DOI: 10.1515/auom-2015-0013
An. Şt. Univ. Ovidius Constanţa
VERSITA
Vol. 23(1),2015, 199–212

Nonuniform exponential stability for evolution families on the real line

Claudia Isabela Morariu and Petre Preda

Abstract

The purpose of the present paper is to investigate the problem of nonuniform exponential stability of evolution families on the real line using the input-output technique known in the literature as the Perron method for the study of exponential stability. In this manuscript we describe an evolution family on the real line and we present sufficient conditions for the nonuniform exponential stability of an evolution family on the real line that does not have exponential growth.

1 Indroduction

One of the most important asymptotic properties of a differential system is the exponential dichotomy, notion introduced by O. Perron in 1930 in [14].

J.L. Daleckij and M.G. Krein in [5], J.L. Massera and J.J. Schäffer in [11, Chapter 8] have obtained dichotomy results for differential equations on $\mathbb{R}$, for the infinite dimensional case and W.A. Coppel in [4] and P. Hartman in [6] for the finite dimensional case.

In 1974 M.G. Krein and J.L. Daleckij in [5, Theorem 4.1, p. 81] shows that if $A \in \mathcal{B}(X)$ then $\sigma(A) \cap i\mathbb{R} = \emptyset$ if and only if the differential equation $\dot{x}(t) = Ax(t) + f(t)$ has an unique solution $x \in \mathcal{C}$, for all $f \in \mathcal{C}$, where $\mathcal{C}$ represents the Banach space of the continuous and bounded functions on $\mathbb{R}$ and $\sigma(A)$ represent the spectrum of the operator A.

Key Words: Evolution family on $\mathbb{R}$, nonuniform exponential stability, Perron condition.
2010 Mathematics Subject Classification: Primary 34D05; Secondary 34C35, 47B48.
Received: 27 April, 2014.
Revised: 15 June, 2014.
Accepted: 29 June, 2014.

An important contribution in the study of the asymptotic behaviour of the dynamical systems described by the evolution families is represented by [3], published in 1999 by C. Chicone and Y. Latushkin. Another important results in the study of the evolution equations were obtained by B.M. Levitan and V.V. Zhikov in [9] and A. Pazy in [13]. Some of the results were extended for the evolution families with nonuniform exponential growth by L. Barreira and C. Valls in [1] and [2].

In 1998 Y. Latushkin, T. Randolph and R. Schnaubelt in [8] study the dichotomy on $\mathbb{R}$ for the evolution families with uniform exponential growth through the assigned evolution semigroup. The dichotomy on $\mathbb{R}_+$ was studied by N. Van Minh, F. Räbiger and R. Schnaubelt in [12] and N.T. Huy in [7].

Similar results for the dichotomy on the real line were obtained by A.L. Sasu and B. Sasu in [15] and A.L. Sasu in [16]. All the above results are obtained for $t_0 \in \mathbb{R}_+$. In [15] and [16] are considered systems described by evolution families with exponential growth on the real line. It can also be mentioned the results obtained by M. Marin and O. Florea in [10] as well as K. Sharma and M. Marin in [17].

It is known that the exponential dichotomy is a generalization of the exponential stability, so it is expected that the above results in more stringent conditions should imply the exponential stability on $\mathbb{R}$.

The main purpose of this paper is to give a sufficient condition for the nonuniform exponential stability of an evolution family without exponential growth on the real line using the concept of *Perron condition*.

Section 2 is devoted to the preliminaries while Section 3 is dedicated to the main results. First there are specified the following concepts: evolution family on $\mathbb{R}$, nonuniform exponentially stable evolution family and uniformly stable evolution family. In Definition 3.1 we describe when we say that an evolution family satisfies the Perron condition. The Theorem 3.3 will be used in the demonstration of one of the most important result of this paper, namely Theorem 3.4.

Further in Definition 3.5 it is specified when an evolution family satisfies the $(p, \infty)-$ Perron condition, where $p > 1$ and in Theorem 3.7 is presented another important result related to the nonuniform exponential stability of an evolution family. For the last result, considering $p = 1$, we obtain a characterization for the uniform exponential stability of an evolution family on the real line.

2 Preliminaries

Let X be a Banach space and $\mathcal{B}(X)$ the Banach algebra of all linear and bounded operators acting on X. We will denote by $\|\cdot\|$ the norm on X and

$\mathcal{B}(X)$ and $\Delta = \{(t, t_0) \in \mathbb{R}^2 : t \geq t_0\}$.

Definition 2.1. An application $\Phi : \Delta \to \mathcal{B}(X), \Phi = \{\Phi(t, t_0)\}_{t \geq t_0}$, is called an *evolution family on* $\mathbb{R}$ if it satisfies the following properties:

(i) $\Phi(t, t) = I$, for all $t \in \mathbb{R}$, where I is the identity operator on X;

(ii) $\Phi(t, \tau)\Phi(\tau, t_0) = \Phi(t, t_0)$, for all $t \geq \tau \geq t_0$;

(iii) The map $\Phi(\cdot, t_0)x$ is continuous on $[t_0, \infty)$, for all $x \in X$ and $\Phi(t, \cdot)x$ is continuous on $(-\infty, t]$, for all $x \in X$.

We mention the following function spaces:

$$\mathcal{C} = \{f : \mathbb{R} \to \mathbb{R} : \text{ f is continuous and bounded}\},$$
$$\mathcal{C}_{00} = \{f \in \mathcal{C} : \lim_{t \to -\infty} f(t) = \lim_{t \to \infty} f(t) = 0\},$$

$$L^p(X) = \{f : \mathbb{R} \to X : \text{ f is measurable and } \int_{-\infty}^{\infty} \|f(t)\|^p dt < \infty\}, \text{ where } p \in [1, \infty)$$

and

$$L^\infty(X) = \{f : \mathbb{R} \to X : \text{ f is measurable and } ess \sup_{t \in \mathbb{R}} \|f(t)\| < \infty\}.$$

The norm on $\mathcal{C}$ and $\mathcal{C}_{00}$ is $|||f||| = \sup_{t \in \mathbb{R}} \|f(t)\|$.

The norms on $L^p(X)$ and $L^\infty(X)$ are denoted by

$$\|f\|_p = \left(\int_{-\infty}^{\infty} \|f(t)\|^p dt \right)^{\frac{1}{p}}, \quad \text{respectively} \quad \|f\|_\infty = ess \sup_{t \in \mathbb{R}} \|f(t)\|.$$

Let $\{\Phi(t, t_0)\}_{t \geq t_0}$ be an evolution family on $\mathbb{R}$.

Definition 2.2. We say that the evolution family $\{\Phi(t, t_0)\}_{t \geq t_0}$ is *nonuniform exponentially stable* if there exists $N : \mathbb{R} \to \mathbb{R}_+^*$ and $\nu > 0$ such that

$$\|\Phi(t, t_0)x\| \leq N(t_0) e^{-\nu(t - t_0)} \|x\|, \text{ for all } t \geq t_0 \text{ and } x \in X.$$

Definition 2.3. We say that the evolution family $\{\Phi(t, t_0)\}_{t \geq t_0}$ is *uniformly stable* if there exists a constant $N > 0$ such that

$$\|\Phi(t, t_0)x\| \leq N \|x\|, \text{ for all } t \geq t_0 \text{ and } x \in X.$$

3 Main results

Definition 3.1. We say that the evolution family $\{\Phi(t,t_0)\}_{t\geq t_0}$ *satisfies the Perron condition* if:

(i) For all $f \in \mathcal{C}_{00}$ it results that $x_f \in \mathcal{C}$, where $x_f(t) = \int\limits_{-\infty}^{t} \Phi(t,\tau)f(\tau)d\tau$;

(ii) If there is $w \in \mathcal{C}$ such that $w(t) = \Phi(t,s)w(s)$, for all $t \geq s$, it results that $w = 0$.

Remark 3.2. *If $\{\Phi(t,t_0)\}_{t\geq t_0}$ satisfies the Perron condition then*

$$x_f(t) = \Phi(t,s)x_f(s) + \int\limits_s^t \Phi(t,\tau)f(\tau)d\tau, \text{ for all } t \geq s.$$

Theorem 3.3. *If $\{\Phi(t,t_0)\}_{t\geq t_0}$ satisfies the Perron condition then there is a constant $K > 0$ such that*

$$|||x_f||| \leq K|||f|||, \text{ for all } f \in \mathcal{C}_{00}.$$

Proof. Let $\mathcal{U} : \mathcal{C}_{00} \to \mathcal{C}$, defined by $\mathcal{U}f = x_f$. We notice that $\mathcal{U}$ is a linear operator and we will prove that it is closed.

Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{C}_{00}$, $f \in \mathcal{C}_{00}$ and $g \in \mathcal{C}$ such that

$$f_n \to f \text{ in } \mathcal{C}_{00} \text{ and } \mathcal{U}f_n \to g \text{ in } \mathcal{C}.$$

We have that

$$\mathcal{U}f_n(t) = x_{f_n}(t) = \Phi(t,s)x_{f_n}(s) + \int\limits_s^t \Phi(t,\tau)f_n(\tau)d\tau, \text{ for all } t \geq s \tag{3.1}$$

and

$$\left\| \int\limits_s^t \Phi(t,\tau)f_n(\tau)d\tau - \int\limits_s^t \Phi(t,\tau)f(\tau)d\tau \right\| \leq \int\limits_s^t \|\Phi(t,\tau)(f_n(\tau) - f(\tau))\|d\tau. \tag{3.2}$$

Since the function $\tau \mapsto \Phi(t,\tau)x : [s,t] \to X$ is continuous, so it is bounded, we have that there is $M_{s,t,x} > 0$ such that

$$\|\Phi(t,\tau)x\| \leq M_{s,t,x}, \text{ for all } t \in \mathbb{R} \text{ and } x \in X$$

and from the Uniform Boundedness Principle it results that there is $M_{s,t} > 0$ such that

$$\|\Phi(t,\tau)x\| \leq M_{s,t}\|x\|, \text{ for all } t \in \mathbb{R} \text{ and } x \in X.$$

From the relation (3.2) it follows that

$$\int_s^t \|\Phi(t,\tau)(f_n(\tau) - f(\tau))\|d\tau \leq M_{s,t} \int_s^t \|f_n(\tau) - f(\tau)\|d\tau$$

$$\leq M_{s,t}(t-s)|||f_n - f||| \xrightarrow[n\to\infty]{} 0.$$

From the relation (3.1), for $n \to \infty$, it results that

$$g(t) = \Phi(t,s)g(s) + \int_s^t \Phi(t,\tau)f(\tau)d\tau, \text{ for all } t \geq s.$$

We consider now $w(t) = g(t) - x_f(t) = \Phi(t,s)w(s)$, for all $t \geq s$, which implies that $w = 0$ because $w \in \mathcal{C}$. It follows that $g = x_f = \mathcal{U}_f$.

We obtain that $\mathcal{U}$ is a closed operator and by the Closed Graph Theorem it is also bounded. Therefore there is $K > 0$ such that

$$|||x_f||| \leq K|||f|||, \text{ for all } f \in \mathcal{C}_{00}.$$

□

Theorem 3.4. *If* $\{\Phi(t,t_0)\}_{t\geq t_0}$ *satisfies the Perron condition then* $\{\Phi(t,t_0)\}_{t\geq t_0}$ *is nonuniform exponentially stable.*

Proof. Let $x \in X,\ t_0 \in \mathbb{R},\ \delta > 0$ and

$$\chi_{t_0}^\delta : \mathbb{R} \to \mathbb{R}_+,\ \chi_{t_0}^\delta(t) = \begin{cases} 0,\ t < t_0 \\ \frac{4}{\delta}(t - t_0),\ t_0 \leq t < t_0 + \frac{\delta}{2} \\ 4 - \frac{4}{\delta}(t-t_0),\ t_0 + \frac{\delta}{2} \leq t < t_0 + \delta \\ 0,\ t \geq t_0 + \delta. \end{cases}$$

It follows that $\chi_{t_0}^\delta \in \mathcal{C}_{00}$ and $|||\chi_{t_0}^\delta||| = 2$.

Now we consider $f : \mathbb{R} \to X,\ f(t) = \chi_{t_0}^1(t)\Phi(t,t_0)x$. Thus $f \in \mathcal{C}_{00}$ and $|||f||| \leq 2\sup_{t\in[t_0,t_0+1]}\|\Phi(t,t_0)\|\|x\| = 2M(t_0)\|x\|$, where $M(t_0) = \sup_{t\in[t_0,t_0+1]}\|\Phi(t,t_0)\|$.

We obtain that

$$x_f(t) = \int_{-\infty}^{t} \Phi(t,\tau)f(\tau)d\tau = \int_{t_0}^{t} \chi_{t_0}^{1}(\tau)d\tau\Phi(t,t_0)x = \begin{cases} 0,\ t < t_0 \\ (t-t_0)\Phi(t,t_0)x,\ t \in [t_0, t_0+1) \\ \Phi(t,t_0)x,\ t \geq t_0+1. \end{cases}$$

But $x_f \in \mathcal{C}$ and from Theorem 3.3 it results that there is $K > 0$ such that

$$\|\Phi(t,t_0)x\| \leq K|||f||| \leq 2KM(t_0)\|x\|,\ \text{for all } t \geq t_0+1 \text{ and } x \in X.$$

For $t \in [t_0, t_0+1)$ we have that $\|\Phi(t,t_0)x\| \leq M(t_0)\|x\|$. Therefore

$$\|\Phi(t,t_0)x\| \leq L(t_0)\|x\|,\ \text{for all } t \geq t_0 \text{ and } x \in X,$$

where $L(t_0) = M(t_0)\max\{1, 2K\}$.

We set now $x \in X,\ t_0 \in \mathbb{R},\ \delta > 0$ and

$$g : \mathbb{R} \to X,\ g(t) = \chi_{t_0}^{\delta}(t)\Phi(t,t_0)x.$$

It follows that $g \in \mathcal{C}_{00}$ and $|||g||| \leq 2L(t_0)\|x\|$.

We obtain that

$$x_g(t) = \int_{-\infty}^{t} \Phi(t,\tau)g(\tau)d\tau = \int_{t_0}^{t} \chi_{t_0}^{\delta}(\tau)d\tau\Phi(t,t_0)x = \begin{cases} 0,\ t < t_0 \\ (t-t_0)\Phi(t,t_0)x,\ t \in [t_0, t_0+\delta) \\ \delta\Phi(t,t_0)x,\ t \geq t_0+\delta. \end{cases}$$

But $x_g \in \mathcal{C}$ and from Theorem 3.3 it results that there is $K > 0$ such that

$$\delta\|\Phi(t,t_0)x\| \leq K|||g||| \leq 2KL(t_0)\|x\|,\ \text{for all } t \geq t_0+\delta,\ x \in X \text{ and } \delta > 0.$$

For $\delta = t - t_0$ it follows that

$$(t-t_0)\|\Phi(t,t_0)x\| \leq 2KL(t_0)\|x\|,\ \text{for all } t \geq t_0 \text{ and } x \in X.$$

Let now $x \in X,\ n \in \mathbb{N}^*,\ t_0 \in \mathbb{R},\ \delta > 0$ and

$$y : \mathbb{R} \to X,\ y(t) = \begin{cases} 0,\ t < t_0 \\ (t-t_0)\Phi(t,t_0)x,\ t \in [t_0, t_0+\delta] \\ \delta(1 - nt + nt_0 + n\delta)\Phi(t,t_0)x,\ t \in (t_0+\delta, t_0+\delta+\frac{1}{n}] \\ 0,\ t > t_0+\delta+\frac{1}{n}. \end{cases}$$

It follows that $y \in \mathcal{C}_{00}$ and $|||y||| \leq 2KL(t_0)\|x\|$.

We obtain that

$$x_y(t) = \int_{-\infty}^{t} \Phi(t,\tau)y(\tau)d\tau = \int_{t_0}^{t} \Phi(t,\tau)y(\tau)d\tau = \frac{(t-t_0)^2}{2!}\Phi(t,t_0)x,$$

$$\text{for all } t \in [t_0, t_0+\delta] \text{ and } \delta > 0.$$

But $x_y \in \mathcal{C}$ and from Theorem 3.3 it results that there is $K > 0$ such that

$$\frac{(t-t_0)^2}{2!}\|\Phi(t,t_0)x\| \leq 2K^2 L(t_0)\|x\|, \text{ for all } t \geq t_0 \text{ and } x \in X.$$

Let $x \in X,\ t_0 \in \mathbb{R},\ n \in \mathbb{N}^*,\ \delta > 0$ and

$$h : \mathbb{R} \to X,\ h(t) = \begin{cases} 0,\ t < t_0 \\ \dfrac{(t-t_0)^2}{2!}\Phi(t,t_0)x,\ t \in [t_0, t_0+\delta] \\ \dfrac{\delta^2}{2!}(1 - nt + nt_0 + n\delta)\Phi(t,t_0)x,\ t \in (t_0+\delta, t_0+\delta+\frac{1}{n}] \\ 0,\ t > t_0 + \delta + \frac{1}{n}. \end{cases}$$

It results that $h \in \mathcal{C}_{00}$ and $|||h||| \leq 2K^2 L(t_0)\|x\|$.
We obtain that

$$x_h(t) = \int_{-\infty}^{t} \Phi(t,\tau)h(\tau)d\tau = \int_{t_0}^{t} \Phi(t,\tau)h(\tau)d\tau = \frac{(t-t_0)^3}{3!}\Phi(t,t_0)x,$$

$$\text{for all } t \in [t_0, t_0+\delta] \text{ and } \delta > 0.$$

But $x_h \in \mathcal{C}$ and from Theorem 3.3 it follows that there is $K > 0$ such that

$$\frac{(t-t_0)^3}{3!}\|\Phi(t,t_0)x\| \leq 2K^3 L(t_0)\|x\|, \text{ for all } t \geq t_0 \text{ and } x \in X.$$

Inductively we obtain that

$$\frac{(t-t_0)^n}{n!}\|\Phi(t,t_0)x\| \leq 2K^n L(t_0)\|x\|, \text{ for all } t \geq t_0,\ x \in X \text{ and } n \in \mathbf{N}.$$

Sharing with $2^n K^n$ it results that

$$\frac{(t-t_0)^n}{2^n K^n n!}\|\Phi(t,t_0)x\| \leq 2\frac{1}{2^n}L(t_0)\|x\|, \text{ for all } t \geq t_0 \text{ and } x \in X.$$

We have that

$$\sum_{n=0}^{\infty} \frac{(t-t_0)^n}{2^n K^n n!} \|\Phi(t,t_0)x\| \leq 2\sum_{n=0}^{\infty} \frac{1}{2^n} L(t_0)\|x\|, \text{ for all } t \geq t_0 \text{ and } x \in X,$$

or equivalently

$$e^{\frac{(t-t_0)}{2k}} \|\Phi(t,t_0)x\| \leq 4L(t_0)\|x\|, \text{ for all } t \geq t_0 \text{ and } x \in X.$$

So

$$\|\Phi(t,t_0)x\| \leq 4L(t_0)e^{-\frac{(t-t_0)}{2k}} \|x\|, \text{ for all } t \geq t_0 \text{ and } x \in X.$$

Denoting by $N(t_0) = 4L(t_0)$ and $\nu = \dfrac{1}{2K}$ we will obtain that there exists $N : \mathbb{R} \to \mathbb{R}_+^*$ and $\nu > 0$ such that

$$\|\Phi(t,t_0)x\| \leq N(t_0)e^{-\nu(t-t_0)}\|x\|, \text{ for all } t \geq t_0 \text{ and } x \in X,$$

so $\{\Phi(t,t_0)\}_{t\geq t_0}$ is nonuniform exponentially stable. □

Definition 3.5. We say that the evolution family $\{\Phi(t,t_0)\}_{t\geq t_0}$ *satisfies the* (p,∞)*-Perron condition* if:

(i) For all $f \in L^p(X)$ it results that $x_f \in L^\infty(X)$, where

$$x_f(t) = \int_{-\infty}^{t} \Phi(t,\tau)f(\tau)d\tau;$$

(ii) If there is $w \in L^\infty(X)$ such that $w(t) = \Phi(t,s)w(s)$, for all $t \geq s$, it results that $w = 0$.

Theorem 3.6. *If* $\{\Phi(t,t_0)\}_{t\geq t_0}$ *satisfies the* (p,∞)*-Perron condition there is a constant* $K > 0$ *such that*

$$\|x_f\|_\infty \leq K\|f\|_p, \text{ for all } f \in L^p(X).$$

Proof. Let $\mathcal{U} : L^p(X) \to L^\infty(X)$, defined by $\mathcal{U}f = x_f$. We notice that $\mathcal{U}$ is a linear operator and we will prove that it is closed.

Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $L^p(X)$, $f \in L^p(X)$ and $g \in L^\infty(X)$ such that

$$f_n \to f \text{ in } L^p(X) \text{ and } \mathcal{U}f_n \to g \text{ in } L^\infty(X).$$

Using the same technique as in Theorem 3.3 we obtain that

$$g(t) = \Phi(t,s)g(s) + \int_s^t \Phi(t,\tau)f(\tau)d\tau, \text{ for all } t \geq s.$$

Considering $w(t) = g(t) - x_f(t) = \Phi(t,s)w(s),\ \forall\ t \geq s$, it follows that $w = 0$ because $w \in L^\infty(X)$. It results that $g = x_f$ a.e and thus $g = x_f = \mathcal{U}_f$ in $L^\infty(X)$.

We obtain that $\mathcal{U}$ is a closed operator and by the Closed Graph Theorem it is also bounded. Therefore there is $K > 0$ such that

$$||x_f||_\infty \leq K||f||_p, \text{ for all } f \in L^p(X).$$

□

Theorem 3.7. *If $\{\Phi(t,t_0)\}_{t \geq t_0}$ satisfies the (p,∞)-Perron condition, $p > 1$ then there exists $N : \mathbb{R} \to \mathbb{R}_+^*$ and $\nu > 0$ such that*

$$\|\Phi(t,t_0)x\| \leq N(t_0)e^{-\nu(t-t_0)^{1-\frac{1}{p}}}\|x\|, \text{ for all } t \geq t_0 \text{ and } x \in X.$$

Proof. Let $x \in X,\ t_0 \in \mathbb{R}$ and

$$f : \mathbb{R} \to X,\ f(t) = \varphi_{[t_0,t_0+1]}(t)\Phi(t,t_0)x,$$

where $\varphi_{[t_0,t_0+1]}$ denotes the characteristic function of the interval $[t_0, t_0+1]$.

It results that

$$f \in L^p(X) \text{ and } \|f\|_p \leq M(t_0)\|x\|, \text{where } M(t_0) = \sup_{t \in [t_0,t_0+1]} \|\Phi(t,t_0)\|.$$

We have that

$$x_f(t) = \int_{-\infty}^t \Phi(t,\tau)f(\tau)d\tau =$$

$$= \int_{t_0}^t \varphi_{[t_0,t_0+1]}(\tau)d\tau\Phi(t,t_0)x = \begin{cases} 0,\ t < t_0 \\ (t-t_0)\Phi(t,t_0)x,\ t \in [t_0,t_0+1) \\ \Phi(t,t_0)x,\ t \geq t_0+1. \end{cases}$$

But $x_f \in L^\infty(X)$ and from Theorem 3.4 it results that there is $K > 0$ such that

$$\|\Phi(t,t_0)x\| \leq K\|f\|_p \leq KM(t_0)\|x\|, \text{ for all } t \geq t_0+1 \text{ and } x \in X.$$

For $t \in [t_0, t_0+1)$ we have that $\|\Phi(t,t_0)x\| \le M(t_0)\|x\|$. Therefore

$$\|\Phi(t,t_0)x\| \le L(t_0)\|x\|, \text{ for all } t \ge t_0 \text{ and } x \in X,$$

where $L(t_0) = M(t_0)\max\{1, K\}$.

Let now $x \in X,\ t_0 \in \mathbb{R},\ \delta > 0$ and

$$g : \mathbb{R} \to X,\ g(t) = \varphi_{[t_0,t_0+\delta]}(t)\Phi(t,t_0)x.$$

It results that $g \in L^p(X)$ and $\|g\|_p \le \delta^{\frac{1}{p}} L(t_0)\|x\|$.

We have that

$$x_g(t) = \int_{-\infty}^{t} \Phi(t,\tau)g(\tau)d\tau =$$

$$= \int_{t_0}^{t} \varphi_{[t_0,t_0+\delta]}(\tau)d\tau\Phi(t,t_0)x = \begin{cases} 0,\ t < t_0 \\ (t-t_0)\Phi(t,t_0)x,\ t \in [t_0, t_0+\delta) \\ \delta\Phi(t,t_0)x,\ t \ge t_0+\delta. \end{cases}$$

But $x_g \in L^\infty(X)$ and from Theorem 3.4 it results that there is $K > 0$ such that

$$\delta\|\Phi(t,t_0)x\| \le K\|g\|_p \le KL(t_0)\delta^{\frac{1}{p}}\|x\|, \text{ for all } t \ge t_0+\delta,\ x \in X \text{ and } \delta > 0.$$

If we put $\delta = t - t_0$ then we obtain that

$$(t-t_0)\|\Phi(t,t_0)x\| \le KL(t_0)(t-t_0)^{\frac{1}{p}}\|x\|, \text{ for all } t \ge t_0 \text{ and } x \in X,$$

or equivalently

$$(t-t_0)^{1-\frac{1}{p}}\|\Phi(t,t_0)x\| \le KL(t_0)\|x\|, \text{ for all } t \ge t_0 \text{ and } x \in X.$$

Now we consider

$$h : \mathbb{R} \to X,\ h(t) = \varphi_{[t_0,t_0+\delta]}(t)(t-t_0)^{1-\frac{1}{p}}\Phi(t,t_0)x.$$

It results that $h \in L^p(X)$ and $\|h\|_p \le KL(t_0)\delta^{\frac{1}{p}}\|x\|$.

We have that

$$x_h(t) = \int_{-\infty}^{t} \Phi(t,\tau)h(\tau)d\tau = \int_{t_0}^{t} (\tau-t_0)^{1-\frac{1}{p}}\varphi_{[t_0,t_0+\delta]}(\tau)d\tau\Phi(t,t_0)x.$$

If $t \geq t_0 + \delta$ then $x_h(t) = \dfrac{(t-t_0)^{2-\frac{1}{p}}}{2-\frac{1}{p}}\Phi(t,t_0)x.$

But $x_h \in L^\infty(X)$ and from Theorem 3.4 there is $K > 0$ such that

$$\frac{(t-t_0)^{2-\frac{1}{p}}}{2!}\|\Phi(t,t_0)x\| \leq K\|h\|_p \leq K^2L(t_0)(t-t_0)^{\frac{1}{p}}\|x\|, \text{ for all } t \geq t_0 \text{ and } x \in X.$$

Inductively we obtain that

$$\frac{(t-t_0)^{n(1-\frac{1}{p})}}{n!}\|\Phi(t,t_0)x\| \leq K^nL(t_0)\|x\|.$$

By sharing with 2^nK^n it results that

$$\frac{(t-t_0)^{n(1-\frac{1}{p})}}{2^nK^nn!}\|\Phi(t,t_0)x\| \leq \frac{L(t_0)}{2^n}\|x\|.$$

Thus

$$\sum_{n=0}^{\infty}\frac{(t-t_0)^{n(1-\frac{1}{p})}}{2^nK^nn!}\|\Phi(t,t_0)x\| \leq \sum_{n=0}^{\infty}\frac{L(t_0)}{2^n}\|x\|,$$

or equivalently

$$e^{\frac{1}{2K}(t-t_0)^{1-\frac{1}{p}}}\|\Phi(t,t_0)x\| \leq 2L(t_0)\|x\|.$$

We have that

$$\|\Phi(t,t_0)x\| \leq 2L(t_0)e^{-\frac{1}{2K}(t-t_0)^{1-\frac{1}{p}}}\|x\|, \text{ for all } t \geq t_0 \text{ and } x \in X.$$

Denoting by $N(t_0) = 2L(t_0)$ and $\nu = \frac{1}{2K}$ we will obtain that there exists $N : \mathbb{R} \to \mathbb{R}_+^*$ and $\nu > 0$ such that

$$\|\Phi(t,t_0)x\| \leq N(t_0)e^{-\nu(t-t_0)^{1-\frac{1}{p}}}\|x\|, \text{ for all } t \geq t_0 \text{ and } x \in X.$$

□

Further we will analyze the case $(1,\infty)$.

Theorem 3.8. *$\{\Phi(t,t_0)\}_{t\geq t_0}$ satisfies the $(1,\infty)$-Perron condition if and only if $\{\Phi(t,t_0)\}_{t\geq t_0}$ is uniformly stable.*

Proof. Necessity. Let $t_0 \in \mathbb{R},\ \delta > 0,\ x \in X$ such that $\Phi(t,t_0)x \neq 0$, for all $t \geq t_0$ and

$$f : \mathbb{R} \to X,\ f(t) = \varphi_{[t_0,t_0+\delta]}(t)\frac{\Phi(t,t_0)x}{\|\Phi(t,t_0)x\|},$$

where $\varphi_{[t_0,t_0+\delta]}$ denotes the characteristic function of the interval $[t_0, t_0+\delta]$.

It results that $f \in L^1(X)$ and $\|f\|_1 = \delta$.

We have that

$$x_f(t) = \int_{-\infty}^{t} \Phi(t,\tau)f(\tau)d\tau = \int_{t_0}^{t_0+\delta} \frac{d\tau}{\|\Phi(\tau,t_0)x\|}\Phi(t,t_0)x, \text{ for all } t \geq t_0+\delta.$$

But $x_f \in L^\infty(X)$ and by Theorem 3.4 it results that there is $K > 0$ such that

$$\frac{1}{\delta}\int_{t_0}^{t_0+\delta} \frac{d\tau}{\|\Phi(\tau,t_0)x\|}\|\Phi(t,t_0)x\| \leq K, \text{ for all } x \in X \text{ and } \delta > 0.$$

For $\delta \to 0$ we obtain that

$$\|\Phi(t,t_0)x\| \leq K\|x\|, \text{ for all } t \geq t_0 \text{ and } x \in X.$$

Let now $t_0 \in \mathbb{R},\ x \in X$ and $t_1 > t_0$ such that $\Phi(t_1,t_0)x = 0$. It implies that

$$\Phi(t,t_0)x = 0, \text{ for all } t \geq t_1.$$

Denoting by $\sigma = \inf\{t \geq t_0 : \Phi(t,t_0)x = 0\}$ it follows that $\Phi(\sigma,t_0)x = 0$, or equivalently $\Phi(t,t_0)x \neq 0$, for all $t \in [t_0,\sigma)$. Therefore

$$\|\Phi(t,t_0)x\| \leq K\|x\|, \text{ for all } t \geq t_0 \text{ and } x \in X.$$

Sufficiency. We consider $f \in L^1(X)$ and $x_f(t) = \int_{-\infty}^{t} \Phi(t,\tau)f(\tau)d\tau$. We have that

$$\|x_f(t)\| \leq \int_{-\infty}^{t} N\|f(\tau)\|d\tau \leq N\|f\|_1 < \infty, \text{ for all } t \in \mathbb{R}.$$

It result that $x_f \in L^\infty(X)$.

Let now $w \in L^\infty(X)$ such that $w(t) = \Phi(t,s)w(s),$ for all $t \geq s$. We also set $t \in \mathbb{R}$ and $s \in [t-1,t]$. It follows that

$$\|w(t)\| \leq N\|w(s)\|, \text{ for all } s \in [t-1,t].$$

We obtain that

$$\|w(t)\| \leq N\int_{t-1}^{t} \|w(s)\|ds \leq N\|w\|_\infty < \infty, \text{ for all } t \geq s,$$

thus $\|w(t)\| = 0,$ for all $t \in \mathbb{R}$, so $w = 0$.

In this way we obtain that the evolution family is uniformly stable. □

Acknowledgment. This work was supported by the strategic grant POSDRU/159/1.5/S/137750, Project "Doctoral and Postdoctoral programs support for increased competitiveness in Exact Sciences research" cofinanced by the European Social Found within the Sectorial Operational Program Human Resources Development 2007-2013.

References

[1] Barreira, L., Valls, C., *Admissibility for nonuniform exponential contractions*, Journal of Differential Equations **249** (2010), 2889-2904.

[2] Barreira, L., Valls, C., *Nonuniform exponential dichotomies and admissibility*, Discrete and Continuous Dynamical Systems **30** (2011), 39-53.

[3] Chicone, C., Latushkin, Y., *Evolution semigroups in dynamical systems and differential equations*, Mathematical Surveys and Monographs, vol. **70**, Providence, RO: American Mathematical Society, 1999.

[4] Coppel, W.A., *Dichotomies in Stability Theory*, Lecture Notes in Math., vol. **629**, Springer-Verlag, New York, 1978.

[5] Daleckij, J.L., Krein, M.G., *Stability of Differential Equations in Banach Space*, Amer. Math. Soc., Providence, RI, 1974.

[6] Hartman, P. *Ordinary Differential Equations*, Wiley, New York/London/Sidney, 1964.

[7] Huy, N.T., *Invariant manifolds of admissibile classes for semi-linear evolution equations*, J. Differential Equations **246** (2009), 1820-1844.

[8] Latushkin, Y., Randolph, T., Schnaubelt, R., *Exponential dichotomy and mild solutions of non-autonomous equations in Banach spaces*, J. Dynam. Differential Equations **3** (1998), 489-510.

[9] Levitan, B.M., Zhikov, V.V., *Almost Periodic Functions and Differential Equations*, Cambridge Univ. Press, Cambridge, 1982.

[10] Marin, M., Florea, O., *On temporal behaviour of solutions in Thermoelasticity of porous micropolar bodies*, An. Sti. Univ. Ovidius Constanţa, Vol. **22**, issue 1, (2014) 169-188.

[11] Massera, J.L., Schäffer, J.J., *Linear Differential Equations and Function Spaces*, Academic Press, New York, 1966.

[12] Van Minh, N., Räbiger, F., Schnaubelt, R., *Exponential stability, exponential expansiveness and exponential dichotomy of evolution equations on the half-line*, Integral Equations Operator Theory **32** (1998), 332-353.

[13] Pazy, A., *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, New York, 1983.

[14] Perron, O., *Die Stabilitätsfrage bei Differentialgleichungen*, Math. Z. **32** (1930), 703-728.

[15] Sasu, A.L., Sasu, B. *Exponential dichotomy on the real line and admissibility of function spaces*, Integr. equ. oper. theory **54** (2006), 113-130.

[16] Sasu, A.L., *Integral equations on function spaces and dichotomy on the real line*, Integr. equ. oper. theory **58** (2007), 133-152.

[17] Sharma, K., Marin, M., *Reflection and transmission of waves from imperfect boundary between two heat conducting micropolar thermoelastic solids*, An. Sti. Univ. Ovidius Constanţa, Vol. **22**, issue 2, (2014) 151-175.

Claudia Isabela MORARIU,
Department of Mathematics,
Faculty of Mathematics and Computer Science,
West University of Timişoara,
Vasile Pârvan Blvd. No. 4, 300223 Timişoara, Romania.
Email: claudia_morariu@yahoo.com

Petre PREDA,
Department of Mathematics,
Faculty of Mathematics and Computer Science,
West University of Timişoara,
Vasile Pârvan Blvd. No. 4, 300223 Timişoara, Romania.
Email: preda@math.uvt.ro