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# The Relationships Between $p-$valent Functions and Univalent Functions

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**Abstract**

In this paper, we obtain some sufficient conditions for general $p-$valent integral operators to be the $p-$th power of a univalent functions in the open unit disk.

## 1 Introduction

Let $\mathcal{A}(p)$ be the class of functions of the form:

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \qquad (p \in \mathbb{N} = \{1, 2, ...\}) \tag{1}$$

which are analytic and $p-$valent in the unit disc $\mathcal{U} = \{z : |z| < 1\}$. A function $f(z) \in \mathcal{A}(p)$ is called $p-$valent starlike of order $\gamma$ if $f(z)$ satisfies

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > \gamma \tag{2}$$

for $0 \leq \gamma < p$, $p \in \mathbb{N}$ and $z \in \mathcal{U}$. By $\mathcal{S}^*(p, \gamma)$, we denote the class of all $p-$valent starlike functions of order $\gamma$. By $\mathcal{S}_p^*(\gamma)$ denote the subclass of $\mathcal{S}^*(p, \gamma)$ consisting of functions $f(z) \in \mathcal{A}(p)$ for which

$$\left|\frac{zf'(z)}{f(z)} - p\right| < p - \gamma \tag{3}$$

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for $0 \leq \gamma < p$, $p \in \mathbb{N}$ and $z \in \mathcal{U}$. Also a function $f(z) \in \mathcal{A}(p)$ is called $p-$valent convex of order $\gamma$ if $f(z)$ satisfies

$$\Re\left\{1+\frac{zf''(z)}{f'(z)}\right\} > \gamma \tag{4}$$

for $0 \leq \gamma < p$, $p \in \mathbb{N}$ and $z \in \mathcal{U}$. By $\mathcal{C}(p,\gamma)$, we denote the class of all $p-$valent convex functions of order $\gamma$. It follows from (2) and (4) that

$$f(z) \in \mathcal{C}(p,\gamma) \Longleftrightarrow \frac{zf'(z)}{p} \in \mathcal{S}^*(p,\gamma). \tag{5}$$

Also $\mathcal{C}_p(\gamma)$ denote the subclass of $\mathcal{C}(p,\gamma)$ consisting of functions $f(z) \in \mathcal{A}(p)$ for which

$$\left|\frac{zf''(z)}{f'(z)} - (p-1)\right| < p-\gamma \tag{6}$$

for $0 \leq \gamma < p$, $p \in \mathbb{N}$ and $z \in \mathcal{U}$.

We define the following general $p-$valent integral operators:

The first family of $p-$valent integral operators has the following form:

$$\mathcal{F}^p_{\alpha_1,\ldots,\alpha_n,\beta}(z) = \left\{\beta p \int_0^z u^{\beta p-1} \prod_{i=1}^{n} \left(\frac{f_i(u)}{u^p}\right)^{1/\alpha_i} du\right\}^{1/\beta}, \tag{7}$$

where the functions $f_i$ for all $i = \overline{1,n}$, $n \in \mathbb{N} = \{1,2,...\}$ belongs to the class $\mathcal{A}(p)$ and the parameters $\beta$ and $\alpha_i$ $(\alpha_i \neq 0)$ for all $i = \overline{1,n}$ are complex numbers such that the integral operators in (7) exist.

**Remark 1:** (7) $p-$valent integral operator, for $p = 1$, was studied by Seenivasagan and Breaz (see [14]) (see also the recent investigations on this subject by Baricz and Frasin [2] and Srivastava, Deniz and Orhan [15]). We note that if $\alpha_i = \alpha$ for all $i = \overline{1,n}$ and $p = 1$, then the $p-$valent integral operator $\mathcal{F}^p_{\alpha_1,\ldots,\alpha_n,\beta}(z)$ reduces to the operator $\mathcal{F}^1_{\alpha,\beta}(z)$ which is related closely to some known integral operators investigated earlier in Geometric Functions Theory (see, for details, [16]). The operators $\mathcal{F}^1_{\alpha,\beta}(z)$ and $\mathcal{F}^1_{\alpha,\alpha}(z)$ were studied by Breaz and Breaz (see [4]) and Pescar (see [12]), respectively. Upon setting $\beta = 1$ and $\alpha = \beta = 1$ in $\mathcal{F}^1_{\alpha,\beta}(z)$, we can obtain the operators $\mathcal{F}^1_{\alpha,1}(z)$ and $\mathcal{F}^1_{1,1}(z)$ which were studied by Breaz and Breaz (see [3]) and Alexander (see [1]), respectively. Furthermore, in their special cases when $p = n = \beta = 1$, and $1/\alpha$ instead of $\alpha_i = \alpha$ for all $i = \overline{1,n}$, the $p-$valent integral operator in (7) would obviously reduce to the operator $\mathcal{F}^1_{1/\alpha,1}(z)$ which was studied Pescar and Owa (see [13]), for $\alpha \in [0,1]$ special case of the operator $\mathcal{F}^1_{1/\alpha,1}(z)$ was studied by Miller, Mocanu and Reade (see [11]). Recently, Bulut (see

[6]) introduced this operator for $p = 1$ by using the generalized Al-Oboudi differential operator.

The second family of $p-$valent integral operators has the following form:

$$\mathcal{G}^{p}_{\alpha_1,\ldots,\alpha_n,\beta}(z) = \left\{ \beta p \int_0^z u^{\beta p-1} \prod_{i=1}^{n} \left( \frac{f_i'(u)}{pu^{p-1}} \right)^{\alpha_i} du \right\}^{1/\beta} \tag{8}$$

where the functions $f_i \in \mathcal{A}(p)$ for all $i = \overline{1,n}$ and the parameters $\beta$ and $\alpha_i$ for all $i = \overline{1,n}$ are complex numbers such that the $p-$valent integral operators in (8) exist.

**Remark 2:** For $p = 1$, $\mathcal{G}^{1}_{\alpha_1,\ldots,\alpha_n,\beta}(z)$ was introduced by Breaz and Breaz (see [5]). Additionally, for $\beta = 1$, $\mathcal{G}^{p}_{\alpha_1,\ldots,\alpha_n,1}(z)$ was studied by Frasin (see [9]).

Hallenbeck and Livingston (see [10]) defined $p-$subordination chains and they obtained some results for $f \in \mathcal{A}(p)$ to be the $p-$th power of a univalent functions in $\mathcal{U}$. Recently, Deniz (see [7]) investigated some results for $f \in \mathcal{A}(p)$ to be the $p-$th power of a univalent functions in $\mathcal{U}$ by using $p-$subordination chains method. Also, Deniz et al. (see [8]) submitted a paper which includes sufficient conditions for a $p-$valent integral operator to be the $p-$th power of a univalent function in $\mathcal{U}$ by using $p-$subordination chains method.

In our present investigation, we need one of sufficient conditions which we recall here as Theorem 1.1 below. This theorem is of fundemental importance to our investigation.

**Theorem 1.1.** *([8]) Let $f \in \mathcal{A}(p)$ and $\alpha$ complex number such that $\Re(\alpha) > 0$. Suppose that*

$$\frac{1 - |z|^{2p\Re(\alpha)}}{\Re(\alpha)} \left| \frac{zf''(z)}{f'(z)} - (p-1) \right| \leqslant p \tag{9}$$

*is true for all $z \in \mathcal{U}$, then the integral operator*

$$\mathcal{H}_{\alpha}(z) = \left[ \alpha \int_0^z u^{p(\alpha-1)} f'(u) du \right]^{1/\alpha} \tag{10}$$

*is the $p-th$ power of a univalent function in $\mathcal{U}$ where the principal branch is considered.*

## 2 Main Results

Firstly, we obtain sufficient conditions for (7) $p-$valent integral operator to be the $p-$th power of a univalent function in $\mathcal{U}$.

**Theorem 2.1.** *Let $f_i \in \mathcal{S}^*(p,\gamma_i)$ $(0 \leq \gamma_i < p)$ for all $i=\overline{1,n}$. If the parameters $\beta$ and $\alpha_i$ $(\alpha_i \neq 0)$ for all $i=\overline{1,n}$ are complex numbers and*

$$p\Re(\beta) \geq 1-p+\sum_{i=1}^{n}\frac{(p-\gamma_i)}{|\alpha_i|} \tag{11}$$

*then the integral operator $\mathcal{F}^p_{\alpha_1,\ldots,\alpha_n,\beta}$ defined by (7) is the $p-th$ power of a univalent function in $\mathcal{U}$ where the principal branch is considered.*

*Proof.* Define the function

$$h(z)=\int_0^z \prod_{i=1}^{n}\left(\frac{f_i(u)}{u^p}\right)^{1/\alpha_i} du. \tag{12}$$

It is easy to see that

$$h'(z)=\prod_{i=1}^{n}\left(\frac{f_i(z)}{z^p}\right)^{1/\alpha_i}. \tag{13}$$

Differentiating (13) logarithmically and multiplying by $z$, we obtain

$$\frac{zh''(z)}{h'(z)}=\sum_{i=1}^{n}\frac{1}{\alpha_i}\left(\frac{zf_i'(z)}{f_i(z)}-p\right). \tag{14}$$

In the light of the hypothesis of Theorem 2.1, we have

$$\begin{aligned}
&\frac{1-|z|^{2p\Re(\beta)}}{\Re(\beta)}\left|1-p+\frac{zh''(z)}{h'(z)}\right| \\
=\ &\frac{1-|z|^{2p\Re(\beta)}}{\Re(\beta)}\left|1-p+\sum_{i=1}^{n}\frac{1}{\alpha_i}\left(\frac{zf_i'(z)}{f_i(z)}-p\right)\right| \\
\leq\ &\frac{1-|z|^{2p\Re(\beta)}}{\Re(\beta)}\left[1-p+\sum_{i=1}^{n}\frac{1}{|\alpha_i|}\left|\frac{zf_i'(z)}{f_i(z)}-p\right|\right] \\
\leq\ &\frac{1}{\Re(\beta)}\left[1-p+\sum_{i=1}^{n}\frac{1}{|\alpha_i|}(p-\gamma_i)\right] \leq p.
\end{aligned}$$

Hence by Theorem 1.1, we get the integral operator $\mathcal{F}^p_{\alpha_1,\ldots,\alpha_n,\beta}$ defined by (7) *is the $p-th$* power of a univalent function in $\mathcal{U}$ where the principal branch is considered. This completes the proof. □

Letting $\beta=n=1$, $\frac{1}{\alpha_1}=\frac{1}{\alpha}$, $\gamma_1=\gamma$ and $f_1=f$ in Theorem 2.1, we obtain following corollary.

**Corollary 2.2.** *Let $f \in \mathcal{S}^*(p,\gamma)$. If the parameter $\alpha \left(|\alpha| \neq \frac{1}{2}\right)$ is complex number and*

$$p \geq \frac{|\alpha| - \gamma}{2|\alpha| - 1} \tag{15}$$

*then the integral operator*

$$\mathcal{F}_{\alpha}^{p}(z) = \int_0^z p u^{p-1} \left(\frac{f(u)}{u^p}\right)^{1/\alpha} du \tag{16}$$

*is the $p-th$ power of a univalent function in $\mathcal{U}$ where the principal branch is considered.*

By taking $\alpha = 1$ in Corollary 2.2, we get:

**Corollary 2.3.** *Let $f \in \mathcal{S}^*(p,\gamma)$. If*

$$p \geq 1 - \gamma \tag{17}$$

*then the integral operator*

$$\mathcal{F}^{p}(z) = p \int_0^z \frac{f(u)}{u} du \tag{18}$$

*is the $p-th$ power of a univalent function in $\mathcal{U}$ where the principal branch is considered.*

In the next theorem, we derive another sufficient condition for (8) $p-$valent integral operator to be the $p-$th power of a univalent function in $\mathcal{U}$.

**Theorem 2.4.** *Let $f_i \in \mathcal{C}(p,\gamma_i)$ $(0 \leq \gamma_i < p)$ for all $i = \overline{1,n}$. If the parameters $\beta$ and $\alpha_i$ for all $i = \overline{1,n}$ are complex numbers and*

$$p\Re(\beta) \geq 1 - p + \sum_{i=1}^{n} |\alpha_i| (p - \gamma_i) \tag{19}$$

*then the integral operator $\mathcal{G}^{p}_{\alpha_1,\ldots,\alpha_n,\beta}$ defined by (8) is the $p-th$ power of a univalent function in $\mathcal{U}$ where the principal branch is considered.*

*Proof.* We define

$$g(z) = \int_0^z \prod_{i=1}^{n} \left(\frac{f_i'(u)}{pu^{p-1}}\right)^{\alpha_i} du. \tag{20}$$

It is easy to see that

$$g'(z) = \prod_{i=1}^{n} \left(\frac{f_i'(z)}{pz^{p-1}}\right)^{\alpha_i}. \tag{21}$$

Differentiating (21) logarithmically and multiplying by $z$, we obtain

$$\frac{zg''(z)}{g'(z)} = \sum_{i=1}^{n} \alpha_i \left( \frac{zf_i''(z)}{f_i'(z)} - (p-1) \right). \tag{22}$$

In the light of the hypothesis of Theorem 2.4, we have

$$\begin{aligned}
&\frac{1-|z|^{2p\Re(\beta)}}{\Re(\beta)} \left| 1-p+\frac{zg''(z)}{g'(z)} \right| \\
= \quad &\frac{1-|z|^{2p\Re(\beta)}}{\Re(\beta)} \left| 1-p+\sum_{i=1}^{n} \alpha_i \left( \frac{zf_i''(z)}{f_i'(z)} - (p-1) \right) \right| \\
\leq \quad &\frac{1-|z|^{2p\Re(\beta)}}{\Re(\beta)} \left[ 1-p+\sum_{i=1}^{n} |\alpha_i| \left| \frac{zf_i''(z)}{f_i'(z)} - (p-1) \right| \right] \\
\leq \quad &\frac{1}{\Re(\beta)} \left[ 1-p+\sum_{i=1}^{n} |\alpha_i| (p-\gamma_i) \right] \leq p.
\end{aligned}$$

Consequently, by Theorem 1.1, we get the integral operator $\mathcal{G}^p_{\alpha_1,\ldots,\alpha_n,\beta}$ defined by (8) *is the* $p-th$ power of a univalent function in $\mathcal{U}$ where the principal branch is considered. Thus, the proof is completed. □

For $\beta = n = 1$, $\alpha_1 = \alpha$, $\gamma_1 = \gamma$ and $f_1 = f$ in Theorem 2.4, we get following corollary.

**Corollary 2.5.** *Let $f \in \mathcal{C}(p,\gamma)$. If the parameter $\alpha$ ($|\alpha| \neq 2$) is complex number and*

$$p \geq \frac{1-\gamma|\alpha|}{2-|\alpha|} \tag{23}$$

*then the integral operator*

$$\mathcal{G}^p_{\alpha}(z) = \int_0^z pu^{p-1} \left( \frac{f'(u)}{pu^{p-1}} \right)^{\alpha} du \tag{24}$$

*is the $p-th$ power of a univalent function in $\mathcal{U}$ where the principal branch is considered.*

If we choose $\alpha = 1$ in Corollary 2.5, we have:

**Corollary 2.6.** *Let $f \in \mathcal{C}(p,\gamma)$. If*

$$p \geq 1-\gamma \tag{25}$$

*then the integral operator*

$$\mathcal{G}^p(z) = \int_0^z f'(u)du \tag{26}$$

*is the $p-th$ power of a univalent function in $\mathcal{U}$ where the principal branch is considered.*

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