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Weaker assumptions for convergence of extended block Kaczmarz and Jacobi projection algorithms

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Abstract

Recent developments in the field of image reconstruction have given rise to the use of projective iterative methods, such as Kaczmarz and Jacobi, when solving inconsistent linear least squares problems. In this paper we try to generalize previous results concerning extended block versions of these two algorithms. We replace the inverse operator with the Moore-Penrose pseudoinverse and try to prove convergence under weaker assumptions. In order to accomplish this task, we show that these algorithms are special cases of a general iterative process for which convergence is already established.

1 Introduction

Many important real-world problems gives after appropriate discretization techniques large, sparse and ill-conditioned systems of linear equations

$$Ax = b, \tag{1}$$

where A is an $m \times n$ real matrix and $b \in I\!R^m$. But, because of both discretization errors and measurements, the right hand side b can be affected by noise and the system (1) becomes inconsistent. For this it must be reformulated and

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a new (more general) type of solution defined. In order to present these constructions we will introduce the following notations which will be used on the whole paper. Therefore $A^T, A_i, A^j, \mathcal{R}(A), \mathcal{N}(A)$, P_V will denote the transpose, i-th row, j-th column, range and null space of A, the projection onto a vector subspace V; also $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ will denote the Euclidean scalar product and norm and all the vectors appearing in the paper will be considered as column vectors. We will assume in the rest of the paper that

$$A_i \neq 0, A^j \neq 0, \text{ for all } i \in \{1, 2, \dots m\}, \text{ and } j \in \{1, 2, \dots n\}. \tag{2}$$

The inconsistency of the system (1) means that $b \notin \mathcal{R}(A)$, i.e.

$$P_{\mathcal{N}(A^T)}(b) = b - P_{\mathcal{R}(A)}(b) \neq 0, \tag{3}$$

thus

$$||Ax - b||^2 = ||Ax - P_{\mathcal{R}(A)}(b)||^2 + ||P_{\mathcal{N}(A^T)}(b)||^2 > 0, \forall x \in I\!\!R^n. \tag{4}$$

Thus, instead of $Ax = b \Leftrightarrow ||Ax - b|| = 0$ we will reformulate the system (1) in the form: find $x \in I\!\!R^n$ such that

$$\|Ax - b\| = \min\{\|Az - b\|, z \in I\!\!R^n\}, \tag{5}$$

i.e. as a linear least squares problem. From (4) it results that the problem (5) is equivalent with the system

$$Ax = P_{\mathcal{R}(A)}(b), \tag{6}$$

which tells us that it has always solutions. These more general solutions will be called *least squares solutions* and their set denoted by $LSS(A; b)$. Moreover, it can be proved that it exists a unique element $x_{LS} \in LSS(A; b)$ with the properties that it is orthogonal on $\mathcal{N}(A^T)$ and has minimal Euclidean norm among the other solutions. It is called the *minimal norm solution*.

In the present paper we are concerned with the approximation of solutions of the problem (5), using block-type extended Kaczmarz and Jacobi projection methods.

Block Kaczmarz and Jacobi projection methods for solving inconsistent linear least squares problems as (5) have been first considered in [4] and extended for weighted formulation of (5) in [3]. But, both of these papers analyse block versions of original Kaczmarz or Jacobi projection methods. Extended block versions of these two algorithms were introduced in [6] and convergence was proved under nonsingularity assumptions on matrices resulted from row and column block decompositions. These assumptions are rather restrictive in practical applications, and the purpose of this paper is to eliminate them from the algorithms. In this respect, we will use the results from the paper [5], in which the authors proposed a general iterative procedure for finding a numerical solution of (5), which includes the algorithms of Kaczmarz, Cimmino, Jacobi Projective and Diagonal Weighting (see, e.g., [9], [5], [4] and [7], respectively). Moreover, they propose and theoretically analyse an extended version of this method, which generates a sequence of approximations convergent to an element from $LSS(A;b)$.

In this paper we show that, after replacing the inverse operator with the Moore-Penrose pseudoinverse, the earlier results on this topic, see [6, Theorem 3.4 and Theorem 6.7], still hold without the nonsingularity hypotheses. This is accomplished by firstly proving that the two algorithms are special cases of the extended iterative method from [5] above mentioned.

The paper is organized as follows: in Section 2 we briefly present Extended Block Kaczmarz (EBK) and Extended Block Jacobi with Relaxation Parameters (EBJRP) algorithms illustrated in [6] and the Extended General (EGEN) procedure from [5]; in Section 3 we show that the EBK and EBJRP solvers, defined with the Moore-Penrose pseudoinverses, belong to the family of EGEN methods.

2 Solving the inconsistent linear least squares problem

We consider block row decompositions of the matrix A and corresponding vector b. In this respect, let $p \geq 2$, $1 \leq m_i \leq m$, with $i \in \{1, 2, \ldots, p\}$, such that $m_1 + m_2 + \cdots + m_p = m$,

$$A = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_p \end{bmatrix}, \text{ and } b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_p \end{bmatrix}, \tag{7}$$

where A_i are $m_i \times n$ matrices and $b_i \in I\!R^{m_i}$.

Similarly, for $q \geq 2$ and $n = n_1 + n_2 + \cdots + n_q$, with $1 \leq n_j < n$ for any $j \in \{1, 2, \ldots, q\}$, the block column decomposition of A is given by

$$A^T = \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_q \end{bmatrix}, \tag{8}$$

where B_j are $n_j \times m$ real matrices.

In [6] the author introduced extended block versions of the Kaczmarz and Jacobi with relaxation parameters algorithms. They proved that under the hypotheses

$$\det(A_i A_i^T) \neq 0,\ \forall\, i \in \{1, 2, \ldots, p\} \tag{9}$$

and

$$\det(B_j B_j^T) \neq 0,\ \forall\, j \in \{1, 2, \ldots, q\} \tag{10}$$

these methods converge to an element of the linear least square solutions set of the problem (5).

Let the linear applications $f_0^i(b; \cdot), F_0(b; \cdot) : I\!R^n \to I\!R^n$ be defined by

$$f_0^i(b; x) = x + A_i^T (A_i A_i^T)^{-1} (b_i - A_i x),\ \forall\, i \in \{1, 2, \ldots, p\}, \tag{11}$$

$$F_0(b; x) = \left(f_0^1 \circ f_0^2 \circ \cdots \circ f_0^p\right)(b; x) \tag{12}$$

and the linear mapping

$$\Phi_0 = \prod_{j=1}^{q} I - B_j^T (B_j B_j^T)^{-1} B_j. \tag{13}$$

The extended block version of the Kaczmarz Algorithm defined according to [6] now follows.

Extended Block Kaczmarz Algorithm(EBK)

Initialization: $x^0 \in R^n$ is arbitrary and $y^0 = b$.

Iterative step For every $k \geq 0$,

$$y^{k+1} = \Phi_0 y^k, \tag{14}$$

$$b^{k+1} = b - y^{k+1}, \tag{15}$$

$$x^{k+1} = F_0(b^{k+1}; x^k). \tag{16}$$

Now, for the real parameters $\omega, \alpha \neq 0$, consider

$$Q_0^\omega = I - \omega \sum_{i=1}^{p} A_i^T (A_i A_i^T)^{-1} A_i, \tag{17}$$

$$R_0^\omega = \omega\, \mathrm{col}\left[\ A_1^T (A_1 A_1^T)^{-1} \mid A_2^T (A_2 A_2^T)^{-1} \mid \ \ldots \ \mid A_p^T (A_p A_p^T)^{-1}\ \right] \tag{18}$$

and

$$\Phi_0^\alpha = I - \alpha \sum_{j=1}^{q} B_j^T (B_j B_j^T)^{-1} B_j. \tag{19}$$

An extended version of the Jacobi method with relaxation parameters was given in [6].

Extended Block Jacobi Algorithm with Relaxation Parameters (EBJRP)

Initialization: $x^0 \in R^n$ is arbitrary and $y^0 = b$.

Iterative step For every $k \geq 0$,

$$y^{k+1} = \Phi_0^{\alpha} y^k, \tag{20}$$

$$b^{k+1} = b - y^{k+1}, \tag{21}$$

$$x^{k+1} = Q_0^{\omega} x^k + R_0^{\omega} b^{k+1}. \tag{22}$$

Unfortunately, in real examples, conditions of the type (9) - (10) are usually not true or hard to verify. Following the considerations from [6], we will show that if we use the Moore-Penrose pseudoinverse rather than the inverse operator, the convergence results remain true without the assumptions (9) - (10). This is accomplished by proving that the two algorithms are particular cases of a general projection method, described below.

For the matrices Q, R and U of dimensions $n \times n$, $n \times m$ and $m \times m$, respectively, the authors defined in [5] the following extended general iterative method.

Extended General Algorithm (EGEN)

Initialization: $x^0 \in R^n$ is arbitrary and $y^0 = b$.

Iterative step For every $k \geq 0$,

$$y^{k+1} = U y^k, \tag{23}$$

$$b^{k+1} = b - y^{k+1}, \tag{24}$$

$$x^{k+1} = Q x^k + R b^{k+1}. \tag{25}$$

Let us suppose that Q, R and U satisfy the following general assumptions

$$Q + RA = I, \tag{26}$$

$$\forall y \in I\!R^m, \; Ry \in \mathcal{R}(A^T), \tag{27}$$

$$\text{if } \tilde{Q} = Q P_{\mathcal{R}(A^T)} \text{ then } \left\| \tilde{Q} \right\| < 1, \tag{28}$$

$$\text{if } x \in \mathcal{N}(A^T) \text{ then } Ux = x, \tag{29}$$

$$\text{if } x \in \mathcal{R}(A) \text{ then } Ux \in \mathcal{R}(A), \tag{30}$$

$$\text{if } \tilde{U} = UP_{\mathcal{R}(A)} \text{ then } \left\| \tilde{U} \right\| < 1, \tag{31}$$

where by $\| T \| = \sup_{x \neq 0} \frac{\|Tx\|}{\|x\|}$ we denoted the spectral norm of a matrix T. The authors proved in [5] the next convergence result for the EGEN algorithm.

Theorem 1. *[5, Theorem 2.6] Let us suppose that the matrices Q and R satisfy equations (26) - (28) and for U the properties (29) - (31) hold. Then, for any $x^0 \in I\!R^n$, the sequence $(x^k)_{k \geq 0}$ generated with the algorithm EGEN converges and*

$$\lim_{k \to \infty} x^k = P_{\mathcal{N}(A)}(x^0) + x_{LS}. \tag{32}$$

It is known (for details and proofs see, e.g., [8]) that $LSS(A; b) = \mathcal{N}(A) + x_{LS}$, therefore, the EGEN procedure is a solver for the inconsistent problem (5).

3 Main results

The next result provides us with important information about the properties of the Moore-Penrose pseudoinverse (for details and proofs see e.g. [1, 2]).

Lemma 1. *(i) The following relations hold for the Moore-Penrose pseudoinverse of an $m \times n$ real matrix A, denoted by $A^\dagger$.*

$$A^\dagger = A^T \left(AA^T \right)^\dagger, \; A^T = A^\dagger AA^T, \quad \textit{and } A^T = A^T AA^\dagger. \tag{33}$$

(ii) The orthogonal projectors $P_{\mathcal{R}(A^T)}$ and $P_{\mathcal{N}(A)}$ are given by

$$P_{\mathcal{R}(A^T)} = A^\dagger A, \; P_{\mathcal{N}(A)} = I - A^\dagger A. \tag{34}$$

In the rest of the paper we will denote by f^i, F, Φ, Q^ω, R^ω and Φ^α, the linear operators f_0^i, F_0, Φ_0, Q_0^ω, R_0^ω and Φ_0^α defined according to (11) - (13) and (17) - (19), respectively, in which we replaced the inverse mapping with the Moore-Penrose pseudoinverse. For any $m \times n$ matrix M, let $M^\dagger$ be its (unique) $n \times m$ pseudoinverse.

For every $i \in \{1, 2, \ldots, p\}$ and $j \in \{1, 2, \ldots, q\}$, we define the matrices

$$P_i = I - A_i^T (A_i A_i^T)^\dagger A_i, \; \phi_j = I - B_j^T (B_j B_j^T)^\dagger B_j, \tag{35}$$

$$\bar{P}_i = A_i^T (A_i A_i^T)^\dagger A_i, \; \bar{\phi}_j = B_j^T (B_j B_j^T)^\dagger B_j \tag{36}$$

and make the following remarks. Equations (35), (33) and (34) yield that P_i is the orthogonal projector on $\mathcal{N}(A_i)$. From (35), (33) and (34), it results that ϕ_j is the orthogonal projector on $\mathcal{N}(B_j)$. Similarly, (36), (33) and (34) give that $\bar{P}_i$ is the orthogonal projector on $\mathcal{R}(A_i^T)$. Using (36), (33) and (34), we obtain that $\bar{\phi}_j$ is the orthogonal projector on $\mathcal{R}(B_j^T)$. Also, the following properties hold

$$P_i^2 = P_i,\ P_i^T = P_i \text{ and } \|P_i\| = 1;\ \phi_j^2 = \phi_j,\ \phi_j^T = \phi_j \text{ and } \|\phi_j\| = 1; \quad (37)$$

$$\bar{P}_i^2 = \bar{P}_i,\ \bar{P}_i^T = \bar{P}_i \text{ and } \left\|\bar{P}_i\right\| = 1;\ \bar{\phi}_j^2 = \bar{\phi}_j,\ \bar{\phi}_j^T = \bar{\phi}_j \text{ and } \left\|\bar{\phi}_j\right\| = 1. \quad (38)$$

We will now consider the corresponding linear mappings

$$\Delta_i = A_i^T (A_i A_i^T)^\dagger,\ Q_i = P_1 P_2 \dots P_i,\ \forall\, i \in \{1, 2, \dots, p\}, \quad (39)$$

$$Q = P_1 P_2 \dots P_p, \quad (40)$$

$$R = \operatorname{col}[\ \Delta_1 \mid Q_1\Delta_2 \mid \ \dots \ \mid Q_{p-1}\Delta_p\ ], \quad (41)$$

$$\Phi_j = \phi_1\phi_2 \dots \phi_j,\ \forall\, j \in \{1, 2, \dots, q\}, \quad (42)$$

$$\tilde{Q}^\omega = Q^\omega P_{\mathcal{R}}(A^T) \text{ and } \tilde{\Phi}^\alpha = \Phi^\alpha P_{\mathcal{R}}(A). \quad (43)$$

Consequently, from (13), (37), (17), (38), (18), (39), (19) we have

$$\Phi = \phi_1\phi_2 \dots \phi_q, \quad (44)$$

$$Q^\omega = I - \omega \sum_{i=1}^{p} \bar{P}_i, \quad (45)$$

$$R^\omega = \omega \operatorname{col}[\ \Delta_1 \mid \Delta_2 \mid \ \dots \ \mid \Delta_p\ ] \quad (46)$$

and

$$\Phi^\alpha = I - \alpha \sum_{j=1}^{q} \bar{\phi}_j. \quad (47)$$

We will require the next three results to prove that the extended block versions of the Kaczmarz and Jacobi algorithms are special cases of the EGEN method.

Lemma 2. *If the $n \times n$ matrix Q and the $n \times m$ matrix R satisfy (26) - (27), the following properties are true.*

$$\text{if } x \in \mathcal{N}(A) \text{ then } Qx = x \quad (48)$$

and

$$\text{if } x \in \mathcal{R}(A^T) \text{ then } Qx \in \mathcal{R}(A^T). \quad (49)$$

Lemma 3. *The following equality holds*

$$\mathcal{N}(A) = \bigcap_{i=1}^{p} \{x \in I\!R^n, P_i x = x\}, \tag{50}$$

Proof. If $x \in \mathcal{N}(A)$, then, for any $i \in \{1, 2, \ldots, p\}$, $x \in \mathcal{N}(A_i)$ and from (35) we have that $P_i x = x$.

Let us now take $x \in I\!R^n$ such that $P_i x = x$ for every $i \in \{1, 2, \ldots, p\}$. We then have $x - A_i^T (A_i A_i^T)^\dagger A_i x = x$, which together with (33) and (34) yields $P_{\mathcal{R}(A^T)}(x) = 0$. Hence, $x \in \mathcal{N}(A_i)$ for every $x \in \{1, 2, \ldots, p\}$ and the proof is complete. □

Lemma 4. *[9, Lemma 2, Corollary 3] If $P_1, P_2, \ldots P_p$ are the orthogonal projectors defined according to (35) and Q is the linear mapping from (40), then*

$$\|Qx\| = \|x\| \text{ if and only if } x \in \mathcal{N}(A) \tag{51}$$

and

$$\|Q\| \leq 1. \tag{52}$$

The following theorem ensures the convergence of the extended block Kaczmarz algorithm defined using the Moore-Penrose pseudoinverse, without the assumptions (9) - (10).

Theorem 2. *If Q, R and Φ are linear applications defined according to (40), (41) and (44), respectively, then*

(i) we have the equality

$$F(b; x) = Qx + Rb, \tag{53}$$

(ii) Q and R satisfy (26) - (28),

(iii) for the matrix Φ the properties (29) - (31) hold.

Proof. (i) For any $i \in \{1, 2, \ldots, n\}$, (35) and (39) yield

$$f_i(b; x) = (I - A_i^T (A_i A_i^T)^\dagger A_i)x + A_i^T (A_i A_i^T)^\dagger b_i = P_i x + \Delta_i b_i. \tag{54}$$

Hence,

$$f_1 \circ f_2 = P_1(P_2 x + \Delta_2 b_2) + \Delta_1 b_1 = Q_2 x + Q_1 \Delta_2 b_2 + \Delta_1 b_1. \tag{55}$$

Using a recursive argument we obtain from (41) that

$$f_1 \circ f_2 \circ \cdots \circ f_p = Qx + \sum_{i=1}^{p} Q_{i-1} \Delta_i b_i = Qx + Rb, \tag{56}$$

where $Q_0 = I$, proving the equality (53).

(ii) In order to prove the assumption (26), from (41), (40), (35) and (39) we will write

$$RA = \sum_{i=1}^{p} Q_{i-1}\Delta_i A_i = \sum_{i=1}^{p} P_1 P_2 \dots P_{i-1} A_i^T (A_i A_i^T)^\dagger A_i$$
$$= \sum_{i=1}^{p} P_1 P_2 \dots P_{i-1}(I - P_i) = I - Q. \tag{57}$$

Using (41) it may be easily proved that

$$\mathcal{R}(R) = \sum_{i=1}^{p} \mathcal{R}(Q_{i-1}\Delta_i). \tag{58}$$

If $y \in \mathcal{R}(Q_{i-1}\Delta_i)$, where $i \in \{1, 2, \dots, p\}$, there exists $z \in I\!R^m$ such that $Q_{i-1}\Delta_i z = y$. Since $P_1, P_2, \dots, P_{i-1}$ are orthogonal projectors, for any $x \in \mathcal{N}(A)$, the relations (40), (35), (39) and (50) give us

$$\langle y, x\rangle = \langle Q_{i-1}\Delta_i z, x\rangle = \langle \Delta_i z, P_{i-1}^T \dots P_1^T x\rangle$$
$$= \langle A_i^T (A_i A_i^T)^\dagger z, x\rangle = \langle (A_i A_i^T)^\dagger z, A_i x\rangle = 0. \tag{59}$$

Since $x \in \mathcal{N}(A)$ was arbitrarily chosen, we get $y \in \mathcal{R}(A^T)$, which, together with (58), implies that $\mathcal{R}(R) \subset \mathcal{R}(A^T)$.

Using the definition $\tilde{Q} = QP_{\mathcal{R}(A^T)}$ and (52) we obtain

$$\begin{aligned}
\left\|\tilde{Q}\right\| &= \sup\{\left\|\tilde{Q}x\right\|, \text{ for all } x \in I\!R^n \text{ with } \|x\| = 1\} \\
&= \sup\{\left\|\tilde{Q}P_{\mathcal{R}(A^T)}(x) + \tilde{Q}P_{\mathcal{N}(A)}(x)\right\|, \text{ for all } x \in I\!R^n \text{ with } \|x\| = 1\} \\
&= \sup\{\left\|\tilde{Q}P_{\mathcal{R}(A^T)}(x)\right\|, \text{ for all } x \in I\!R^n \text{ with } \left\|P_{\mathcal{R}(A^T)}(x)\right\| \le 1\} \\
&= \sup\{\left\|QP_{\mathcal{R}(A^T)}(x)\right\|, \text{ for all } x \in I\!R^n \text{ with } \left\|P_{\mathcal{R}(A^T)}(x)\right\| = 1\} \\
&\le 1.
\end{aligned} \tag{60}$$

Let us suppose that $\left\|\tilde{Q}\right\| = 1$. From the above inequality it exists $x \in \mathcal{R}(A^T)$ such that $\|Qx\| = \|x\|$. Consequently, (51) yields $x \in \mathcal{N}(A)$, which implies that $x = 0$, in contradiction with the assumption that $\|x\| = 1$. Therefore, $\left\|\tilde{Q}\right\| < 1$.

(iii) We will firstly observe that, given the linear system of equations $A^T y = 0$ and the block column decomposition (8) of A, the construction of the matrix Φ is equivalent to that of Q for the system $Ax = b$. Likewise, the matrix $S = \text{col}\left[\ B_1^T(B_1B_1^T)^\dagger \mid \Phi_1 B_2^T(B_2B_2^T)^\dagger \mid \ \ldots \ \mid \Phi_{q-1}B_q^T(B_qB_q^T)^\dagger\ \right]$ corresponds to R. Hence, from (ii) and Lemma 2, the matrix Φ satisfies (29) - (31) with A^T instead of A and the proof is complete. □

In the case of the EBJRP algorithm we will confirm similar statements. The assumptions (28) and (31) will be proved using results from [4, 6].

Theorem 3. *The following properties are true*
(i) Q^ω and R^ω satisfy (26) - (27),
(ii) for the matrix Φ^α the assumptions (29) - (30) hold.

Proof. (i) Using (46), (39), (36) and (45) we obtain

$$R^\omega A = \omega \sum_{i=1}^{p} \Delta_i A_i = \omega \sum_{i=1}^{p} \bar{P}_i = I - Q^\omega. \tag{61}$$

The definition of (39) yields $\mathcal{R}(R^\omega) = \sum_{i=1}^{p} \mathcal{R}(\Delta_i)$. Now, for any $i \in \{1, 2, \ldots, p\}$, let $y \in \mathcal{R}(\Delta_i)$ and $x \in \mathcal{N}(A)$. It results that there exists $z \in I\!R^m$ such that $\Delta_i z = y$ and from (39) we have $\langle y, x\rangle = \langle \Delta_i z, x\rangle = \langle (A_iA_i^T)^\dagger z, A_i x\rangle = 0$. Hence, $y \in \mathcal{R}(A^T)$, and $\mathcal{R}(R^\omega) \subset \mathcal{R}(A^T)$.

(ii) Following the reasoning from Theorem 2, the previous part of this result and Lemma 2 imply that the matrix Φ^α satisfies the properties (29) - (30). □

We will use the following results from [6] and [4] to prove that the assumptions (28) and (31) are true for $\tilde{Q}^\omega$ and $\tilde{\Phi}^\omega$, respectively.

Lemma 5. *[6] The matrices $\tilde{Q}^\omega$ and $\tilde{\Phi}^\alpha$ are normal matrices.*

Theorem 4. *[4, Theorem 9] $0 < \omega < \frac{2}{\rho(E)}$ if and only if $\rho(\tilde{Q}^\omega) < 1$, where $E = \frac{1}{\omega} R^\omega A$.*

Corollary 1. *[6] If $0 < \omega < \frac{2}{\rho(E)}$ and $0 < \alpha < \frac{2}{\rho(D)}$, where $D = \frac{1}{\alpha}(I - \Phi^\omega)$, then $\tilde{Q}^\omega$ and $\tilde{\Phi}^\alpha$ satisfy the assumptions (28) and (31), respectively.*

Proof. The results in Theorem 4 are also true for $\tilde{Q}^\omega$ and D. From Lemma 5 we have that $\tilde{Q}^\omega$ and $\tilde{\Phi}^\alpha$ are normal matrices, which implies that $\left\|\tilde{Q}^\omega\right\| = \rho(\tilde{Q}^\omega)$ and $\left\|\tilde{\Phi}^\alpha\right\| = \rho(\tilde{\Phi}^\alpha)$. Consequently, Theorem 4 yields that $\left\|\tilde{Q}^\omega\right\| < 1$ and $\left\|\tilde{\Phi}^\alpha\right\| < 1$. □

From Theorem 3 and Corollary 1 it results that the Extended Block Jacobi with Relaxation Parameters algorithm is a particular case of the EGEN method.

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