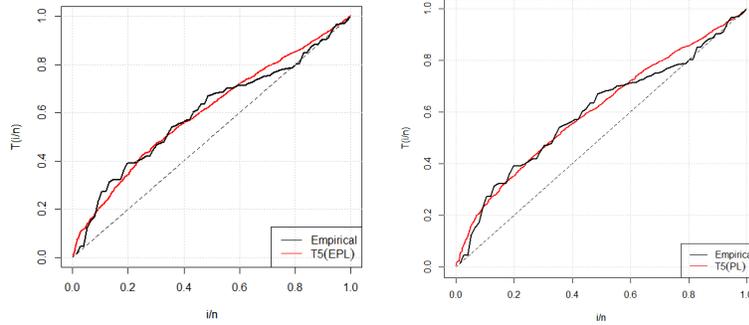


Figure 5: The TTT-transform plot for example 1.

	\hat{a}	$\hat{\lambda}_0$	$\hat{\lambda}_1$	$\hat{\lambda}_2$
$T_3(Pareto, \lambda_0, \lambda_1, \lambda_2)$	0.9456290	-0.7394268	-0.7784058	-0.8363968
$T_2(Pareto, \lambda_0, \lambda_1)$	0.7755660	-0.8469511	-0.8843003	-
$T_1(Pareto, \lambda_0)$	0.5857841	-0.9103600	-	-

Table 4: The MLE's for data set 2.

Appendix A

Proof of Theorem 2

Proof. For $n = 1$, we have

$$\lim_{t \rightarrow \infty} \frac{\bar{F}_1(t+x)}{\bar{F}_1(t)} = \lim_{t \rightarrow \infty} \frac{\bar{F}(t+x)[1 - \lambda_0 + \lambda_0 \bar{F}(t+x)]}{\bar{F}(t)[1 - \lambda_0 + \lambda_0 \bar{F}(t)]} = l(x) < \infty, \forall x,$$

where $l(x) = \lim_{t \rightarrow \infty} \frac{\bar{F}(t+x)}{\bar{F}(t)}$.

	AIC	BIC	K-S	p-value
$T_3(Pareto, \lambda_0, \lambda_1, \lambda_2)$	764.6231	771.2773	0.10025	0.7914
$T_2(Pareto, \lambda_0, \lambda_1)$	767.3783	772.369	0.15253	0.2936
$T_1(Pareto, \lambda_0)$	774.6983	778.0254	0.235	0.02232

Table 5: The criteria comparison for data set 2.

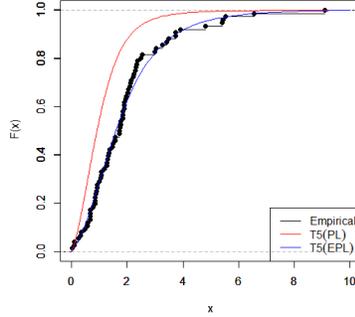


Figure 6: The empirical cdf for example 1.

	w	p-value
$T_3(\text{Pareto}, \lambda_0, \lambda_1, \lambda_2)$ vs $T_2(\text{Pareto}, \lambda_0, \lambda_1)$	7.658383	0.005650913
$T_3(\text{Pareto}, \lambda_0, \lambda_1, \lambda_2)$ vs $T_1(\text{Pareto}, \lambda_0)$	12.56532	0.001868428

Table 6: Comparison of submodels for data set 2

Now, for $(n - 1) \rightarrow n$, we have

$$\lim_{t \rightarrow \infty} \frac{\bar{F}_n(t+x)}{\bar{F}_n(t)} = \lim_{t \rightarrow \infty} \frac{\bar{F}_{n-1}(t+x)[1 - \lambda_{n-1} + \lambda_{n-1}\bar{F}_{n-1}(t+x)]}{\bar{F}_{n-1}(t)[1 - \lambda_{n-1} + \lambda_{n-1}\bar{F}_{n-1}(t)]} = l(x).$$

By mathematical induction, we conclude the proof. □

Proof of Theorem 3

Proof. Rewriting the corresponding density functions of F_1, F_2, \dots, F_n in terms of survival functions, we get

$$\begin{aligned} f_1(x) &= f(x)[1 - \lambda_0 + 2\lambda_0\bar{F}(x)] \\ f_2(x) &= f_1(x)[1 - \lambda_1 + 2\lambda_1\bar{F}_1(x)] \\ f_3(x) &= f_2(x)[1 - \lambda_2 + 2\lambda_2\bar{F}_2(x)] \\ &\dots \\ f_n(x) &= f_{n-1}(x)[1 - \lambda_{n-1} + 2\lambda_{n-1}\bar{F}_{n-1}(x)] \end{aligned}$$

For $n = 1$, we have

$$\begin{aligned} \frac{d}{dx} \frac{1}{h_1(x)} &= \frac{d}{dx} \frac{\bar{F}_1}{f_1(x)} = \frac{d}{dx} \frac{\bar{F}(x)[1 - \lambda_0 + \lambda_0 \bar{F}(x)]}{f(x)[1 - \lambda_0 + 2\lambda_0 \bar{F}(x)]} \\ &= - \left(\frac{d}{dx} \frac{\bar{F}(x)}{f(x)} \right) \frac{1 - \lambda_0 + \lambda_0 \bar{F}(x)}{1 - \lambda_0 + 2\lambda_0 \bar{F}(x)} + \frac{\lambda_0 \bar{F}(x)[\lambda_0 - 1]}{[1 - \lambda_0 + 2\lambda_0 \bar{F}(x)]^2} \end{aligned}$$

Because $F \in \mathcal{C}_3$, we have

$$\lim_{x \rightarrow \infty} \frac{1 - \lambda_0 + \lambda_0 \bar{F}(x)}{1 - \lambda_0 + 2\lambda_0 \bar{F}(x)} = 1 \quad \text{and} \quad \lim_{x \rightarrow \infty} \frac{\lambda_0 \bar{F}(x)[\lambda_0 - 1]}{[1 - \lambda_0 + 2\lambda_0 \bar{F}(x)]^2} = 0$$

Therefore $\lim_{x \rightarrow \infty} \frac{d}{dx} \frac{1}{h_1(x)} = 0$.

Now, for $(n - 1) \rightarrow n$, we have

$$\begin{aligned} \frac{d}{dx} \frac{1}{h_n(x)} &= \frac{d}{dx} \frac{\bar{F}_n(x)}{f_n(x)} = \frac{d}{dx} \frac{\bar{F}_{n-1}(x)[1 - \lambda_{n-1} + \lambda_{n-1} \bar{F}_{n-1}(x)]}{f_{n-1}(x)[1 - \lambda_{n-1} + 2\lambda_{n-1} \bar{F}_{n-1}(x)]} \\ &= \left(\frac{d}{dx} \frac{\bar{F}_{n-1}(x)}{f_{n-1}(x)} \right) \frac{1 - \lambda_{n-1} + \lambda_{n-1} \bar{F}_{n-1}(x)}{1 - \lambda_{n-1} + 2\lambda_{n-1} \bar{F}_{n-1}(x)} \\ &\quad + \frac{\lambda_{n-1} \bar{F}_{n-1}(x)[\lambda_{n-1} - 1]}{[1 - \lambda_{n-1} + 2\lambda_{n-1} \bar{F}_{n-1}(x)]^2} \end{aligned}$$

Because $F_{n-1} \in \mathcal{C}_3$, we have

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{1 - \lambda_{n-1} + \lambda_{n-1} \bar{F}_{n-1}(x)}{1 - \lambda_{n-1} + 2\lambda_{n-1} \bar{F}_{n-1}(x)} &= 1 \quad \text{and} \\ \lim_{x \rightarrow \infty} \frac{\lambda_{n-1} \bar{F}_{n-1}(x)[\lambda_{n-1} - 1]}{[1 - \lambda_{n-1} + 2\lambda_{n-1} \bar{F}_{n-1}(x)]^2} &= 0 \end{aligned}$$

Therefore, we get that $\lim_{x \rightarrow \infty} \frac{d}{dx} \frac{1}{h_n(x)} = 0$.

By mathematical induction, we conclude the proof. \square

Appendix B

The second partial derivatives of the log-likelihood function are

$$\begin{aligned} \frac{\partial^2 \mathcal{L}}{\partial \theta_s \partial \theta_j} &= \sum_{i=1}^n \frac{1}{[f_{n-1}(x_i; \rho_{n-2})]^2} \left\{ f_{n-1}(x_i; \rho_{n-2}) \frac{\partial^2}{\partial \theta_s \partial \theta_j} f_{n-1}(x_i; \rho_{n-2}) \right. \\ &\quad \left. - \left[\frac{\partial}{\partial \theta_s} f_{n-1}(x_i; \rho_{n-2}) \right] \left[\frac{\partial}{\partial \theta_j} f_{n-1}(x_i; \rho_{n-2}) \right] \right\} \\ &\quad - \frac{1}{[1 + \lambda_{n-1} - 2\lambda_{n-1} F_{n-1}(x_i; \rho_{n-2})]^2} \\ &\quad \times \left\{ 2\lambda_{n-1} \frac{\partial^2}{\partial \theta_s \partial \theta_j} F_{n-1}(x_i; \rho_{n-2}) \right. \\ &\quad \left. + 4\lambda_{n-1}^2 \left[\frac{\partial}{\partial \theta_s} F_{n-1}(x_i; \rho_{n-2}) \right] \left[\frac{\partial}{\partial \theta_j} F_{n-1}(x_i; \rho_{n-2}) \right] \right\}, j, s = \overline{1, k} \end{aligned}$$

$$\begin{aligned} \frac{\partial^2 \mathcal{L}}{\partial \theta_s \partial \lambda_j} &= \sum_{i=1}^n \frac{1}{[f_{n-1}(x_i; \rho_{n-2})]^2} \left\{ f_{n-1}(x_i; \rho_{n-2}) \frac{\partial^2}{\partial \theta_s \partial \lambda_j} f_{n-1}(x_i; \rho_{n-2}) \right. \\ &\quad \left. - \left[\frac{\partial}{\partial \theta_s} f_{n-1}(x_i; \rho_{n-2}) \right] \left[\frac{\partial}{\partial \lambda_j} f_{n-1}(x_i; \rho_{n-2}) \right] \right\} \\ &\quad - \frac{1}{[1 + \lambda_{n-1} - 2\lambda_{n-1} F_{n-1}(x_i; \rho_{n-2})]^2} \left\{ 2\lambda_{n-1} \frac{\partial^2}{\partial \theta_s \partial \lambda_j} F_{n-1}(x_i; \rho_{n-2}) \right. \\ &\quad \left. + 4\lambda_{n-1}^2 \left[\frac{\partial}{\partial \theta_s} F_{n-1}(x_i; \rho_{n-2}) \right] \left[\frac{\partial}{\partial \lambda_j} F_{n-1}(x_i; \rho_{n-2}) \right] \right\}, \end{aligned}$$

$$j = \overline{0, n-2}, s = \overline{1, k}$$

where

$$\begin{aligned} \frac{\partial^2}{\partial \theta_s \partial \lambda_j} f_n(x_i; \rho_{n-1}) &= -2\lambda_{n-1} \frac{\partial}{\partial \theta_s} F_{n-1}(x_i; \rho_{n-2}) \frac{\partial}{\partial \lambda_j} f_{n-1}(x_i; \rho_{n-2}) \\ &\quad + [1 + \lambda_{n-1} - 2\lambda_{n-1} F_{n-1}(x_i; \rho_{n-2})] \frac{\partial^2}{\partial \lambda_j \partial \theta_s} f_{n-1}(x_i; \rho_{n-2}) \\ &\quad - 2\lambda_{n-1} \left[\frac{\partial}{\partial \theta_s} f_{n-1}(x_i; \rho_{n-2}) \right] \frac{\partial}{\partial \lambda_j} F_{n-1}(x_i; \rho_{n-2}) \\ &\quad + f_{n-1}(x_i; \rho_{n-2}) \frac{\partial^2}{\partial \theta_s \partial \lambda_j} F_{n-1}(x_i; \rho_{n-2}) \end{aligned}$$

and

$$\begin{aligned} \frac{\partial^2}{\partial \theta_s \partial \lambda_j} F_n(x_i; \rho_{n-1}) &= -2\lambda_{n-1} \frac{\partial}{\partial \theta_s} F_{n-1}(x_i; \rho_{n-2}) \frac{\partial}{\partial \lambda_j} F_{n-1}(x_i; \rho_{n-2}) \\ &+ [1 + \lambda_{n-1} - 2\lambda_{n-1} F_{n-1}(x_i; \rho_{n-2})] \frac{\partial^2}{\partial \theta_s \partial \lambda_j} F_{n-1}(x_i; \rho_{n-2}), \quad j \neq n-1 \end{aligned}$$

$$\begin{aligned} \frac{\partial^2 \mathcal{L}}{\partial \theta_s \partial \lambda_{n-1}} &= \sum_{i=1}^n \frac{\frac{\partial}{\partial \theta_s} F_{n-1}(x_i; \rho_{n-2})}{[1 + \lambda_{n-1} - 2\lambda_{n-1} F_{n-1}(x_i; \rho_{n-2})]^2} \\ &\times \left\{ 1 - \lambda_{n-1} + 2\lambda_{n-1} F_{n-1}(x_i; \rho_{n-2}) \right\}, \quad s = \overline{1, k} \end{aligned}$$

$$\begin{aligned} \frac{\partial^2 \mathcal{L}}{\partial \lambda_r \partial \lambda_j} &= \sum_{i=1}^n \frac{1}{[f_{n-1}(x_i; \rho_{n-2})]^2} \left\{ f_{n-1}(x_i; \rho_{n-2}) \frac{\partial^2}{\partial \lambda_r \partial \lambda_j} f_{n-1}(x_i; \rho_{n-2}) \right. \\ &- \left. \left[\frac{\partial}{\partial \lambda_r} f_{n-1}(x_i; \rho_{n-2}) \right] \left[\frac{\partial}{\partial \lambda_j} f_{n-1}(x_i; \rho_{n-2}) \right] \right\} \\ &- \frac{1}{[1 + \lambda_{n-1} - 2\lambda_{n-1} F_{n-1}(x_i; \rho_{n-2})]^2} \left\{ 2\lambda_{n-1} \frac{\partial^2}{\partial \lambda_r \partial \lambda_j} F_{n-1}(x_i; \rho_{n-2}) \right. \\ &+ \left. 4\lambda_{n-1}^2 \left[\frac{\partial}{\partial \lambda_r} F_{n-1}(x_i; \rho_{n-2}) \right] \left[\frac{\partial}{\partial \lambda_j} F_{n-1}(x_i; \rho_{n-2}) \right] \right\}, \quad j, r = \overline{0, n-2} \end{aligned}$$

where

$$\begin{aligned} \frac{\partial^2}{\partial \lambda_r \partial \lambda_j} f_n(x_i; \rho_{n-1}) &= -2\lambda_{n-1} \frac{\partial}{\partial \lambda_r} F_{n-1}(x_i; \rho_{n-2}) \frac{\partial}{\partial \lambda_j} f_{n-1}(x_i; \rho_{n-2}) \\ &+ [1 + \lambda_{n-1} - 2\lambda_{n-1} F_{n-1}(x_i; \rho_{n-2})] \frac{\partial^2}{\partial \lambda_j \partial \lambda_r} f_{n-1}(x_i; \rho_{n-2}) \\ &- 2\lambda_{n-1} \left[\frac{\partial}{\partial \lambda_r} f_{n-1}(x_i; \rho_{n-2}) \frac{\partial}{\partial \lambda_j} F_{n-1}(x_i; \rho_{n-2}) + f_{n-1}(x_i; \rho_{n-2}) \right. \\ &\times \left. \frac{\partial^2}{\partial \lambda_r \partial \lambda_j} F_{n-1}(x_i; \rho_{n-2}) \right], \quad r = \overline{0, n-2} \end{aligned}$$

and

$$\begin{aligned} \frac{\partial^2}{\partial \lambda_r \partial \lambda_j} F_n(x_i; \rho_{n-1}) &= -2\lambda_{n-1} \frac{\partial}{\partial \lambda_r} F_{n-1}(x_i; \rho_{n-2}) \frac{\partial}{\partial \lambda_j} F_{n-1}(x_i; \rho_{n-2}) \\ &+ [1 + \lambda_{n-1} - 2\lambda_{n-1} F_{n-1}(x_i; \rho_{n-2})] \frac{\partial^2}{\partial \lambda_r \partial \lambda_j} F_{n-1}(x_i; \rho_{n-2}), \quad k \neq n-1 \end{aligned}$$

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \lambda_j \partial \lambda_{n-1}} &= \sum_{i=1}^m \frac{-2 \frac{\partial}{\partial \lambda_j} F_{n-1}(x_i; \rho_{n-2})}{[1 + \lambda_{n-1} - 2\lambda_{n-1} F_{n-1}(x_i; \rho_{n-2})]^2} \\ &\times \left\{ 1 + 2\lambda_{n-1} - 4\lambda_{n-1} F_{n-1}(x_i; \rho_{n-2}) \right\}, j \neq n-1 \end{aligned}$$

$$\frac{\partial^2 \mathcal{L}}{\partial^2 \lambda_{n-1}} = \sum_{i=1}^m \frac{-[1 - 2F_{n-1}(x_i; \rho_{n-2})]^2}{[1 + \lambda_{n-1} - 2\lambda_{n-1} F_{n-1}(x_i; \rho_{n-2})]^2}$$

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