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Some remarks on a fractional integro-differential inclusion with boundary conditions

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Abstract

We study the existence of solutions for fractional integrodifferential inclusions of order $q \in (1, 2]$ with families of mixed, closed, strip and integral boundary conditions. We establish Filippov type existence results in the case of nonconvex set-valued maps.

1 Introduction

Differential equations with fractional order have recently proved to be strong tools in the modelling of many physical phenomena. As a consequence there was an intensive development of the theory of differential equations and inclusions of fractional order ([13, 15, 16] etc.). Applied problems require definitions of fractional derivative allowing the utilization of physically interpretable initial conditions. Caputo's fractional derivative, originally introduced in [6] and afterwards adopted in the theory of linear visco elasticity, satisfies this demand. Very recently several qualitative results for fractional integro-differential equations were obtained in [1, 10, 12, 14, 17, 18] etc.

This paper is concerned with the following fractional integro-differential inclusion

$$D_c^q x(t) \in F(t, x(t), V(x)(t)) \quad a.e.\ ([0,T]), \tag{1}$$

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where $q \in (1,2]$, D_c^q is the Caputo fractional derivative, $F : [0,T] \times \mathbf{R} \times \mathbf{R} \to \mathcal{P}(\mathbf{R})$ is a set-valued map and $V : C([0,T],\mathbf{R}) \to C([0,T],\mathbf{R})$ is a nonlinear Volterra integral operator defined by $V(x)(t) = \int_0^t k(t,s,x(s))ds$ with $k(.,.,.) : [0,T] \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ a given function.

We study (1) subject to four families of boundary conditions:

i) Mixed boundary conditions

$$Tx'(0) = -ax(0) - bx(T), \quad Tx'(T) = bx(0) + dx(T). \tag{2}$$

ii) Closed boundary conditions

$$x(T) = \alpha x(0) + \beta Tx'(0), \quad Tx'(T) = \gamma x(0) + \delta Tx'(0), \tag{3}$$

where $a, b, d, \alpha, \beta, \gamma, \delta \in \mathbf{R}$ are given constants.

iii) Strip boundary conditions

$$x(0) = \sigma \int_\alpha^\beta x(s)ds, \quad x(1) = \eta \int_\gamma^\delta x(s)ds, \tag{4}$$

where $\sigma, \eta \in \mathbf{R}$ and $0 < \alpha < \beta < \gamma < \delta < 1$.

iv) Nonlocal Riemann-Liouville type integral boundary conditions

$$x(0) = aI^\omega x(\mu), \quad x(1) = bI^\nu x(\theta), \tag{5}$$

where $a, b \in \mathbf{R}$, $\nu, \omega, \mu, \theta \in (0,1)$ and $I^q x(.)$ is the Riemann-Liouville fractional integral of order q.

The aim of this note is to show that Filippov's ideas ([7]) can be suitably adapted in order to obtain the existence of solutions for problems (1)-(2), (1)-(3), (1)-(4) and (1)-(5). Recall that for a differential inclusion defined by a lipschitzian set-valued map with nonconvex values, Filippov's theorem ([7]) consists in proving the existence of a solution starting from a given "quasi" solution. Moreover, the result provides an estimate between the "quasi" solution and the solution obtained.

We note that in the case when F does not depend on the last variable, existence results for problems (1)-(2), (1)-(3), (1)-(4) and (1)-(5) may be found in [7,8,9]. In fact, the results in the present paper extend the main results in [7,8,9] to fractional integrodifferential inclusions.

The paper is organized as follows: in Section 2 we recall some preliminary results that we need in the sequel and in Section 3 we prove our main results.

2 Preliminaries

Let (X,d) be a metric space. Recall that the Pompeiu-Hausdorff distance of the closed subsets $A, B \subset X$ is defined by

$$d_H(A,B) = \max\{d^*(A,B), d^*(B,A)\},\ d^*(A,B) = \sup\{d(a,B); a \in A\},$$

where $d(x,B) = \inf_{y\in B} d(x,y)$.

Let $I = [0,T]$, we denote by $C(I,\mathbf{R})$ the Banach space of all continuous functions from I to $\mathbf{R}$ with the norm $||x(.)||_C = \sup_{t\in I} |x(t)|$ and $L^1(I,\mathbf{R})$ is the Banach space of integrable functions $u(.) : I \to \mathbf{R}$ endowed with the norm $||u(.)||_1 = \int_0^T |u(t)|dt$.

Definition 2.1. a) *The fractional integral of order* $\alpha > 0$ of a Lebesgue integrable function $f : (0,\infty) \to \mathbf{R}$ is defined by

$$I^\alpha f(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)\mathrm{d}s,$$

provided the right-hand side is pointwise defined on $(0,\infty)$ and $\Gamma(.)$ is the (Euler's) Gamma function defined by $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1}e^{-t}\mathrm{d}t$.

b) *The Caputo fractional derivative of order* $\alpha > 0$ of a function $f : [0,\infty) \to \mathbf{R}$ is defined by

$$\mathrm{D}_c^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{-\alpha+n-1} f^{(n)}(s)\mathrm{d}s,$$

where $n = [\alpha] + 1$. It is assumed implicitly that f is n times differentiable whose n-th derivative is absolutely continuous.

We recall (e.g., [13]) that if $\alpha > 0$ and $f \in C(I,\mathbf{R})$ or $f \in L^\infty(I,\mathbf{R})$ then $(\mathrm{D}_c^\alpha I^\alpha f)(t) \equiv f(t)$.

The next two technical results are proved in [2].

Lemma 2.2. *The unique solution* $x(.) \in C(I,\mathbf{R})$ *of problem*

$$D_c^q x(t) = f(t) \quad a.e.\ ([0,T]), \tag{6}$$

with boundary conditions (2) is given by $x(t) = \int_0^T G_1(t,s)f(s)ds$, *where* $G_1(.,.)$ *is the Green function defined by*

$$G_1(t,s) := \begin{cases} \frac{(t-s)^{q-1}}{\Gamma(q)} - \frac{1}{\Delta_1}\big(\frac{[T(b+d)+(b^2-ad)t](T-s)^{q-1}}{T\Gamma(q)} + \\ + \frac{[(a+b)t-(1+b)T](T-s)^{q-2}}{\Gamma(q-1)}\big) \quad if \quad 0 \le s < t \le T, \\ -\frac{1}{\Delta_1}\big(\frac{[T(b+d)+(b^2-ad)t](T-s)^{q-1}}{T\Gamma(q)} + \frac{[(a+b)t-(1+b)T](T-s)^{q-2}}{\Gamma(q-1)}\big) \\ if \quad 0 \le t < s \le T, \end{cases}$$

with $\Delta_1 = (1+b)(b+d) - (a+b)(d-1) \neq 0$.

Note that

$$|G_1(t,s)| \le \frac{T^{q-1}}{\Gamma(q)} \cdot \big(1 + \frac{|b+d+b^2-ad| + (q-1)|a-1|}{|\Delta_1|}\big) =: M_1 \quad \forall t,s \in I.$$

Lemma 2.3. *The unique solution $x(.) \in C(I, \mathbf{R})$ of problem (6)-(3) is given by $x(t) = \int_0^T G_2(t,s)f(s)ds$, where the Green function defined by*

$$G_2(t,s) := \begin{cases} \frac{(t-s)^{q-1}}{\Gamma(q)} - \frac{1}{\Delta_2}(\frac{[T(1-\delta)+\gamma t](T-s)^{q-1}}{T\Gamma(q)} + \\ + \frac{[(1-\alpha)t-(1-\beta)T](T-s)^{q-2}}{\Gamma(q-1)}) \quad if \quad 0 \le s < t \le T, \\ \frac{1}{\Delta_2}(\frac{[T(1-\delta)+\gamma t](T-s)^{q-1}}{T\Gamma(q)} + \frac{[(1-\alpha)t-(1-\beta)T](T-s)^{q-2}}{\Gamma(q-1)}) \\ if \quad 0 \le t < s \le T, \end{cases}$$

with $\Delta_2 = \gamma(1-\beta) + (1-\alpha)(1-\delta) \neq 0$.

Note that

$$|G_2(t,s)| \le \frac{T^{q-1}}{\Gamma(q)} \cdot (1 + \frac{|1-\delta+\gamma| + (q-1)|\alpha-\beta|}{|\Delta_2|}) =: M_2 \quad \forall t,s \in I.$$

For simplicity, in the following results $T = 1$.

The next result is proved in [3].

Lemma 2.4. *For a given function $f(.) \in C(I, \mathbf{R})$ the unique solution of problem (6)-(4) is given by*

$$\begin{aligned} &x(t) = \tfrac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s)ds + \tfrac{\sigma}{\Delta}[-(\tfrac{\eta}{2}(\delta^2-\gamma^2)-1)+ \\ &t(\eta(\delta-\gamma)-1)] \int_\alpha^\beta (\int_0^s \tfrac{(s-m)^{q-1}}{\Gamma(q)} f(m)dm)ds + \tfrac{1}{\Delta}[\tfrac{\sigma}{2}(\beta^2-\alpha^2)- \\ &(\sigma(\beta-\alpha)-1)t][\eta \int_\gamma^\delta (\int_0^s \tfrac{(s-m)^{q-1}}{\Gamma(q)} f(m)dm)ds - \int_0^1 \tfrac{(1-s)^{q-1}}{\Gamma(q)} f(s)ds], \end{aligned}$$

where

$$\Delta = [\frac{\eta}{2}(\delta^2-\gamma^2)-1)][\sigma(\beta-\alpha)-1] - [\frac{\sigma}{2}(\beta^2-\alpha^2)][\eta(\delta-\gamma)-1] \neq 0.$$

Denote $A(t,s) = \frac{(t-s)^{q-1}}{\Gamma(q)}\chi_{[0,t]}(s)$, $B(t,s) = \frac{\sigma}{\Delta\Gamma(q)}[-(\frac{\eta}{2}(\delta^2-\gamma^2)-1) + t(\eta(\delta-\gamma)-1)]\frac{1}{q}[(\beta-s)^q\chi_{[0,\beta]}(s)-(\alpha-s)^q\chi_{[0,\alpha]}(s)]$, $C(t,s) = \frac{1}{\Delta\Gamma(q)}[\frac{\sigma}{2}(\beta^2-\alpha^2) - (\sigma(\beta-\alpha)-1)t]\frac{\eta}{q}[(\delta-s)^q\chi_{[0,\delta]}(s) - (\gamma-s)^q\chi_{[0,\gamma]}(s)]$, $D(t,s) = -\frac{1}{\Delta\Gamma(q)}[\frac{\sigma}{2}(\beta^2 - \alpha^2)-(\sigma(\beta-\alpha)-1)t](1-s)^{q-1}$ and $G_3(t,s) = A(t,s)+B(t,s)+C(t,s)+D(t,s)$, where $\chi_S(.)$ is the characteristic function of the set S. Then the solution $x(.)$ in Lemma 3 may be written as $x(t) = \int_0^1 G_3(t,s)f(s)ds$. Moreover, for any $t,s \in I$ we have

$$|G_3(t,s)| \le \frac{1}{\Gamma(q)} + \frac{\sigma}{|\Delta|\Gamma(q)}[|\frac{\eta}{2}(\delta^2-\gamma^2)-1| + |\eta(\delta-\gamma)-1|]\frac{\beta^q+\alpha^q}{q} +$$

$$\frac{1}{|\Delta|\Gamma(q)}[|\frac{\sigma}{2}(\beta^2-\alpha^2)| + |\sigma(\beta-\alpha)-1|][\frac{\eta}{q}(\delta^q+\gamma^q)+1] =: M_3.$$

The proof of the following lemma may be found in [4].

Lemma 2.5. *For a given $f(.) \in C(I, \mathbf{R})$ the unique solution of the problem (6)-(5) is given by*

$$x(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s)ds + (c_1 - tc_4) \int_0^\mu \frac{(\mu-s)^{q+\omega-1}}{\Gamma(q+\omega)} f(s)ds + \\ (c_2 + c_3 t)[b \int_0^\theta \frac{(\theta-s)^{q+\nu-1}}{\Gamma(q+\nu)} f(s)ds - \frac{1}{\Gamma(q)} \int_0^1 (1-s)^{q-1} f(s)ds],$$

where

$$c_1 = \frac{a}{c}(1 - \frac{b\theta^{\nu+1}}{\Gamma(\nu+2)}),\ c_2 = \frac{a\mu^{\omega+1}}{c\Gamma(\omega+2)},\ c_3 = \frac{1}{c}(1 - \frac{a\mu^\omega}{\Gamma(\omega+1)}),\ c_4 = \frac{a}{c}(1-$$

$$-\frac{b\theta^\nu}{\Gamma(\nu+1)}),\ c = (1 - \frac{a\mu^\omega}{\Gamma(\omega+1)})(1 - \frac{b\theta^{\nu+1}}{\Gamma(\nu+2)}) + \frac{a\mu^{\omega+1}}{\Gamma(\omega+2)}(1 - \frac{b\theta^\nu}{\Gamma(\nu+1)}).$$

It is implicitly assumed that $c \neq 0$.

Denote $A_1(t,s) = \frac{(t-s)^{q-1}}{\Gamma(q)} \chi_{[0,t]}(s)$, $B_1(t,s) = (c_1 - tc_4) \frac{(\mu-s)^{q+\omega-1}}{\Gamma(q+\omega)} \chi_{[0,\mu]}(s)$, $C_1(t,s) = b(c_2 + c_3 t) \frac{(\theta-s)^{q+\nu-1}}{\Gamma(q+\nu)} \chi_{[0,\theta]}(s)$, $D_1(t,s) = -(c_2 + c_3 t) \frac{(1-s)^{q-1}}{\Gamma(q)}$ and $G_4(t,s) = A_1(t,s) + B_1(t,s) + C_1(t,s) + D_1(t,s)$, then the solution $x(.)$ in Lemma 4 may be written as $x(t) = \int_0^1 G_4(t,s) f(s)ds$. Moreover, for any $t, s \in I$ we have

$$|G_4(t,s)| \leq \frac{1}{\Gamma(q)} + (|c_1| + |c_4|) \frac{\mu^{q+\omega-1}}{\Gamma(q+\omega)} + |b|(|c_2| + |c_3|) \frac{\theta^{q+\nu-1}}{\Gamma(q+\nu)} + \\ (|c_2| + |c_3|) \frac{1}{\Gamma(q)} =: M_4$$

3 The main results

In order to prove our results we need the following hypotheses.

Hypothesis. i) $F(.,.) : I \times \mathbf{R} \times \mathbf{R} \to \mathcal{P}(\mathbf{R})$ *has nonempty closed values and is $\mathcal{L}(I) \otimes \mathcal{B}(\mathbf{R} \times \mathbf{R})$ measurable.*

ii) *There exists $L(.) \in L^1(I, (0, \infty))$ such that, for almost all $t \in I$, $F(t,.,.)$ is $L(t)$-Lipschitz in the sense that*

$$\mathrm{d}_H(F(t, x_1, y_1), F(t, x_2, y_2)) \leq L(t)(|x_1 - x_2| + |y_1 - y_2|) \quad \forall\ x_1, x_2, y_1, y_2 \in \mathbf{R}.$$

iii) $k(.,.,.) : I \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ *is a function such that $\forall x \in \mathbf{R}$, $(t,s) \to k(t,s,x)$ is measurable.*

iv) $|k(t,s,x) - k(t,s,y)| \leq L(t)|x-y|$ *a.e.* $(t,s) \in I \times I$, $\forall\, x, y \in \mathbf{R}$.

We use next the following notations

$$M(t) := L(t)(1 + \int_0^t L(u)du), \quad t \in I, \quad M_0 = \int_0^T M(t)dt.$$

Theorem 3.1. *Assume that Hypothesis is satisfied and* $M_1M_0 < 1$. *Let* $y(.) \in C(I, \mathbf{R})$ *be such that* $Ty'(0) = -ay(0) - by(T)$, $Ty'(T) = by(0) + dy(T)$ *and there exists* $p(.) \in L^1(I, \mathbf{R}_+)$ *with* $d(D_c^q y(t), F(t, y(t), V(y)(t))) \le p(t)$ *a.e.* (I).

Then there exists $x(.) \in C(I, \mathbf{R})$ *a solution of problem (1)-(2) satisfying for all* $t \in I$

$$|x(t) - y(t)| \le \frac{M_1}{1 - M_1M_0} \int_0^T p(t)dt. \tag{7}$$

Proof. The set-valued map $t \to F(t, y(t), V(y)(t))$ is measurable with closed values and

$$F(t, y(t), V(y)(t)) \cap \{D_c^q y(t) + p(t)[-1, 1]\} \neq \emptyset \quad a.e.\ (I).$$

It follows (e.g., Theorem 1.14.1 in [5]) that there exists a measurable selection $f_1(t) \in F(t, y(t), V(y)(t))$ *a.e.* (I) such that

$$|f_1(t) - D_c^q y(t)| \le p(t) \quad a.e.\ (I) \tag{8}$$

Define $x_1(t) = \int_0^T G_1(t, s) f_1(s)ds$ and one has

$$|x_1(t) - y(t)| \le M_1 \int_0^T p(t)dt.$$

We claim that it is enough to construct the sequences $x_n(.) \in C(I, \mathbf{R})$, $f_n(.) \in L^1(I, \mathbf{R})$, $n \ge 1$ with the following properties

$$x_n(t) = \int_0^T G_1(t, s) f_n(s)ds, \quad t \in I, \tag{9}$$

$$f_n(t) \in F(t, x_{n-1}(t), V(x_{n-1})(t)) \quad a.e.\ (I), \tag{10}$$

$$|f_{n+1}(t) - f_n(t)| \le L(t)(|x_n(t) - x_{n-1}(t)| + \int_0^t L(s)|x_n(s) - x_{n-1}(s)|ds) \quad a.e.\ (I) \tag{11}$$

If this construction is realized then from (8)-(11) we have for almost all $t \in I$

$$|x_{n+1}(t) - x_n(t)| \le M_1(M_1M_0)^n \int_0^T p(t)dt \quad \forall n \in \mathbf{N}.$$

Indeed, assume that the last inequality is true for $n - 1$ and we prove it for n. One has

$$|x_{n+1}(t) - x_n(t)| \le \int_0^T |G_1(t, t_1)|.|f_{n+1}(t_1) - f_n(t_1)|dt_1 \le$$

$$M_1\int_0^T L(t_1)[|x_n(t_1)-x_{n-1}(t_1)|+\int_0^{t_1}L(s)|x_n(s)-x_{n-1}(s)|ds]dt_1\le M_1$$

$$\int_0^T L(t_1)(1+\int_0^{t_1}L(s)ds)dt_1.M_1^nM_0^{n-1}\int_0^T p(t)dt=M_1(M_1M_0)^n\int_0^T p(t)dt$$

Therefore $\{x_n(.)\}$ is a Cauchy sequence in the Banach space $C(I,\mathbf{R})$, hence converging uniformly to some $x(.)\in C(I,\mathbf{R})$. Therefore, by (11), for almost all $t\in I$, the sequence $\{f_n(t)\}$ is Cauchy in $\mathbf{R}$. Let $f(.)$ be the pointwise limit of $f_n(.)$.

Moreover, one has

$$\begin{aligned}&|x_n(t)-y(t)|\le|x_1(t)-y(t)|+\textstyle\sum_{i=1}^{n-1}|x_{i+1}(t)-x_i(t)|\le\\&M_1\textstyle\int_0^T p(t)dt+\sum_{i=1}^{n-1}(M_1\int_0^T p(t)dt)(M_1M_0)^i=\frac{M_1\int_0^T p(t)dt}{1-M_1M_0}.\end{aligned}\tag{12}$$

On the other hand, from (8), (11) and (12) we obtain for almost all $t\in I$

$$\begin{aligned}&|f_n(t)-D_c^qy(t)|\le\textstyle\sum_{i=1}^{n-1}|f_{i+1}(t)-f_i(t)|+|f_1(t)-D_c^qy(t)|\le\\&L(t)\frac{M_1\int_0^T p(t)dt}{1-M_1M_0}+p(t).\end{aligned}$$

Hence the sequence $f_n(.)$ is integrably bounded and therefore $f(.)\in L^1(I,\mathbf{R})$.

Using Lebesgue's dominated convergence theorem and taking the limit in (9), (10) we deduce that $x(.)$ is a solution of (1)-(2). Finally, passing to the limit in (12) we obtained the desired estimate on $x(.)$.

It remains to construct the sequences $x_n(.),f_n(.)$ with the properties in (9)-(11). The construction will be done by induction.

Since the first step is already realized, assume that for some $N\ge1$ we already constructed $x_n(.)\in C(I,\mathbf{R})$ and $f_n(.)\in L^1(I,\mathbf{R})$, $n=1,2,...N$ satisfying (9), (11) for $n=1,2,...N$ and (10) for $n=1,2,...N-1$. The set-valued map $t\to F(t,x_N(t),V(x_N)(t))$ is measurable. Moreover, the map $t\to L(t)(|x_N(t)-x_{N-1}(t)|+\int_0^t L(s)|x_N(s)-x_{N-1}(s)|ds)$ is measurable. By the lipschitzianity of $F(t,.)$ we have that for almost all $t\in I$

$$\begin{aligned}&F(t,x_N(t))\cap\{f_N(t)+L(t)(|x_N(t)-x_{N-1}(t)|+\\&\textstyle\int_0^t L(s)|x_N(s)-x_{N-1}(s)|ds)[-1,1]\}\ne\emptyset.\end{aligned}$$

Theorem 1.14.1 in [5] yields that there exist a measurable selection $f_{N+1}(.)$ of $F(.,x_N(.),V(x_N)(.))$ such that for almost all $t\in I$

$$|f_{N+1}(t)-f_N(t)|\le L(t)(|x_N(t)-x_{N-1}(t)|+\int_0^t L(s)|x_N(s)-x_{N-1}(s)|ds).$$

We define $x_{N+1}(.)$ as in (9) with $n=N+1$. Thus $f_{N+1}(.)$ satisfies (10) and (11) and the proof is complete. $\square$

The proofs of the next three theorems are similar to the proof of Theorem 3.1.

Theorem 3.2. *Assume that Hypothesis is satisfied and* $M_2M_0 < 1$. *Let* $y(.) \in C(I, \mathbf{R})$ *be such that* $y(T) = \alpha y(0) + \beta T y'(0)$, $Ty'(T) = \gamma y(0) + \delta T y'(0)$ *and there exists* $p(.) \in L^1(I, \mathbf{R})$ *with* $d(D_c^q y(t), F(t, y(t, V(y)(t)))) \leq p(t)$ *a.e.* (I).

Then there exists $x(.) \in C(I, \mathbf{R})$ *a solution of problem (1)-(3) satisfying for all* $t \in I$

$$|x(t) - y(t)| \leq \frac{M_2}{1 - M_2M_0} \int_0^T p(t)dt.$$

Theorem 3.3. *Assume that Hypothesis is satisfied and* $M_3M_0 < 1$. *Let* $y(.) \in C(I, \mathbf{R})$ *be such that* $y(0) = \sigma \int_\alpha^\beta y(s)ds$, $y(1) = \eta \int_\gamma^\delta y(s)ds$ *and there exists* $p(.) \in L^1(I, \mathbf{R}_+)$ *with* $d(D_c^q y(t), F(t, y(t), V(y)(t))) \leq p(t)$ *a.e.* (I).

Then there exists $x(.) \in C(I, \mathbf{R})$ *a solution of problem (1)-(4) satisfying for all* $t \in I$

$$|x(t) - y(t)| \leq \frac{M_3}{1 - M_3M_0} \int_0^1 p(t)dt.$$

Theorem 3.4. *Assume that Hypothesis is satisfied and* $M_4M_0 < 1$ *Let* $y(.) \in C(I, \mathbf{R})$ *be such that* $y(0) = aI^\omega y(\mu)$, $y(1) = bI^\nu y(\theta)$ *and there exists* $p(.) \in L^1(I, \mathbf{R}_+)$ *with* $d(D_c^q y(t), F(t, y(t), V(y)(t))) \leq p(t)$ *a.e.* (I).

Then there exists $x(.) \in C(I, \mathbf{R})$ *a solution of problem (1)-(5) satisfying for all* $t \in I$

$$|x(t) - y(t)| \leq \frac{M_4}{1 - M_4M_0} \int_0^1 p(t)dt.$$

Remark 3.5. If $F(., ., .)$ does not depend on the last variable the fractional integrodifferential inclusion (1) reduces to

$$D_c^q x(t) \in F(t, x(t)) \quad a.e. \ (I)$$

and Theorem 3.1 yields Theorem 3.3 in [7], Theorem 3.2 yields Theorem 3.4 in [7], Theorem 3.3 yields Theorem 3.4 in [8] and Theorem 3.4 yields Theorem 3.5 in [9].

References

[1] Aghajani, A., Jalilian, Y., Trujillo, J.J., *On the existence of solutions of fractional integro-differential equations*, Fract. Calc. Appl. Anal. **15** (2012), 44-69.

[2] Ahmad, B., Nieto, J.J., Pimentel, J., *Some boundary value problems of fractional differential equations and inclusions*, Computers and Mathematics with Applications **62** (2011), 1238-1250.

[3] Ahmad, B., Ntouyas, S.K., *Existence results for nonlocal boundary value problems of fractional differential equations and inclusions with strip conditions*, Boundary Value Problems **55** (2012), 1-21.

[4] Ahmad, B., Ntouyas, S.K., *Existence of solutions for fractional differential inclusions with four-point nonlocal Riemann-Liouville type integral boundary conditions*, Filomat **27** (2013), 1027-1036.

[5] Aubin, J.P., Cellina, A., *Differential Inclusions*, Springer, Berlin, 1984.

[6] Caputo, M., *Elasticità e Dissipazione*, Zanichelli, Bologna, 1969.

[7] Cernea, A., *A note on the existence of solutions for some boundary value problems of fractional differential inclusions*, Fract. Calc. Appl. Anal. **15** (2012), 183-194.

[8] Cernea, A., *On a fractional differential inclusion with strip boundary conditions*, J. Fract. Calc. Appl. **4** (2013), 169-176.

[9] Cernea, A., *On a fractional differential inclusion with nonlocal Riemann-Liouville type integral boundary conditions*, Libertas Math. **33** (2013), 37-46.

[10] Chalishajar, D.N., Karthikeyan, K., *Existence and uniqueness results for boundary value problems of higher order fractional integro-differential equations involving Gronwall's inequality in Banach spaces*, Acta Math. Scientia **33B** (2013), 758-772.

[11] Filippov, A.F., *Classical solutions of differential equations with multivalued right hand side*, SIAM J. Control **5** (1967), 609-621.

[12] Karthikeyan, K., Trujillo, J.J., *Existence and uniqueness results for fractional integro-differential equations with boundary value conditions*, Commun Nonlinear Sci. Numer. Simulat. **17** (2012), 4037-4043.

[13] Kilbas, A., Srivastava, H.M., Trujillo, J.J., *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam, 2006.

[14] Marin, M., Florea, O., *On temporal behavior of solutions in Thermoelasticity of porous micropolar bodies*, An. Şt. Univ. Ovidius Constanţa **22** (2014), 169-188.

[15] Miller, K., Ross, B., *An Introduction to the Fractional Calculus and Differential Equations*, John Wiley, New York, 1993.

[16] Podlubny, I., *Fractional Differential Equations*, Academic Press, San Diego, 1999.

[17] Sharma, K., Marin, M., *Reflection and transmision of waves from imperfect boundary between two heat conducting micropolar thermoelastic solids*, An. Şt. Univ. Ovidius Constanţa **22** (2014), 151-175.

[18] Wang, J.R., Wei, W., Yang, Y., *Fractional nonlocal integro-differential equations of mixed type with time varying generating operators and optimal control*, Opuscula Math. **30** (2010), 361-381.

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