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APPROXIMATING FIXED POINTS OF NONSELF CONTRACTIVE TYPE MAPPINGS IN BANACH SPACES ENDOWED WITH A GRAPH

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Dedicated to Professor Ravi P. Agarwal

Abstract

Let K be a non-empty closed subset of a Banach space X endowed with a graph G. We obtain fixed point theorems for nonself G-contractions of Chatterjea type. Our new results complement and extend recent related results [Berinde, V., Păcurar, M., *The contraction principle for nonself mappings on Banach spaces endowed with a graph*, J. Nonlinear Convex Anal. **16** (2015), no. 9, 1925–1936; Balog, L., Berinde, V., *Fixed point theorems for nonself Kannan type contractions in Banach spaces endowed with a graph*, Carpathian J. Math. **32** (2016), no. 3 (in press)] and thus provide more general and flexible tools for studying nonlinear functional equations.

1 Introduction

Let X, Y be linear spaces and $F : D \subset X \to Y$ be a nonlinear mapping. One of the most effective ways to solve the equation

$$F(x) = 0,\ x \in D, \tag{1.1}$$

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is to convert it equivalently into a fixed point problem of the form

$$x = T(x),\ x \in K, \tag{1.2}$$

where $T : K \subset X \to X$ is a mapping constructed by a certain scheme.

For example, in the case of the well-known Newton method, considered here for the sake of simplicity in $X = \mathbb{R}$, the iteration function F involved in (1.2) is given by:

$$Tx = x - F(x)/F'(x), x \in K.$$

The equivalent form (1.2) of equation (1.1) is extremely important for at least two major reasons:

1. Problem (1.2) can be solved by applying a suitable fixed point theorem, thus obtaining an existence or an existence and uniqueness result for the original problem (1.1);

2. The particular form of problem (1.2) now provides a direct way to construct a simple iterative scheme to approximate the solution (s) of (1.1), i.e.,

$$x_{n+1} = Tx_n,\ n \geq 0,$$

with $x_0 \in K$ the starting value.

One of the most important and flexible tools in nonlinear analysis to deal with a problem of the form (1.2) is the well-known Banach contraction principle, stated here in its complete form, see for example [19].

Theorem 1. *Let (X, d) be a complete metric space and $T : X \to X$ a strict contraction, i.e., a map satisfying*

$$d(Tx, Ty) \leq a\, d(x, y)\,, \quad \text{for all } x, y \in X\,, \tag{1.3}$$

where $0 \leq a < 1$ is constant. Then:

$(p1)$ T has a unique fixed point p in X (i.e., $Tp = p$);

$(p2)$ The Picard iteration $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1} = Tx_n\,, \quad n = 0, 1, 2, \ldots \tag{1.4}$$

converges to p, for any $x_0 \in X$.

$(p3)$ The following estimate holds

$$d(x_{n+i-1}, p) \leq \frac{a^i}{1-a}\, d(x_n, x_{n-1})\,, \quad n = 0, 1, 2, \ldots;\ i = 1, 2, \ldots \tag{1.5}$$

As it can be seen from (1.3), Theorem 1 can be applied only to nonlinear equations (1.2) with T a *continuous self mapping.*

But, most of the concrete problems of the form (1.1) or (1.2) we may encounter in pure and applied mathematics involve generally *discontinuous* and/or *non-self* mappings T. This demand motivated authors to search for more general and more flexible fixed point tools that could be applied to such general nonlinear problems.

Kannan [42] has been the first one to consider in this context *discontinuous self* mappings T, by considering instead of (1.3) the following alternative and independent contractive condition: there exists a constant $a \in \left[0, \dfrac{1}{2}\right)$ such that

$$d(Tx, Ty) \leq a\big[d(x, Tx) + d(y, Ty)\big], \quad \text{for all } x, y \in X. \tag{1.6}$$

On the other hand, the study of non-self mappings started with the paper by Caristi [28], in the case of *nonself single-valued* contractions, and with the paper by Assad and Kirk [13], for *non-self multi-valued* contractive mappings $T : K \to \mathcal{P}(X)$, where (X, d) is a convex metric space in the sense of Menger and K is a non-empty closed subset of X.

For some recent and more general results on this topic we refer to [16], [17], [19], [3], [4] and references therein. In a recent paper [20], the second author and M. Păcurar established two fixed point theorems for non self contractions defined on Banach spaces endowed with a graph, while very recently [15], the present authors extended these results to non-self Kannan type contractions $T : X \to X$ on Banach spaces endowed with a graph.

The main aim of the present work is to extend the results in [20] and [15] to the case of mappings satisfying a dual condition of (1.6) which is due to Chatterjea [30] and is independent of both contractive condition (1.3) and (1.6): there exists $a \in (0, \frac{1}{2})$ such that

$$d(Tx, Ty) \leq a\big[d(x, Ty) + d(y, Tx)\big], \quad \text{for all } x, y \in X. \tag{1.7}$$

To accomplish this task, we need some basic prerequisites related to fixed point theorems for self and non self contractions in Banach spaces or convex metric spaces endowed with a graph, basically taken from [20] and [15], and which are presented in the next section.

2 Metric spaces endowed with a graph

Let (X, d) be a metric space and let Δ denote the diagonal of the Cartesian product $X \times X$. Consider now a directed simple graph $G = (V(G), E(G))$

such that the set of its vertices, $V(G)$, coincides with X and $E(G)$, the set of its edges, contains all loops, i.e., $\Delta \subset E(G)$.

By G^{-1} we denote the *converse graph* of G, i.e., the graph obtained by G by reversing its edges, i.e.,

$$E(G^{-1}) = \{(y,x) \in X \times X : (x,y) \in E(G)\}.$$

If x, y are vertices in the graph G, then a *path* from x to y of length N is a sequence $\{x_i\}_{i=1}^N$ of $N+1$ vertices of G such that

$$x_0 = x,\ x_N = y \text{ and } (x_{i-1}, x_i) \in E(G),\ i = 1, 2, \ldots, N.$$

A graph G is said to be connected if there is at least a path between any two vertices. If $G = (V(G), E(G))$ is a graph and $H \subset V(G)$, then the graph $(H, E(H))$ with $E(H) = E(G) \cap (H \times H)$ is called the *subgraph of* G *determined by* H. Denote it by G_H.

If $\tilde{G} = (X, E(\tilde{G}))$ is the symmetric graph obtained by putting together the vertices of both G and G^{-1}, i.e.,

$$E(\tilde{G}) = E(G) \cup E(G^{-1}),$$

then G is called *weakly connected* if $\tilde{G}$ is connected.

A mapping $T : X \to X$ is said to be (well) defined on a metric space endowed with a graph G if it has the property

$$\forall x, y \in X,\ (x,y) \in E(G) \text{ implies } (Tx, Ty) \in E(G). \tag{2.1}$$

According to [41], a mapping $T : X \to X$, which is well defined on a metric space endowed with a graph G, is called a *G-contraction* if there exists a constant $\alpha \in (0,1)$ such that for all $x, y \in X$ with $(x,y) \in E(G)$ we have

$$d(Tx, Ty) \leq \alpha \cdot d(x,y). \tag{2.2}$$

Example 1. *If G_0 is the complete graph on X, that is, $E(G_0) = X \times X$, then a G_0-contraction is a usual contraction in the sense of Banach, i.e., it satisfies condition* (1.3)*, while a G_0-Kannan contraction is a usual Kannan contraction, i.e., it satisfies condition* (1.6)*.*

3 Main results

Let X be a Banach space, K a nonempty closed subset of X and $T : K \to X$ a non-self mapping. If $x \in K$ is such that $Tx \notin K$, then we can always choose an $y \in \partial K$ (the boundary of K) such that $y = (1-\lambda)x + \lambda Tx\,(0 < \lambda < 1)$, which actually expresses the fact that

$$d(x, Tx) = d(x, y) + d(y, Tx),\ y \in \partial K, \tag{3.1}$$

where we denoted $d(x, y) = \|x - y\|$.

In general, the set Y of points y satisfying condition (3.1) above may contain more than one element. We suppose Y is always nonempty.

In this context we shall need the following important concept first introduced and used in [19].

Definition 1. *Let X be a Banach space, K a nonempty closed subset of X and $T : K \to X$ a non-self mapping. Let $x \in K$ with $Tx \notin K$ and let $y \in \partial K$ be the corresponding elements given by (3.1). If, for any such elements x, we have*

$$d(y, Ty) \leq d(x, Tx), \tag{3.2}$$

for all corresponding $y \in Y$, then we say that T has property (M).

Note that the non-self mapping T in the next example has property (M).

Example 2. *([20], Example 4) Let $X = [0, 1] \cup \{3\}$ be endowed with the usual norm and let $K = \{0, 1, 3\}$. Consider the function $T : K \to X$, defined by $Tx = 0$, for $x \in \{0, 1\}$ and $T3 = 0.5$. As the only value $x \in K$ with $Tx \notin K$ is $x = 3$ and to it corresponds the set $Y = \{1\}$, and since*

$$d(y, Ty) = d(1, T1) = |1 - 0| < |3 - 0.5| = d(3, T3) = d(xTx),$$

property (M) obviously holds.

A condition quite similar to (3.2), called inward condition, has been used by Caristi [28] to obtain a generalization of contraction mapping principle for non self mappings. The inward condition is more general than property (M) since it does not require y in (3.1) to belong to ∂K, see also [37] (this has been communicated to us by Professor Rus [69]).

Note also that, in general, the set Y of points y satisfying condition (3.1) above may contain more than one element.

For a non self mapping $T : K \to X$ we shall say that it is (well) defined on the Banach space X endowed with the graph G if it has this property for the subgraph of G induced by K, that is,

$$(x, y) \in E(G) \text{ with } Tx, Ty \in K \text{ implies } (Tx, Ty) \in E(G) \cap (K \times K), \tag{3.3}$$

for all $x, y \in K$.

The next theorem establishes a fixed point theorem for non self Chatterjea contractions defined on a Banach space endowed with a graph.

Theorem 2. *Let (X, d, G) be a Banach space endowed with a simple directed and weakly connected graph G such that the property (L) holds, i.e., for any sequence $\{x_n\}_{n=1}^{\infty} \subset X$ with $x_n \to x$ as $n \to \infty$ and $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$, there exists a subsequence $\{x_{k_n}\}_{n=1}^{\infty}$ satisfying*

$$(x_{k_n}, x) \in E(G),\ \forall n \in \mathbb{N}. \tag{3.4}$$

Let K be a nonempty closed subset of X andlet $T : K \to X$ be a Chatterjea contraction, i.e., a mapping for which there exists a constant $a \in [0, 1/2)$ such that

$$d(Tx, Ty) \leq a\big[d(x, Ty) + d(y, Tx)\big], \text{ for all } (x, y) \in E(G_K), \tag{3.5}$$

where G_K is the subgraph of G determined by K.

If $K_T := \{x \in \partial K : (x, Tx) \in E(G)\} \neq \emptyset$, T has property (M) satisfies Rothe's boundary condition

$$T(\partial K) \subset K, \tag{3.6}$$

then

(i) $Fix\,(T) = \{x^*\}$;

(ii) Picard iteration $\{x_n = T^n x_0\}_{n=1}^{\infty}$ converges to x^*, for all $x_0 \in K_T$, and the following estimate holds

$$d(x_n, x^*) \leq \frac{\delta^{[n/2]}}{1-\delta}\, \max\{d(x_0, x_1), d(x_1, x_2)\}\,, \quad n = 0, 1, 2, \ldots \tag{3.7}$$

where $\delta = \dfrac{a}{1-a}$.

Proof. If $T(K) \subset K$, then T is actually a self mapping of the closed set K and the conclusion follows by Chatterjea fixed point theorem [30] with $X = K$. Therefore, in the following we consider only the case $T(K) \not\subset K$. Let $x_0 \in K_T$. This means that $(x_0, Tx_0) \in E(G)$ and in view of (2.1), we have

$$(T^n x_0, T^{n+1} x_0) \in E(G),\ \forall n \in \mathbb{N}. \tag{3.8}$$

Denote $y_n := T^n x_0$, for all $n \in \mathbb{N}$.

By (3.6) it also follows that $Tx_0 \in K$.

Denote $x_1 := y_1 = Tx_0$. Now, if $Tx_1 \in K$, set $x_2 := y_2 = Tx_1$. If $Tx_1 \notin K$, we can choose an element x_2 on the segment $[x_1, Tx_1]$ which also belong to ∂K, that is,

$$x_2 = (1 - \lambda)x_1 + \lambda Tx_1\ (0 < \lambda < 1).$$

Continuing in this way we obtain two sequences $\{x_n\}$ and $\{y_n\}$ whose terms satisfy one of the following properties:

i) $x_n := y_n = Tx_{n-1}$, if $Tx_{n-1} \in K$;
ii) $x_n = (1-\lambda)x_{n-1} + \lambda Tx_{n-1} \in \partial K\,(0 < \lambda < 1)$, if $Tx_{n-1} \notin K$.

To simplify the argumentation in the proof, let us denote

$$P = \{x_k \in \{x_n\} : x_k = y_k = Tx_{k-1}\}$$

and

$$Q = \{x_k \in \{x_n\} : x_k \neq Tx_{k-1}\}.$$

Note that $\{x_n\} \subset K$ for all $n \in \mathbb{N}$ and that, if $x_k \in Q$, then both x_{k-1} and x_{k+1} belong to the set P.

Moreover, by virtue of (3.6), we cannot have two consecutive terms of $\{x_n\}$ in the set Q (but we can have two consecutive terms of $\{x_n\}$ in the set P) .

We claim that $\{x_n\}$ is a Cauchy sequence.

To prove this, we must discuss three different cases: Case I. $x_n,\, x_{n+1} \in P$; Case II. $x_n \in P,\ x_{n+1} \in Q$; Case III. $x_n \in Q,\ x_{n+1} \in P$;

Case I. $x_n,\, x_{n+1} \in P$.

In this case we have $x_n = y_n = Tx_{n-1}$, $x_{n+1} = y_{n+1} = Tx_n$, and hence

$$d(x_{n+1}, x_n) = d(y_{n+1}, y_n) = d(Tx_n, Tx_{n-1}).$$

Since $\{x_n\} \subset K$ for all $n \in \mathbb{N}$, by (3.8) $(x_n, x_{n-1}) \in E(G_K)$, and so by the contraction condition (3.5), we get

$$d(x_{n+1}, x_n) = d(Tx_n, Tx_{n-1}) \leq a[d(x_n, Tx_{n-1}) + d(x_{n-1}, Tx_n)]$$

$$= ad(x_{n-1}, x_{n+1}) \leq a[d(x_{n-1}, x_n) + d(x_n, x_{n+1})],$$

by triangle inequality, and this leads to

$$d(x_{n+1}, x_n) \leq \delta d(x_n, x_{n-1}), \tag{3.9}$$

where $\delta = \dfrac{a}{1-a}$.

Case II. $x_n \in P,\ x_{n+1} \in Q$.

In this case we have $x_n = y_n = Tx_{n-1}$, but $x_{n+1} \neq y_{n+1} = Tx_n$ and

$$d(x_n, x_{n+1}) + d(x_{n+1}, Tx_n) = d(x_n, Tx_n).$$

Thus $d(x_{n+1}, Tx_n) \neq 0$ and hence

$$d(x_n, x_{n+1}) = d(x_n, Tx_n) - d(x_{n+1}, Tx_n) < d(x_n, Tx_n). \tag{3.10}$$

Now, by a similar argument to that in Case I, $(x_n, x_{n-1}) \in E(G_K)$ and hence by the contraction condition (3.5) we get

$$d(x_n, Tx_n) = d(Tx_{n-1}, Tx_n) \le a[d(x_{n-1}, Tx_n) + d(x_n, Tx_{n-1})]$$

$$= ad(x_{n-1}, Tx_n) \le a[d(x_{n-1}, x_n) + d(x_n, Tx_n)].$$

Thus

$$d(x_n, Tx_n) \le \delta d(x_n, x_{n-1}).$$

and therefore, by means of 3.10,

$$d(x_n, x_{n+1}) < d(x_n, Tx_n) \le \delta d(x_n, x_{n-1}).$$

which is exactly inequality (3.9) obtained in Case I.

Case III. $x_n \in Q,\ x_{n+1} \in P$. In this case we have $x_{n+1} = Tx_n,\ x_n \neq y_n = Tx_{n-1}$ and

$$d(x_{n-1}, x_n) + d(x_n, Tx_{n-1}) = d(x_{n-1}, Tx_{n-1}). \tag{3.11}$$

Hence, by property (M) we get

$$d(x_n, x_{n+1}) = d(x_n, Tx_n) \le d(x_{n-1}, Tx_{n-1}) = d(Tx_{n-2}, Tx_{n-1}).$$

(since $x_n \in Q \Longrightarrow x_{n-1} \in P$). Thus,

$$d(x_n, x_{n+1}) \le d(Tx_{n-2}, Tx_{n-1}).$$

Since, by (3.8), $(y_{n-1}, y_n) \in E(G)$, by the contraction condition (3.5) with $x := x_{n-2}$ and $y := x_{n-1}$ we obtain

$$d(Tx_{n-2}, Tx_{n-1}) \le a[d(x_{n-2}, Tx_{n-1}) + d(x_{n-1}, Tx_{n-2})]$$

$$= ad(x_{n-2}, x_n),$$

since $x_{n-1} = Tx_{n-2}$. Therefore, by triangle inequality,

$$d(x_n, x_{n+1}) \le ad(x_{n-2}, x_n) \le a[d(x_{n-2}, x_{n-1}) + d(x_{n-1}, x_n)]$$

$$= 2a \cdot \frac{d(x_{n-2}, x_{n-1}) + d(x_{n-1}, x_n)}{2} \le 2a \max\{d(x_{n-2}, x_{n-1}),\ d(x_{n-1}, x_n)\}.$$

Since $\max\{2a, \frac{a}{1-a}\} = \frac{a}{1-a} := \delta$, we finally obtain

$$d(x_n, x_{n+1}) \le \delta d(x_{n-2}, x_{n-1}). \tag{3.12}$$

Now, by summaryzing all three cases and using (3.9) and (3.12), it follows that the sequence $\{d(x_n, x_{n-1})\}$ satisfies the inequality

$$d(x_n, x_{n+1}) \leq \delta \max\{d(x_{n-2}, x_{n-1}),\, d(x_{n-1}, x_n)\}, \tag{3.13}$$

for all $n \geq 2$. Now, by induction for $n \geq 2$, from (3.13) one obtains

$$d(x_n, x_{n+1}) \leq \delta^{[n/2]} \max\{d(x_0, x_1),\, d(x_1, x_2)\}, \tag{3.14}$$

where $[n/2]$ denotes the greatest integer not exceeding $n/2$.

Further, for $m > n > N$,

$$d(x_n, x_m) \leq \sum_{i=N}^{\infty} d(x_i, x_{i-1}) \leq 2\frac{\delta^{[N/2]}}{1-\delta} \max\{d(x_0, x_1),\, d(x_1, x_2)\}, \tag{3.15}$$

which shows that $\{x_n\}$ is a Cauchy sequence.

Since $\{x_n\} \subset K$ and K is closed, $\{x_n\}$ converges to some point x^* in K, i.e.,

$$x^* = \lim_{n\to\infty} x_n. \tag{3.16}$$

By property (L), there exists a subsequence $\{x_{k_n}\}_{n=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ satisfying

$$(x_{k_n}, x^*) \in E(G),\ \forall n \in \mathbb{N}.$$

and hence, by the contraction condition (3.5),

$$\begin{aligned} d(x_{k_n+1}, Tx^*) = d(Tx_{k_n}, Tx^*) &\leq a[d(x_{k_n}, Tx^*) + d(x^*, Tx_{k_n}] \\ &\leq a[d(x_{k_n}, Tx_{k_n}) + d(Tx_{k_n}, Tx^*) + d(x^*, x_{k_n+1}]. \\ &= a[d(x_{k_n}, Tx_{k_n}) + d(x_{k_n+1}, Tx^*) + d(x^*, x_{k_n+1}]. \end{aligned}$$

This yields,

$$d(x_{k_n+1}, Tx^*) \leq \delta d(x_{k_n}, Tx_{k_n}) + ad(x^*, x_{k_n+1}],$$

which, by means of (3.15) and by letting $n \to \infty$ shows that the sequence $\{x_{k_n}\}_{n=1}^{\infty}$ converges to Tx^* as $n \to \infty$. By (3.16) and the uniqueness of the limit in a metric space, we infer that $x^* = Tx^*$, i.e., x^* is a fixed point of T.

The uniqueness of x^* immediately follows by the contraction condition (3.5), which implies the uniqueness condition

$$d(Tx, Ty) \leq \delta d(x, y) + 2\delta d(x, Tx), \text{ for all } (x, y) \in E(G_K).$$

In the end, by using the estimate (3.14) and triangle inequality we obtain for any $n, p \in \mathbb{N}^*$

$$d(x_n, x_{n+p}) \leq \delta^{[n/2]} \frac{1-\delta^{[(p+1)/2]}}{1-\delta} \max\{d(x_0, x_1),\, d(x_1, x_2)\},$$

from which, by letting $p \to \infty$, we get exactly the error estimate (3.7). $\square$

A weaker form of Theorem 2 can be stated as follows.

Theorem 3. *Let (X, d, G) be a Banach space endowed with a simple directed and weakly connected graph G. Let K be a nonempty closed subset of X and $T : K \to X$ be a G-Chatterjea contraction on K.*

If $K_T := \{x \in \partial K : (x, Tx) \in E(G)\} \neq \emptyset$, T is orbitally G-continuous and T satisfies Rothe's boundary condition

$$T(\partial K) \subset K,$$

then the conclusion of Theorem 2 remains valid.

4 Conclusions and further study

The Chatterjea-type contractive condition (1.7) (or (3.5) in the non self mapping case) is independent of the Banach type contraction condition (1.3) considered in [20], and of Kannan-type contractive condition (1.6), as shown by the next examples.

Example 3. *([53], Example 1.3.1)* Let $X = [0, 1]$ with the usual norm and $T : [0, 1] \to [0, 1]$ be defined by

$$T(x) = \begin{cases} \dfrac{2}{5}, & x \in \left[0, \dfrac{2}{3}\right) \\[2ex] \dfrac{1}{5}, & x \in \left[\dfrac{2}{3}, 1\right]. \end{cases}$$

Then T is a discontinuous Kannan operator with constant $a = \dfrac{3}{7}$, it is neither a Banach contraction nor a Chatterjea contraction.

Example 4. *([53], Example 1.3.4)* Let $X = [0, 1]$ with the usual norm and $f : [0, 1] \to [0, 1]$ be defined by

$$f(x) = \begin{cases} \dfrac{1}{5}, & x \in \left[0, \dfrac{8}{15}\right) \\[2ex] \dfrac{1}{3}, & x \in \left[\dfrac{8}{15}, 1\right]. \end{cases}$$

Then f is a Chatterjea operator with constant $a = \dfrac{2}{5}$ but f is neither a Banach contraction nor a Kannan contraction (see Example 1.3.7 in [53] for the proof).

This shows that Theorems 2 and 3 established in the present paper are important and very general alternative fixed point theorems for non self mappings in Banach spaces endowed with a graph. They provide effective generalisations and extensions of similar results in literature and subsume several important results in the fixed point theory of self and nonself mappings.

Both Theorem 2 and Theorem 3 were established in Banach spaces endowed with a graph for the sake of simplicity of exposition but they can be transposed in more general settings, like convex metric spaces or $CAT\,(0)$ spaces without any major technical difficulty.

By working on Banach spaces endowed with a graph, our results are valid not only for mappings that satisfy the contraction condition (3.5) for all pairs (x, y) of the space $X \times X$, but only for the pairs (x, y) which are vertices of a simple directed and weakly connected graph $G = (X, E(G))$, with $E(G) \subset X \times X$.

Amongst the most important particular cases of Theorem 2 and Theorem 3, we mention in the following just the following ones:

1. If G is the graph G_0 in Example 1, then by Theorem 2 we obtain an extension of Chatterjea fixed point theorem [30] for non self mappings, restricted here for the reasons mentioned above to Banach spaces instead of usual complete metric spaces.

2. If $K = X$, and G is the graph G_0 in Example 1, then by Theorem 2 we obtain the original Chatterjea fixed point theorem [30] for self mappings, restricted here for the reasons mentioned above to Banach spaces instead of usual complete metric spaces.

For further developments, we have in view considering nonself single-valued as well as multi-valued mappings by starting from the corresponding case of self mappings, see [1]-[4],[5], [21], [22], [25], [32], [33], [38], [39], [40], [43], [44], [50]-[59], [71]-[73], [74]-[77] etc.

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