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$(\alpha, \beta, \lambda, \delta, m, \Omega)_p-$Neighborhood for some families of analytic and multivalent functions

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Abstract

In the present investigation, we give some interesting results related with neighborhoods of $p-$valent functions. Relevant connections with some other recent works are also pointed out.

1 Introduction and Definitions

Let $\mathcal{A}$ demonstrate the family of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disk $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$.

We denote by $\mathcal{A}_p(n)$ the class of functions $f(z)$ normalized by

$$f(z) = z^p + \sum_{k=n}^{\infty} a_{k+p} z^{k+p} \quad (n, p \in \mathbb{N} := \{1, 2, 3, ...\}) \tag{1}$$

which are analytic and p-valent in $\mathcal{U}$.

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Upon differentiating both sides of (1) m times with respect to z, we have

$$f^{(m)}(z) = \frac{p!}{(p-m)!} z^{p-m} + \sum_{k=n}^{\infty} \frac{(k+p)!}{(k+p-m)!} a_{k+p} z^{k+p-m} \tag{2}$$

$$(n, p \in \mathbb{N}; m \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}; p > m).$$

We show by $\mathcal{A}_p(n,m)$ the class of functions of the form (2) which are analytic and p-valent in $\mathcal{U}$.

The concept of neighborhood for $f(z) \in \mathcal{A}$ was first given by Goodman [7]. The concept of δ-neighborhoods $N_\delta(f)$ of analytic functions $f(z) \in \mathcal{A}$ was first studied by Ruscheweyh [8]. Walker [12], defined a neighborhood of analytic functions having positive real part. Later, Owa *et al.*[13] generalized the results given by Walker. In 1996, Altıntaş and Owa [14] gave (n,δ)-neighborhoods for functions $f(z) \in \mathcal{A}$ with negative coefficients. In 2007, (n,δ)-neighborhoods for p-valent functions with negative coefficients were considered by Srivastava *et al.* [4], and Orhan [5]. Very recently, Orhan *et al.*[1], introduced a new definition of (n,δ)-neighborhood for analytic functions $f(z) \in \mathcal{A}$. Orhan *et al.*'s [1] results were generalized for the functions $f(z) \in \mathcal{A}$ and $f(z) \in \mathcal{A}_p(n)$ by many author (see, [6, 9, 10, 15]).

In this paper, we introduce the neighborhoods $(\alpha, \beta, \lambda, m, \delta, \Omega)_p - N(g)$ and $(\alpha, \beta, \lambda, m, \delta, \Omega)_p - M(g)$ of a function $f^{(m)}(z)$ when $f(z) \in \mathcal{A}_p(n)$.

Using the Salagean derivative operator [3]; we can write the following equalities for the function $f^{(m)}(z)$ given by

$$D^0 f^{(m)}(z) = f^{(m)}(z)$$

$$D^1 f^{(m)}(z) = \frac{z}{(p-m)} \left(f^{(m)}(z)\right)'$$

$$= \frac{p!}{(p-m)!} z^{p-m} + \sum_{k=n}^{\infty} \frac{(k+p-m)(k+p)!}{(p-m)(k+p-m)!} a_{k+p} z^{k+p-m}$$

$$D^2 f^{(m)}(z) = D(D f^{(m)}(z))$$

$$= \frac{p!}{(p-m)!} z^{p-m} + \sum_{k=n}^{\infty} \frac{(k+p-m)^2(k+p)!}{(p-m)^2(k+p-m)!} a_{k+p} z^{k+p-m}$$

....

$$D^{\Omega} f^{(m)}(z) = D(D^{\Omega-1} f^{(m)}(z))$$
$$= \frac{p!}{(p-m)!} z^{p-m} + \sum_{k=n}^{\infty} \frac{(k+p-m)^{\Omega}(k+p)!}{(p-m)^{\Omega}(k+p-m)!} a_{k+p} z^{k+p-m}.$$

We define $\mathcal{F} : \mathcal{A}_p(n, m) \to \mathcal{A}_p(n, m)$ such that

$$\mathcal{F}(f^{(m)}(z)) = (1-\lambda)\left(D^{\Omega} f^{(m)}(z)\right) + \frac{\lambda z}{(p-m)} \left(D^{\Omega} f^{(m)}(z)\right)'$$

$$= \frac{p!}{(p-m)!} z^{p-m} + \sum_{k=n}^{\infty} \frac{(k+p)!(k+p-m)^{\Omega}(1+\lambda k(p-m)^{-1})}{(p-m)^{\Omega}(k+p-m)!} a_{k+p} z^{k+p-m} \tag{3}$$

$$(0 \le \lambda \le 1;\ \Omega, m \in \mathbb{N}_0;\ p > m).$$

Let $\mathcal{F}(\lambda, m, \Omega)$ denote class of functions of the form (3) which are analytic in $\mathcal{U}$.

For $f, g \in \mathcal{F}(\lambda, m, \Omega)$, f said to be $(\alpha, \beta, \lambda, m, \delta, \Omega)_p$−neighborhood for g if it satisfies

$$\left| \frac{e^{i\alpha}\mathcal{F}'(f^{(m)}(z))}{z^{p-m-1}} - \frac{e^{i\beta}\mathcal{F}'(g^{(m)}(z))}{z^{p-m-1}} \right| < \delta \quad (z \in \mathcal{U})$$

for some $-\pi \le \alpha - \beta \le \pi$ *and* $\delta > \frac{p!}{(p-m-1)!}\sqrt{2[1-\cos(\alpha-\beta)]}$. We show this neighborhood by $(\alpha, \beta, \lambda, m, \delta, \Omega)_p - N(g)$.

Also, we say that $f \in (\alpha, \beta, \lambda, m, \delta, \Omega)_p - M(g)$ if it satisfies

$$\left| \frac{e^{i\alpha}\mathcal{F}(f^{(m)}(z))}{z^{p-m}} - \frac{e^{i\beta}\mathcal{F}(g^{(m)}(z))}{z^{p-m}} \right| < \delta \quad (z \in \mathcal{U})$$

for some $-\pi \le \alpha - \beta \le \pi$ *and* $\delta > \frac{p!}{(p-m-1)!}\sqrt{2[1-\cos(\alpha-\beta)]}$.

We give some results for functions belonging to $(\alpha, \beta, \lambda, m, \delta, \Omega)_p - N(g)$ and $(\alpha, \beta, \lambda, m, \delta, \Omega)_p - M(g)$.

2 Main Results

Now we can establish our main results.

Theorem 2.1. *If $f \in \mathcal{F}(\lambda, m, \Omega)$ satisfies*

$$\sum_{k=n}^{\infty} \frac{(k+p-m)^{\Omega+1}(k+p)!(1+\lambda k(p-m)^{-1})}{(p-m)^{\Omega}(k+p-m)!} \left|e^{i\alpha}a_{k+p} - e^{i\beta}b_{k+p}\right|$$

$$\leq \delta - \frac{p!}{(p-m-1)!}\sqrt{2[1-\cos(\alpha-\beta)]} \tag{4}$$

for some $-\pi \leq \alpha - \beta \leq \pi$; $p > m$ and $\delta > \frac{p!}{(p-m-1)!}\sqrt{2[1-\cos(\alpha-\beta)]}$, then $f \in (\alpha,\beta,\lambda,m,\delta,\Omega)_p - N(g)$.

Proof. By virtue of (3), we can write

$$\left|\frac{e^{i\alpha}\mathcal{F}'(f^{(m)}(z))}{z^{p-m-1}} - \frac{e^{i\beta}\mathcal{F}'(g^{(m)}(z))}{z^{p-m-1}}\right|$$

$$= \left|\frac{p!(p-m)}{(p-m)!}e^{i\alpha} + e^{i\alpha}\sum_{k=n}^{\infty}\frac{(k+p-m)^{\Omega+1}(k+p)!(1+\lambda k(p-m)^{-1})}{(p-m)^{\Omega}(k+p-m)!}a_{k+p}z^k \right.$$
$$\left. - \frac{p!(p-m)}{(p-m)!}e^{i\beta} - e^{i\beta}\sum_{k=n}^{\infty}\frac{(k+p-m)^{\Omega+1}(k+p)!(1+\lambda k(p-m)^{-1})}{(p-m)^{\Omega}(k+p-m)!}b_{k+p}z^k\right|$$

$$< \frac{p!}{(p-m-1)!}\sqrt{2[1-\cos(\alpha-\beta)]}$$
$$+ \sum_{k=n}^{\infty}\frac{(k+p-m)^{\Omega+1}(k+p)!(1+\lambda k(p-m)^{-1})}{(p-m)^{\Omega}(k+p-m)!}\left|e^{i\alpha}a_{k+p} - e^{i\beta}b_{k+p}\right|.$$

If

$$\sum_{k=n}^{\infty} \frac{(k+p-m)^{\Omega+1}(k+p)!(1+\lambda k(p-m)^{-1})}{(p-m)^{\Omega}(k+p-m)!} \left|e^{i\alpha}a_{k+p} - e^{i\beta}b_{k+p}\right|$$

$$\leq \delta - \frac{p!}{(p-m-1)!}\sqrt{2[1-\cos(\alpha-\beta)]},$$

then we observe that

$$\left| \frac{e^{i\alpha}\mathcal{F}'(f^{(m)}(z))}{z^{p-m-1}} - \frac{e^{i\beta}\mathcal{F}'(g^{(m)}(z))}{z^{p-m-1}} \right| < \delta \quad (z \in \mathcal{U}).$$

Thus, $f \in (\alpha, \beta, \lambda, m, \delta, \Omega)_p - N(g)$. This evidently completes the proof of Theorem 2.1. □

Remark 2.2. *In its special case when*

$$m = \Omega = \lambda = \alpha = 0 \text{ and } \beta = \alpha, \tag{5}$$

in Theorem 2.1 yields a result given earlier by Altuntaş et al. ([9] p.3, Theorem 1).

We give an example for Theorem 2.1.

Example 2.1. *For given*

$$g(z) = \frac{p!}{(p-m)!} z^{p-m} + \sum_{k=n}^{\infty} B_{k+p}(\alpha, \beta, \lambda, m, \delta, \Omega) z^{k+p-m} \in \mathcal{F}(\lambda, m, \Omega)$$

$$(n, p \in \mathbb{N} = \{1, 2, 3, ...\}; \; p > m; \; \Omega, m \in \mathbb{N}_0)$$

we consider

$$f(z) = \frac{p!}{(p-m)!} z^{p-m} + \sum_{k=n}^{\infty} A_{k+p}(\alpha, \beta, \lambda, m, \delta, \Omega) z^{k+p-m} \in \mathcal{F}(\lambda, m, \Omega)$$

$$(n, p \in \mathbb{N} = \{1, 2, 3, ...\}; \; p > m; \; \Omega, m \in \mathbb{N}_0)$$

with

$$A_{k+p} = \frac{(p-m)^{\Omega}\{\delta - \frac{p!}{(p-m-1)!}\sqrt{2[1-\cos(\alpha-\beta)]}\}(k+p-m)!(n+p-1)}{(1+\lambda k(p-m)^{-1})(k+p-m)^{\Omega+1}(k+p-1)!(k+p)^2(k+p-1)} e^{-i\alpha}$$
$$+ e^{i(\beta-\alpha)} B_{k+p}.$$

Then we have that

$$\sum_{k=n}^{\infty} \frac{(k+p-m)^{\Omega+1}(k+p)!(1+\lambda k(p-m)^{-1})}{(p-m)^{\Omega}(k+p-m)!} \left| e^{i\alpha} A_{k+p} - e^{i\beta} B_{k+p} \right|$$

$$= (n+p-1)\left(\delta - \frac{p!}{(p-m-1)!}\sqrt{2[1-\cos(\alpha-\beta)]}\right)\sum_{k=n}^{\infty}\frac{1}{(k+p-1)(k+p)}. \tag{6}$$

Finally, in view of the telescopic series, we obtain

$$\begin{aligned}\sum_{k=n}^{\infty}\frac{1}{(k+p-1)(k+p)} &= \lim_{\zeta\longrightarrow\infty}\sum_{k=n}^{\zeta}\left[\frac{1}{k+p-1}-\frac{1}{k+p}\right] \\ &= \lim_{\zeta\longrightarrow\infty}\left[\frac{1}{n+p-1}-\frac{1}{\zeta+p}\right] \\ &= \frac{1}{n+p-1}.\end{aligned} \tag{7}$$

Using (7) in (6), we get

$$\sum_{k=n}^{\infty}\frac{(k+p-m)^{\Omega+1}(k+p)!(1+\lambda k(p-m)^{-1})}{(p-m)^{\Omega}(k+p-m)!}\left|e^{i\alpha}A_{k+p}-e^{i\beta}B_{k+p}\right|$$

$$=\delta - \frac{p!}{(p-m-1)!}\sqrt{2[1-\cos(\alpha-\beta)]}.$$

Therefore, we say that $f \in (\alpha,\beta,\lambda,m,\delta,\Omega)_p - N(g)$.

Also, Theorem 2.1 gives us the following corollary.

Corollary 2.3. *If $f \in \mathcal{F}(\lambda,m,\Omega)$ satisfies*

$$\sum_{k=n}^{\infty}\frac{(k+p-m)^{\Omega+1}(k+p)!(1+\lambda k(p-m)^{-1})}{(p-m)^{\Omega}(k+p-m)!}\,||a_{k+p}|-|b_{k+p}||$$

$$\leq \delta - \frac{p!}{(p-m-1)!}\sqrt{2[1-\cos(\alpha-\beta)]}$$

for some $-\pi \leq \alpha-\beta \leq \pi$ and $\delta > \frac{p!}{(p-m-1)!}\sqrt{2[1-\cos(\alpha-\beta)]}$, and $\arg(a_{k+p})-\arg(b_{k+p}) = \beta-\alpha$ $(n,p\in\mathbb{N}=\{1,2,3,...\};\, m\in\mathbb{N}_0,\, p>m)$, then $f \in (\alpha,\beta,\lambda,m,\delta,\Omega)_p - N(g)$.

Proof. By Theorem 2.1, we see the inequality (4) which implies that $f \in (\alpha,\beta,\lambda,m,\delta,\Omega)_p - N(g)$.

Since $\arg(a_{k+p})-\arg(b_{k+p}) = \beta-\alpha$, if $\arg(a_{k+p}) = \alpha_{k+p}$, we see $\arg(b_{k+p}) = \alpha_{k+p}-\beta+\alpha$. Therefore,

$$e^{i\alpha}a_{k+p}-e^{i\beta}b_{k+p} = e^{i\alpha}\left|a_{k+p}\right|e^{i\alpha_{k+p}} - e^{i\beta}\left|b_{k+p}\right|e^{i(\alpha_{k+p}-\beta+\alpha)}$$

$$= (|a_{k+p}| - |b_{k+p}|)e^{i(\alpha_{k+p}+\alpha)}$$

implies that

$$\left|e^{i\alpha}a_{k+p} - e^{i\beta}b_{k+p}\right| = ||a_{k+p}| - |b_{k+p}||\,. \tag{8}$$

Using (8) in (4) the proof of the corollary is complete.

□

Next, we can prove the following theorem.

Theorem 2.4. *If $f \in \mathcal{F}(\lambda, m, \Omega)$ satisfies*

$$\sum_{k=n}^{\infty} \frac{(k+p-m)^{\Omega}(k+p)!(1+\lambda k(p-m)^{-1})}{(p-m)^{\Omega}(k+p-m)!}\left|e^{i\alpha}a_{k+p} - e^{i\beta}b_{k+p}\right|$$

$$\leq \delta - \frac{p!}{(p-m)!}\sqrt{2[1-\cos(\alpha-\beta)]} \quad (z \in \mathcal{U})\,.$$

for some $-\pi \leq \alpha - \beta \leq \pi$; $p > m$ and $\delta > \frac{p!}{(p-m)!}\sqrt{2[1-\cos(\alpha-\beta)]}$ then $f \in (\alpha,\beta,\lambda,m,\delta,\Omega)_p - M(g)$.

The proof of this theorem is similar with Theorem 2.1.

Corollary 2.5. *If $f \in \mathcal{F}(\lambda, m, \Omega)$ satisfies*

$$\sum_{k=n}^{\infty} \frac{(k+p-m)^{\Omega}(k+p)!(1+\lambda k(p-m)^{-1})}{(p-m)^{\Omega}(k+p-m)!}\,||a_{k+p}| - |b_{k+p}||$$

$$\leq \delta - \frac{p!}{(p-m)!}\sqrt{2[1-\cos(\alpha-\beta)]} \quad (z \in \mathcal{U})\,.$$

for some $-\pi \leq \alpha - \beta \leq \pi$; $p > m$ and $\delta > \frac{p!}{(p-m)!}\sqrt{2[1-\cos(\alpha-\beta)]}$ and $\arg(a_{k+p}) - \arg(b_{k+p}) = \beta - \alpha$, *then $f \in (\alpha,\beta,\lambda,m,\delta,\Omega)_p - M(g)$.*

Our next result as follows.

Theorem 2.6. *If $f \in (\alpha,\beta,\lambda,m,\delta,\Omega)_p - N(g), 0 \leq \alpha < \beta \leq \pi$; $p > m$ and* $\arg(e^{i\alpha}a_{k+p} - e^{i\beta}b_{k+p}) = k\phi$, *then*

$$\sum_{k=n}^{\infty} \frac{(k+p-m)^{\Omega+1}(k+p)!(1+\lambda k(p-m)^{-1})}{(p-m)^{\Omega}(k+p-m)!}\left|e^{i\alpha}a_{k+p} - e^{i\beta}b_{k+p}\right|$$

$$\leq \delta - \frac{p!}{(p-m-1)!}(\cos\alpha - \cos\beta).$$

Proof. For $f \in (\alpha,\beta,\lambda,m,\delta,\Omega)_p - N(g)$, we have

$$\begin{aligned}
&\left|\frac{e^{i\alpha}\mathcal{F}'(f^{(m)}(z))}{z^{p-m-1}} - \frac{e^{i\beta}\mathcal{F}'(g^{(m)}(z))}{z^{p-m-1}}\right| \\
=& \left|\frac{p!(e^{i\alpha}-e^{i\beta})}{(p-m-1)!} + \sum_{k=n}^{\infty}\frac{(k+p-m)^{\Omega+1}(k+p)!(1+\lambda k(p-m)^{-1})}{(p-m)^{\Omega}(k+p-m)!}(e^{i\alpha}a_{k+p}-e^{i\beta}b_{k+p})z^k\right| \\
=& \left|\frac{p!(e^{i\alpha}-e^{i\beta})}{(p-m-1)!} + \sum_{k=n}^{\infty}\frac{(k+p-m)^{\Omega+1}(k+p)!(1+\lambda k(p-m)^{-1})}{(p-m)^{\Omega}(k+p-m)!}(e^{i\alpha}a_{k+p}-e^{i\beta}b_{k+p})e^{ik\phi}z^k\right| \\
&< \delta.
\end{aligned}$$

Let us consider z such that $\arg z = -\phi$. Then $z^k = |z|^k e^{-ik\phi}$. For such a point $z \in \mathcal{U}$, we see that

$$\begin{aligned}
&\left|\frac{e^{i\alpha}\mathcal{F}'(f(z))}{z^{p-m-1}} - \frac{e^{i\beta}\mathcal{F}'(g(z))}{z^{p-m-1}}\right| \\
=& \left|\frac{p!(e^{i\alpha}-e^{i\beta})}{(p-m-1)!} + \sum_{k=n}^{\infty}\frac{(k+p-m)^{\Omega+1}(k+p)!(1+\lambda k(p-m)^{-1})}{(p-m)^{\Omega}(k+p-m)!}\left|e^{i\alpha}a_{k+p}-e^{i\beta}b_{k+p}\right||z|^k\right| \\
=& \left[\left(\sum_{k=n}^{\infty}\frac{(k+p-m)^{\Omega+1}(k+p)!(1+\lambda k(p-m)^{-1})}{(p-m)^{\Omega}(k+p-m)!}\left|e^{i\alpha}a_{k+p}-e^{i\beta}b_{k+p}\right||z|^k + \frac{p!(\cos\alpha-\cos\beta)}{(p-m-1)!}\right)^2\right. \\
&\left. + \left(\frac{p!(\sin\alpha-\sin\beta)}{(p-m-1)!}\right)^2\right]^{\frac{1}{2}} < \delta.
\end{aligned}$$

This implies that

$$\left(\sum_{k=n}^{\infty}\frac{(k+p-m)^{\Omega+1}(k+p)!(1+\lambda k(p-m)^{-1})}{(p-m)^{\Omega}(k+p-m)!}\left|e^{i\alpha}a_{k+p}-e^{i\beta}b_{k+p}\right||z|^k + \frac{p!(\cos\alpha-\cos\beta)}{(p-m-1)!}\right)^2 < \delta^2,$$

or

$$\frac{p!}{(p-m-1)!}(\cos\alpha-\cos\beta)+\sum_{k=n}^{\infty}\frac{(k+p-m)^{\Omega+1}(k+p)!(1+\lambda k(p-m)^{-1})}{(p-m)^{\Omega}(k+p-m)!}\left|e^{i\alpha}a_{k+p}-e^{i\beta}b_{k+p}\right||z|^k < \delta$$

for $z \in \mathcal{U}$. Letting $|z| \longrightarrow 1^-$, we have that

$$\sum_{k=n}^{\infty}\frac{(k+p-m)^{\Omega+1}(k+p)!(1+\lambda k(p-m)^{-1})}{(p-m)^{\Omega}(k+p-m)!}\left|e^{i\alpha}a_{k+p}-e^{i\beta}b_{k+p}\right| \le \delta - \frac{p!}{(p-m-1)!}(\cos\alpha-\cos\beta).$$

□

Remark 2.7. *Applying the parametric substitutions listed in (5), Theorem 2.4 and 2.6 would yield a set of known results due to Altuntaş et al. ([9] p.5, Theorem 4; p.6, Theorem 7).*

Theorem 2.8. *If* $f \in (\alpha,\beta,\lambda,m,\delta,\Omega)_p - M(g), 0 \le \alpha < \beta \le \pi$ *and* $\arg(e^{i\alpha}a_{k+p} - e^{i\beta}b_{k+p}) = k\phi$, *then*

$$\sum_{k=n}^{\infty} \frac{(k+p-m)^{\Omega}(k+p)!(1+\lambda k(p-m)^{-1})}{(p-m)^{\Omega}(k+p-m)!} \left| e^{i\alpha}a_{k+p} - e^{i\beta}b_{k+p} \right|$$

$$\le \delta + \frac{p!}{(p-m-1)!}(\cos\beta - \cos\alpha).$$

The proof of this theorem is similar with Theorem 2.6.

Remark 2.9. *Taking* $\lambda = \alpha = \Omega = m = 0$, $\beta = \alpha$ *and* $p = 1$,*in Theorem 2.8, we arrive at the following theorem due to Orhan et al.[1].*

Theorem 2.10. *If* $f \in (\alpha,\delta) - N(g)$ *and* $\arg(a_n - e^{i\alpha}b_n) = (n-1)\varphi \quad (n = 2,3,4,...)$, *then*

$$\sum_{n=2}^{\infty} n \left| a_n - e^{i\alpha}b_n \right| \le \delta + \cos\alpha - 1.$$

We give an application of following lemma due to Miller and Mocanu [2] (see also, [11]).

Lemma 2.1. *Let the function*

$$w(z) = b_n z^n + b_{n+1}z^{n+1} + b_{n+2}z^{n+2} + ... \qquad (z \in \mathcal{U})$$

be regular in $\mathcal{U}$ *with* $w(z) \ne 0$, $(n \in \mathcal{U})$. *If* $z_0 = r_0 e^{i\theta_0}$ $(r_0 < 1)$ *and* $|w(z_0)| = \max_{|z| \le r_0} |w(z)|$, *then* $z_0 w'(z_0) = q w(z_0)$ *where* q *is real and* $q \ge n \ge 1$.

Applying the above lemma, we derive

Theorem 2.11. *If* $f \in \mathcal{F}(\lambda,m,\Omega)$ *satisfies*

$$\left| \frac{e^{i\alpha}\mathcal{F}'(f^{(m)}(z))}{z^{p-m-1}} - \frac{e^{i\beta}\mathcal{F}'(g^{(m)}(z))}{z^{p-m-1}} \right| < \delta(p+n-m) - \frac{p!}{(p-m-1)!}\sqrt{2[1-\cos(\alpha-\beta)]}$$

for some $-\pi \le \alpha - \beta \le \pi$; $p > m$ *and* $\delta > \left(\frac{p!}{(p+n-m)(p-m-1)!}\right)\sqrt{2[1-\cos(\alpha-\beta)]}$, *then*

$$\left| \frac{e^{i\alpha}\mathcal{F}(f^{(m)}(z))}{z^{p-m}} - \frac{e^{i\beta}\mathcal{F}(g^{(m)}(z))}{z^{p-m}} \right| < \delta + \frac{p!}{(p-m)!}\sqrt{2[1-\cos(\alpha-\beta)]} \quad (z \in \mathcal{U}).$$

Proof. Let us define $w(z)$ by

$$\frac{e^{i\alpha}\mathcal{F}(f^{(m)}(z))}{z^{p-m}} - \frac{e^{i\beta}\mathcal{F}(g^{(m)}(z))}{z^{p-m}} = \frac{p!}{(p-m)!}(e^{i\alpha}-e^{i\beta}) + \delta w(z). \tag{9}$$

Then $w(z)$ is analytic in $\mathcal{U}$ and $w(0) = 0$. By logarithmic differentiation, we get from (9) that

$$\frac{e^{i\alpha}\mathcal{F}'(f^{(m)}(z)) - e^{i\beta}\mathcal{F}'(g^{(m)}(z))}{e^{i\alpha}\mathcal{F}(f^{(m)}(z)) - e^{i\beta}\mathcal{F}(g^{(m)}(z))} - \frac{p-m}{z} = \frac{\delta w\prime(z)}{\frac{p!}{(p-m)!}(e^{i\alpha}-e^{i\beta}) + \delta w(z)}.$$

Since

$$\frac{e^{i\alpha}\mathcal{F}'(f^{(m)}(z)) - e^{i\beta}\mathcal{F}'(g^{(m)}(z))}{z^{p-m}\left(\frac{p!}{(p-m)!}(e^{i\alpha}-e^{i\beta}) + \delta w(z)\right)} = \frac{p-m}{z} + \frac{\delta w\prime(z)}{\frac{p!}{(p-m)!}(e^{i\alpha}-e^{i\beta}) + \delta w(z)},$$

we see that

$$\frac{e^{i\alpha}\mathcal{F}'(f^{(m)}(z))}{z^{p-m-1}} - \frac{e^{i\beta}\mathcal{F}'(g^{(m)}(z))}{z^{p-m-1}} = \frac{p!}{(p-m-1)!}(e^{i\alpha}-e^{i\beta}) + \delta w(z)\left(p-m+\frac{zw\prime(z)}{w(z)}\right).$$

This implies that

$$\left|\frac{e^{i\alpha}\mathcal{F}'(f^{(m)}(z))}{z^{p-m-1}} - \frac{e^{i\beta}\mathcal{F}'(g^{(m)}(z))}{z^{p-m-1}}\right| = \left|\frac{p!}{(p-m-1)!}(e^{i\alpha}-e^{i\beta}) + \delta w(z)\left(p-m+\frac{zw\prime(z)}{w(z)}\right)\right|.$$

We claim that

$$\left|\frac{e^{i\alpha}\mathcal{F}'(f^{(m)}(z))}{z^{p-m-1}} - \frac{e^{i\beta}\mathcal{F}'(g^{(m)}(z))}{z^{p-m-1}}\right| < \delta(p-m+n) - \frac{p!}{(p-m-1)!}\sqrt{2[1-\cos(\alpha-\beta)]}$$

in $\mathcal{U}$.

Otherwise, there exists a point $z_0 \in \mathcal{U}$ such that $z_0 w'(z_0) = qw(z_0)$ (by Miller and Mocanu's Lemma) where $w(z_0) = e^{i\theta}$ and $q \geq n \geq 1$.

Therefore, we obtain that

$$\begin{aligned}\left|\frac{e^{i\alpha}\mathcal{F}'(f^{(m)}(z))}{z_0^{p-m-1}} - \frac{e^{i\beta}\mathcal{F}'(g^{(m)}(z))}{z_0^{p-m-1}}\right| &= \left|\frac{p!}{(p-m-1)!}(e^{i\alpha}-e^{i\beta}) + \delta e^{i\theta}\left(p-m+q\right)\right| \\ &\geq \delta\left(p+q-m\right) - \left|\frac{p!}{(p-m-1)!}(e^{i\alpha}-e^{i\beta})\right| \\ &\geq \delta\left(p+n-m\right) - \frac{p!}{(p-m-1)!}\sqrt{2[1-\cos(\alpha-\beta)]}.\end{aligned}$$

This contradicts our condition in Theorem 2.11.

Hence, there is no $z_0 \in \mathcal{U}$ such that $|w(z_0)| = 1$. This means that $|w(z)| < 1$ for all $z \in \mathcal{U}$.

Thus, have that

$$\begin{aligned}\left|\frac{e^{i\alpha}\mathcal{F}(f^{(m)}(z))}{z^{p-m}} - \frac{e^{i\beta}\mathcal{F}(g^{(m)}(z))}{z^{p-m}}\right| &= \left|\frac{p!}{(p-m)!}(e^{i\alpha}-e^{i\beta}) + \delta w(z)\right| \\ &\leq \frac{p!}{(p-m)!}\left|e^{i\alpha}-e^{i\beta}\right| + \delta\left|w(z)\right| \\ &< \delta + \frac{p!}{(p-m)!}\sqrt{2[1-\cos(\alpha-\beta)]}.\end{aligned}$$

□

Upon setting $m = 0$, $\alpha = \varphi, \wp = \mathcal{F}$ and $\beta = \alpha$ in Theorem 2.11, we have the following corollary given by Sağsöz et al.[6].

Corollary 2.12. *If $f \in \wp(\Omega,\lambda)$ satisfies*

$$\left|\frac{e^{i\alpha}\wp'(f(z))}{z^{p-1}} - \frac{e^{i\beta}\wp'(g(z))}{z^{p-1}}\right| < \delta(p+n) - p\sqrt{2[1-\cos(\varphi-\alpha)]}$$

for some $-\pi \leq \alpha - \beta \leq \pi$;and $\quad \delta > \left(\frac{p}{(p+n)}\right)\sqrt{2[1-\cos(\alpha-\beta)]}$, then

$$\left|\frac{e^{i\alpha}\wp(f(z))}{z^{p}} - \frac{e^{i\beta}\wp(g(z))}{z^{p}}\right| < \delta + \sqrt{2[1-\cos(\varphi-\alpha)]} \quad (z \in \mathcal{U}).$$

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