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n–ary hyperstructures constructed from binary quasi–ordered semigroups

Michal Novák

Abstract

Based on works by Davvaz, Vougiouklis and Leoreanu-Fotea in the field of n–ary hyperstructures and binary relations we present a construction of n–ary hyperstructures from binary quasi-ordered semigroups. We not only construct the hyperstructures but also study their important elements such as identities, scalar identities or zeros. We also relate the results to earlier results obtained for a similar binary construction and include an application of the results on a hyperstructure of linear differential operators.

1 Introduction

Since its introduction in 1930s, the study of binary hyperstructures has become an established area of research thanks to authors of numerous papers on the topic as well as thanks to standard books which sum up the basic concepts of hyperstructure theory and their applications. Yet the step from binary hyperstructures to *n–ary hyperstructures* has been done only recently by Davvaz and Vougiouklis who in [13] introduced the concept of *n–ary hypergroup* (sometimes called simply *n–hypergroup*) and presented n–ary generalization of some very basic concepts of hyperstructure theory.

Apart from [13] the origins of our paper can be traced back to the issue introduced to hyperstructure theory by Rosenberg, Corsini, Leoreanu-Fotea,

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Chvalina and others in works such as [3, 4, 10, 11, 23], i.e. the relation of hyperstructures and binary relations. Some particular constructions of hyperstructures associated to quasi-ordered single-valued structures introduced by Chvalina in [3, 4] have been studied and developped by Corsini, Davvaz, Heidari, Hošková–Mayerová, Nezhad, and others in works such as [5, 8, 10, 14, 21].

This paper generalizes one of Chvalina's constructions of binary hyperstructures from single-valued quasi-ordered semigroups. Results recently obtained in the area of n–ary generalization of hyperstructures associated to binary relations fall into three groups: some, such as Cristea and Ştefănescu in e.g. [7, 9], generalize the binary relation and construct *binary hyperstructures associated to n–ary relations* while others, such as Leoreanu-Fotea and Davvaz in e.g. [17] generalize the hyperstructure and construct *n–ary hyperstructures associated to binary relations*. Finally, the third approach, presented e.g. in [1] is possible too – as one can study *n–ary hyperstructures associated to n–ary relations*. Out of these three options we develop the approach pioneered by Leoreanu-Fotea and Davvaz in [17].

We make use of n–ary hyperstructure concepts defined in [2, 13, 15]. As far as the basic binary concepts of hyperstructure theory are concerned, we use their definitions and meaning included in [10, 12]. For respective definitions see section 2 or respective places in the paper. Sometimes the definitions are adjusted in order to keep unified form of notation and/or naming throughout the paper. This is especially true for definitions and theorems taken from [2].

Notice that the original contruction, which is generalized in this paper, can be used in a number of contexts including differential equations, integral and integro–differential equations (hyperstructures of linear differential operators, Fredholm and Volterra equations), microeconomics (preference relations), chemistry, genetics, etc. For details cf. references of papers written on the topic by authors such as Chvalina, Hošková–Mayerová, Račková or Novák. Some more examples may be found in [21] and its references.

Finally, notice that the study of n–ary hyperstructures has important implications in the study of fuzzy hyperstructures and that the connection between hypergroups and n–ary hypergroups has been thoroughly studied in [16].

2 Basic notions and concepts

In the paper we work with the generalization of the basic concepts of the hyperstructure theory such as *(binary) hyperoperation*, *semihypergroup* and *hypergroup*. For their definitions cf. e.g. [10, 12]. Further, we work with the following three definitions included in [13] in the following wording:

Definition 2.1. *Let H be a non-empty set and f be a mapping $f : H \times H \to P^*(H)$, where $P^*(H)$ is the set of all non-empty subsets of H. Then f is* *called a binary hyperoperation of H. We denote by H^n the cartesian product $H \times \ldots \times H$, where H appears n times. An element of H^n will be denoted by $(x_1, \ldots, x_n)$, where $x_i \in H$ for any i with $1 \le i \le n$. In general, a mapping $f : H^n \to P^*(H)$ is called an n–ary hyperoperation and n is called the arity of hyperoperation. Let f be an n–ary hyperoperation on H and $A_1, \ldots, A_n$ subsets of H. We define*

$$f(A_1, \ldots, A_n) = \cup\{f(x_1, \ldots, x_n) | x_i \in A_i, i = 1, \ldots, n\}.$$

We shall use the following abbreviated notation: the sequence $x_i, x_{i+1}, \ldots, x_j$ will be denoted by x_i^j. For $j < i, x_i^j$ is the empty set. In this convention

$$f(x_1, \ldots, x_i, y_{i+1}, \ldots, y_j, z_{j+1}, \ldots, z_n)$$

will be written as $f(x_1^i, y_{i+1}^j, z_{j+1}^n)$.

Definition 2.2. *A non-empty set H with an n–ary hyperoperation $f : H^n \to P^*(H)$ will be called an n–ary hypergroupoid and will be denoted by (H, f). An n–ary hypergroupoid (H, f) will be called an n–ary semihypergroupoid if and only if the following associative axiom holds:*

$$f(x_1^{i-1}, f(x_i^{n+i-1}), x_{n+i}^{2n-1}) = f(x_1^{j-1}, f(x_j^{n+j-1}), x_{n+j}^{2n-1}) \qquad (1)$$

for every $i, j \in \{1, 2, \ldots, n\}$ and $x_1, x_2, \ldots, x_{2n-1} \in H$.

Definition 2.3. *An n–ary semihypergroup (H, f) in which the equation*

$$b \in f(a_1^{i-1}, x_i, a_{i+1}^n) \qquad (2)$$

has the solution $x_i \in H$ for every $a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n, b \in H$ and $1 \le i \le n$, is called an n–ary hypergroup.

Notice that [17] uses the names *n–semihypergroup* and *n–hypergroup* instead. With respect to Definition 2.3 also notice that in our paper, especially in Theorem 4.3, we make use of an equivalent definition of the hypergroup by means of generalization of the *reproductive axiom*. For details cf. p. 156 or [13], p. 167.

In the paper we also use generalizations of the concept of *identity*, *scalar identity*, *zero element* and *inverse*. The respective n–ary definitions are included in section 5 of the paper. Notice that in the binary context we use them in the following meaning.

Definition 2.4. *An element $e \in H$, where $(H, *)$ is a hyperstructure, is called an identity if for all $x \in H$ there holds $x * e \ni x \in e * x$. If for all $x \in H$ there* *holds* $x * e = \{x\} = e * x$, *then* $e \in H$ *is called a scalar identity. If* $(H, *)$ *is a hypergroup endowed with at least one identity, then an element* $a' \in H$ *is called an inverse of* $a \in H$ *if there is an identity* $e \in H$ *such that* $a * a' \ni e \in a' * a$. *An element* $0 \in H$ *is called a zero element of* H *if for all* $x \in H$ *there holds* $x * 0 = \{0\} = 0 * x$.

Notice that the zero element of Definition 2.4 is sometimes called *absorbing element* or *zero scalar element* or simply *zero scalar*. Study of elements with the above properties (usually when combined in hyperstructures with two (hyper)operations) is important especially in the context of various types of ring-like hyperstructures or hyperideals. For implications in the area of (binary) EL–hyperstructures cf. [20], for some implications in the theory of hyperideals (in n–ary context) cf. e.g. [2].

3 The binary construction and nature of its n–ary extension

The original construction, which we are going to extend, has first been presented in [4] in the following form.

Lemma 3.1. *([4], Theorem 1.3, p. 146) Let* $(S, \cdot, \leq)$ *be a partially ordered semigroup. Binary hyperoperation* $* : S \times S \to \mathcal{P}'(S)$ *defined by*

$$a * b = [a \cdot b)_{\leq} \tag{3}$$

is associative. The semi-hypergroup $(S, *)$ *is commutative if and only if the semigroup* $(S, \cdot)$ *is commutative.*

□

The hyperstructure $(S, *)$ constructed in this way is usually called the *associated* hyperstructure to the single-valued structure $(S, \cdot)$ or an *"Ends lemma"–based hyperstructure*, or an *EL–hyperstructure* for short. The carrier set is denoted by S if it is a semigroup or H if it is a group.

Lemma 3.2. *([4], Theorem 1.4, p. 147) Let* $(S, \cdot, \leq)$ *be a partially ordered semigroup. The following conditions are equivalent:*

1^0 *For any pair* $(a, b) \in S^2$ *there exists a pair* $(c, c') \in S^2$ *such that* $b \cdot c \leq a$ *and* $c' \cdot b \leq a$

2^0 *The associated semi-hypergroup* $(S, *)$ *is a hypergroup.*

□

Remark 3.3. *If $(S,\cdot,\le)$ is a partially ordered group, then if we take $c = b^{-1}\cdot a$ and $c' = a\cdot b^{-1}$, then condition 1^0 is valid. Therefore, if $(S,\cdot,\le)$ is a partially ordered group, then its associated hyperstructure is a hypergroup.*

Remark 3.4. *The wording of the above lemmas is the exact translation of lemmas from [4]. The respective proofs, however, do not change in any way, if we regard* quasi-ordered *structures instead of* partially ordered *ones as the anti-symmetry of the relation $\le$ is not needed (with the exception of the $\Leftarrow$ implication of the part on commutativity, which does not hold in this case). The often quoted version of the "Ends lemma" is therefore the version assuming quasi–ordered structures.*

Example 3.5. *Regard the set $(\mathbb{R},+,\le)$, i.e. the partially ordered group of real numbers. Obviously, $(R,*)$, where*

$$a*b = [a+b)_{\le} = \{x\in\mathbb{R}; a+b\le x\}$$

for arbitrary real numbers a,b, is a commutative hypergroup.

Example 3.6. *Regard the set $(\mathcal{P}^*(S),\cup,\subseteq)$ of all non-empty subsets of an arbitrary set S. Obviously, $(\mathcal{P}^*(S),\cup,\subseteq)$ is a partially ordered semigroup which is not a group and $(\mathcal{P}^*(S),*)$, where*

$$A*B = [A\cup B)_{\subseteq} = \{X\in\mathcal{P}^*(S); A\cup B\subseteq X\}$$

for arbitrary subsets A,B of S, is a commutative semihypergroup. One can prove that it is not a hypergroup. However, one can prove that by including $\emptyset$ we get a hypergroup.

In other words, EL–hyperstructures are hyperstructures of *arity* 2. It is thus natural to find out whether the construction can be extended to involve more than two elements.

Analogically to (3) we could define an n-ary hyperoperation $*:\underbrace{S\times\ldots\times S}_{n}\to\mathcal{P}^*(S)$ by

$$\underbrace{a_1*\ldots*a_n}_{n} = [\underbrace{a_1\cdot\ldots\cdot a_n}_{n})_{\le} = \{x\in S; \underbrace{a_1\cdot\ldots\cdot a_n}_{n}\le x\} \tag{4}$$

In a standard notation used e.g. by [13] or [17] this would be denoted as a hyperoperation $f: S^n\to\mathcal{P}^*(S)$ (or with H instead of S if we wanted to make use of the distinction semihypergroup vs. hypergroup) defined by

$$f(a_1^n) = [\underbrace{a_1\cdot\ldots\cdot a_n}_{n})_{\le} = \{x\in S; \underbrace{a_1\cdot\ldots\cdot a_n}_{n}\le x\}. \tag{5}$$

The hypergroupoid would be an *n-ary hypergroupoid* and would be denoted in the former case by $(S,*)$ and in the latter case by (S,f).*

However, first of all we need to establish meaning of the very basic concepts used in (4) or (5). The result of the hyperoperation $f(a_1^n)$ applied on elements $a_1,\ldots,a_n$, $n>2$ is the upper end of a *single* element $\underbrace{a_1\cdot\ldots\cdot a_n}_{n}\in S$. (In further text we call such an element as *generating* the upper end.) Yet *how* does one obtain this single element? In other words, *what is the arity of the single-valued operation* $\cdot$? In a general case, $\cdot$ may be a binary operation, an $n-ary$ operation, or a j–ary operation for some special j such that $2<j<n$.

In this paper we suppose that $\cdot$ is a binary operation, i.e. that the product $\underbrace{a_1\cdot\ldots\cdot a_n}_{n}$ is an *iterated binary operation.* This is usually defined in such a way that for $j\geq 1$, $n\geq j$ we denote by a_j^n a sequence of elements a_i, $j\leq i\leq n$ and for the single-valued binary operation s_f we define two new operations s_l^{it} and s_r^{it} in the following way:

$$s_l^{it}(a_1^n)=\begin{cases} a_1 & n=1\\ s_f(s_l^{it}(a_1^{n-1}),a_n) & n>1\end{cases}$$

and

$$s_r^{it}(a_1^n)=\begin{cases} a_1 & n=1\\ s_f(a_n,s_r^{it}(a_1^{n-1})) & n>1\end{cases}$$

Obviously, in a general case $s_l^{it}(a_1^n)\neq s_r^{it}(a_1^n)$. However, if the original binary operation s_f is associative, then the two newly defined operations s_l^{it} and s_r^{it} are equal and we may write s^{it} instead.

In the paper we will use the notation $\underbrace{a_1\cdot\ldots\cdot a_n}_{n}$ in the sense of $s^{it}(a_1^n)$. More precisely we should distinguish between $s_l^{it}(a_1^n)$ and $s_r^{it}(a_1^n)$ but this would be redundant because the construction we have been using and which we attempt to generalize, i.e. Lemma 3.1, assumes asociativity of the single-valued operation.

Remark 3.7. *Notice that the decision on nature of* $\underbrace{a_1\cdot\ldots\cdot a_n}_{n}$ *has a number of implications. If contrary to our assumption one decides to consider this element as a result of an n–ary operation, then all theorems must be adjusted to work with n–ary quasi-ordered (semi)groups. These, however, must first be properly defined. Thus, from a certain point of view, our decision on the nature*

*Further on we will use the standard notation, i.e. define the n–ary hyperoperation using analogies of (5). Analogies of notation (4) will be used only at places where the explicit reference to the binary hyperoperation $*$ makes the understanding more straightforward.

of $\underbrace{a_1 \cdot \ldots \cdot a_n}_{n}$ *is not only naturally following from the context but also easier and more convenient to work with. For details on iterated binary operations, cf. e.g. [18].*

Remark 3.8. *Just as we have considered the meaning of* $\underbrace{a_1 \cdot \ldots \cdot a_n}_{n}$ *and discussed whether it is a result of an* n*–ary or an iterated binary* single-valued operation $\cdot$, *we may discuss the meaning of the symbol* $\underbrace{a_1 * \ldots * a_n}_{n}$. *Again, in a general case it could stand for both an* n*–ary or an iterated binary* hyperoperation. *Yet as has been suggested above, in the case of the hyperoperation we choose the* n*–ary option.*

4 Associativity and commutativity

First, discuss the issue of associativity and commutativity in n–ary hyperstructures defined by (5).

Theorem 4.1. *Let* $(S, \cdot, \leq)$ *be a quasi-ordered semigroup.* n*–ary hyperoperation* $f : S^n \to \mathcal{P}^*(S)$ *defined by (5), i.e. as*

$$f(a_1^n) = [\underbrace{a_1 \cdot \ldots \cdot a_n}_{n})_{\leq} = \{x \in S; \underbrace{a_1 \cdot \ldots \cdot a_n}_{n} \leq x\}.$$

is associative. Furthermore, it is commutative if the semigroup $(S, \cdot)$ *is commutative.*

□

Proof. In order to prove associativity, we will modify the proof of [4], Lemma 1.6, p. 148, which shows that if we start with a partially ordered semigroup $(S, \cdot)$ there holds $a * (b * c) = (a * b) * c = [a \cdot b \cdot c)_{\leq}$.

First of all, suppose the following: $x, y, a_i \in S$, $i = 1, \ldots, n+1$, $x \leq y$ and that $(S, \cdot, \leq)$ is a partially ordered semigroup. This implies that $a_i \cdot x \leq a_i \cdot y$, $x \cdot a_i \leq y \cdot a_i$ and $[a_i \cdot y)_{\leq} \subseteq [a_i \cdot x)_{\leq}$, $[y \cdot a_i)_{\leq} \subseteq [x \cdot a_i)_{\leq}$ for $i = 1, \ldots, n$ (and the same for any product of any number of elements of S in position of a_i – if we keep their order).

Second, notice that obviously for all $x \in S$ such that $a_n \cdot a_{n+1} \leq x$ there is $[\underbrace{a_1 \cdot \ldots \cdot a_{n-1}}_{n-1} \cdot x)_{\leq} \subseteq [\underbrace{a_1 \cdot \ldots \cdot a_{n+1}}_{n+1})_{\leq}$. This is easy to verify because the fact that $y \in [\underbrace{a_1 \cdot \ldots \cdot a_{n-1}}_{n-1} \cdot x)_{\leq}$ is equivalent to the fact that $\underbrace{a_1 \cdot \ldots \cdot a_{n-1}}_{n-1} \cdot$

$x \leq y$. On the other hand, the fact that $a_n \cdot a_{n+1} \leq x$ is equivalent to $\underbrace{a_1 \cdot \ldots \cdot a_{n+1}}_{n+1} \leq \underbrace{a_1 \cdot \ldots \cdot a_{n-1}}_{n-1} \cdot x$, which due to transitivity of the relation $\leq$ means that $\underbrace{a_1 \cdot \ldots \cdot a_{n+1}}_{n+1} \leq y$, i.e. $y \in [\underbrace{a_1 \cdot \ldots \cdot a_{n+1}}_{n+1})_\leq$. Naturally, it is not important whether we multiply by x from left or right, i.e. there is also $[x \cdot \underbrace{a_3 \cdot \ldots \cdot a_{n+1}}_{n-1})_\leq \subseteq [\underbrace{a_1 \cdot \ldots \cdot a_{n+1}}_{n+1})_\leq$ for all $x \in S$ such that $a_1 \cdot a_2 \leq x$.

Then consider that the proof of Lemma 1.6 of [4] goes (using the above considerations for $n = 2$ and notation a, b, c instead of a_i) as follows:

$$a * (b * c) = \bigcup_{x \in b*c} a * x = \bigcup_{x \in [b \cdot c)_\leq} [a \cdot x)_\leq = [a \cdot b \cdot c)_\leq \cup \bigcup_{x > b \cdot c} [a \cdot x)_\leq = [a \cdot b \cdot c)_\leq$$

and similarly

$$(a * b) * c = \bigcup_{x \in [a \cdot b)_\leq} [x \cdot c)_\leq = [a \cdot b \cdot c)_\leq,$$

which combined means that $a * (b * c) = (a * b) * c = a * b * c$. This can be denoted as $f(a, f(b, c)) = f(f(a, b), c)$ or $f(a_1, f(a_2^3)) = f(f(a_1^2), a_3)$ using the notation (5) for any triple of elements of S.

Analogously we prove that $f(a_1, f(a_2^4)) = f(f(a_1^3), a_4) = f(a_1^4)$ for any quadruple of elements of S as well as $f(a_1, f(a_2^5)) = f(f(a_1^4), a_5) = f(a_1^5)$ for any quintuple of elements of S. Thus for arity $n = 3$ we have that

$$f(a_1^{i-1}, f(a_i^{i+2}), a_{i+3}^5) = f(a_1^{j-1}, f(a_j^{i+2}), a_{j+3}^5)$$

for all $i, j \in \{1, 2, 3\}$, which means that associativity in 3–ary EL–hypergroupoids (S, f) is secured. Obviously, this consideration can be repeated for any higher arity n.

Proving commutativity is rather simple: since the single-valued operation $\cdot$ is commutative and as has been shown above also associative, then all permutations $\underbrace{a_1 \cdot \ldots \cdot a_n}_{n}$ are equal. This means that all respective upper ends $[\underbrace{a_1 \cdot \ldots \cdot a_n}_{n})_\leq$ are equal because they are generated always by the same element. In other words, all permutations of the hyperoperation f are equal, i.e. the hyperoperation f is commutative. □

In [22] implications of the converse of Lemma 3.1 have been studied. The fact that commutativity of the binary hyperoperation implies commutativity of the single-valued operation is included already in Lemma 3.1. The same fact on binary associativity was proved in [22] as Theorem 3.1. Notice that in both cases, the relation $\leq$ must be *partial ordering*, i.e. not quasi-ordering only. This follows from the fact that the implication

$$[a)_{\leq} = [b)_{\leq} \Rightarrow a = b \tag{6}$$

is valid only on condition of antisymmetry of the relation $\leq$, and the respective proofs make use of (6). For a counterexample of (6) used in the binary context of Lemma 3.1 cf. e.g. [22], Example 3.15.

Let us now study the converse of Theorem 4.1.

Theorem 4.2. *Let $(S, \cdot)$ be a non-trivial groupoid and $\leq$ a partial ordering on S such that for an arbitrary pair of elements $(a, b) \in S^2$, $a \leq b$, and for an arbitrary $c \in S$ there holds $c \cdot a \leq c \cdot b$, $a \cdot c \leq b \cdot c$. Further define an n–ary hyperoperation f (also denoted by $*$) using notation (5) (or (4)).*

Then if the hyperoperation f (or $*$) is associative, then the single-valued operation $\cdot$ is associative too. Furthermore, if the hyperoperation f (or $*$) is commutative, then the single-valued operation $\cdot$ is commutative too.

□

Proof. The fact that the hyperoperation f (or $*$) is associative, means that all permutations $f(a_1^{i-1}, f(a_i^{n+i-1}), a_{n+i}^{2n-1})$ for an arbitrary $i \in \{1, 2, \ldots, n\}$ are equal, i.e. if an arbitrary element $x \in S$ belongs to one of the permutations $f(a_1^{i-1}, f(a_i^{n+i-1}), a_{n+i}^{2n-1})$, it belongs to all other ones.

Suppose an arbitrary $x \in f(a_1^{i-1}, f(a_i^{n+i-1}), a_{n+i}^{2n-1})$ for some $i \in \{1, 2, \ldots, n\}$, e.g. for $i = 1$. This means that $x \in f(f(a_1^n), a_{n+1}^{2n-1})$, i.e. using the $*$ notation, $x \in \underbrace{a_1 * \ldots * a_n}_{n} * \underbrace{a_{n+1} * \ldots * a_{2n-1}}_{n-1}$. This means that there exists an element $x_1 \in \underbrace{a_1 * \ldots * a_n}_{n}$ such that $x \in x_1 * \underbrace{a_{n+1} * \ldots * a_{2n-1}}_{n-1}$. In other words, for these elements there holds that $\underbrace{a_1 \cdot \ldots \cdot a_n}_{n} \leq x_1$ and $x_1 \cdot \underbrace{a_{n+1} \cdot \ldots \cdot a_{2n-1}}_{n-1} \leq x$. Thanks to the properties assumed in the theorem this – when combined – means that

$$(\underbrace{a_1 \cdot \ldots \cdot a_n}_{n}) \cdot (\underbrace{a_{n+1} \cdot \ldots \cdot a_{2n-1}}_{n-1}) \leq x_1 \cdot (\underbrace{a_{n+1} \cdot \ldots \cdot a_{2n-1}}_{n-1}) \leq x$$

and thanks to assumed transitivity of the relation $\leq$ we get that

$$x \in [(\underbrace{a_1 \cdot \ldots \cdot a_n}_{n}) \cdot (\underbrace{a_{n+1} \cdot \ldots \cdot a_{2n-1}}_{n-1}))_{\leq}. \tag{7}$$

Yet we could have started with any other permutation $f(a_1^{i-1}, f(a_i^{n+i-1}), a_{n+i}^{2n-1})$ and apply analogous reasoning on it. E.g. for $i = 2$ we have that $x \in a_1 * (\underbrace{a_2 * \ldots * a_{n+1}}_{n}) * \underbrace{a_{n+2} * \ldots * a_{2n-1}}_{n-1}$ and conclude that

$$x \in [a_1 \cdot (\underbrace{a_2 \cdot \ldots \cdot a_{n+1}}_{n}) \cdot (\underbrace{a_{n+2} \cdot \ldots \cdot a_{2n-1}}_{n-2}))_{\leq}, \tag{8}$$

and since $f(a_1^{i-1}, f(a_i^{n+i-1}), a_{n+i}^{2n-1})$ are equal for $i = 1$ and $i = 2$ (just as for any other $i \in \{1, 2, \ldots, n\}$) and we supposed an *arbitrary* element $x \in f(a_1^{i-1}, f(a_i^{n+i-1}), a_{n+i}^{2n-1})$, we get that the upper ends in (7) and (8) (just as any other upper end which results from using another i) are equal too.

Since we assume that the relation $\leq$ is antisymmetric, using implication (6) we get that also the elements generating the upper ends are equal. As a result, the single-valued operation $\cdot$ is associative.

Proving commutativity of the single-valued operation $\cdot$ is rather straightforward. If the hyperoperation f is commutative, then $f(a_1^n)$ is the same regardless of the permutation of elements $a_1, \ldots, a_n$. According to definition of the hyperoperation f marked as (5), this means that all upper ends $[\underbrace{a_1 \cdot \ldots \cdot a_n}_{n})_{\leq}$ are the same regardless of the permutation of elements $a_1, \ldots, a_n$. However, on condition of antisymmetry of the relation $\leq$, from (6) we immediately get that also $\underbrace{a_1 \cdot \ldots \cdot a_n}_{n}$ is the same regardless of the permutation of elements $a_1, \ldots, a_n$, which together with already proved associativity means that the single-valued operation $\cdot$ is commutative. □

Now we can proceed to conditions on which an n–ary EL–semihypergroup becomes an n–ary hypergroup. Recall that the concept of n–ary hypergroup may be defined in two equivalent ways: either as Definition 2.3 or by expanding the reproductive axiom, i.e. expanding validity of

$$x * H = H * x = H$$

for all $x \in H$, to the form

$$\underbrace{H * \ldots * H}_{i-1} * x * \underbrace{H * \ldots * H}_{n-i} = H \tag{9}$$

for all $x \in H$ and all $i = \{1, 2, \ldots, n\}$ using notation (4) or

$$f(H^{i-1}, x, H^{n-i}) = H \tag{10}$$

for all $x \in H$ and all $i = \{1, 2, \ldots, n\}$ using notation (5).

Since in the *Ends lemma* context obviously $f(H^{i-1}, x, H^{n-i}) \subseteq H$ for an arbitrary $i \in \{1, 2, \ldots, n\}$, we must concentrate on the other inclusion, i.e. secure that

$$H \subseteq \underbrace{H * \ldots * H}_{i-1} * x * \underbrace{H * \ldots * H}_{n-i}, \tag{11}$$

or $H \subseteq f(H^{i-1}, x, H^{n-i})$, for all $x \in H$ and $i = \{1, 2, \ldots, n\}$.

Theorem 4.3. *Let $(H, \cdot, \leq)$ be a quasi-ordered group. The n–ary EL–semihypergroup constructed using Theorem 4.1 is an n–ary hypergroup.*

□

Proof. As has been suggested above, we need to verify validity of inclusion (11). To do this, suppose an arbitrary element $h \in H$ and first of all suppose that we need to verify that $H \subseteq H * x$ or $H \subseteq x * H$. Obviously, $h \cdot x^{-1} \in H$ and $x^{-1} \cdot h \in H$. Thus we get that $h \cdot x^{-1} \cdot x = h \leq h$ (since $\leq$ is reflexive) and $x \cdot x^{-1} \cdot h = h \leq h$, i.e $h \in [(h \cdot x^{-1}) \cdot x)_{\leq} \subseteq \bigcup\limits_{g \in H} [g \cdot x)_{\leq} = H * x$ as well as $h \in x * H$.

Yet instead of $h \cdot x^{-1} \in H$ we may write $h \cdot h^{-1} \cdot h \cdot x^{-1} \in H * H = \bigcup\limits_{f \in H, g \in H} [f \cdot g)_{\leq}$ (and instead of $x^{-1} \cdot h \in H$ we may write $x^{-1} \cdot h \cdot h^{-1} \cdot h \in H * H$) and we can repeat this for any number of instances of H. □

Remark 4.4. *Securing the existence of elements, which in the proof of Theorem 4.3 provide that an arbitrary element $h \in H$ is in relation with the fixed $x \in H$, i.e. of elements $\underbrace{(h \cdot h^{-1}) \cdot \ldots \cdot (h \cdot h^{-1})}_{n-1} \cdot (h \cdot x^{-1})$ and $(x^{-1} \cdot h) \cdot \underbrace{(h^{-1} \cdot h) \cdot \ldots \cdot (h^{-1} \cdot h)}_{n-1}$, is not a problem in a group. However, in a semigroup, this is not straightforward. Notice that if such elements do exist for a given n, then the n–ary EL–semihypergroup is an n–ary hypergroup even if the underlying single-valued structure is a semigroup. As a special case of this we get the condition used in Lemma 3.2.*

5 Important elements

Papers dealing with various aspects of n–ary hypergroups such as [2, 13, 17] usually need to work with the n–ary generalization of the concept of *identity* element and concepts similar to it. Let us include the respective definitions as well – yet when actually using them we expand them from hypergroups to semihypergroups.

Definition 5.1. *([13], p. 168) Element e of an n–ary hypergroup (H, f) is called a neutral (identity) element if*

$$f(\underbrace{e, \ldots, e}_{i-1}, x, \underbrace{e, \ldots, e}_{n-i})$$

includes x for all $x \in H$ and all $1 \leq i \leq n$.

Regarding such elements (with the novelty of expanding the above definition onto semihypergroups) we might prove the following in the *Ends lemma* context.

Theorem 5.2. *Let (S, f) be an n–ary EL–semihypergroup associated to a quasi–ordered monoid $(S, \cdot, \leq)$ with the identity u. Then*

1. *If $e \in S$ is an identity of (S, f), then $\underbrace{e \cdot \ldots \cdot e}_{n-1} \leq u$.*
2. *If $e \leq u$ for some $e \in S$, then e is an identity of (S, f).*

□

Proof. In order to prove part 1 suppose that $e \in S$ is an identity of (S, f), i.e. that $x \in f(\underbrace{e, \ldots, e}_{i-1}, x, \underbrace{e, \ldots, e}_{n-i})$ for all $x \in H$ and all i such that $1 \leq i \leq n$. In the context of definition of the hyperoperation f – see (5) – the inclusion means that $x \in [\underbrace{e \cdot \ldots \cdot e}_{i-1}, x, \underbrace{e \cdot \ldots \cdot e}_{n-i})_{\leq}$, i.e. $\underbrace{e \cdot \ldots \cdot e}_{i-1} \cdot x \cdot \underbrace{e \cdot \ldots \cdot e}_{n-i} \leq x$. Since this holds for all $x \in S$, we may e.g. set $x = u$, where u is the identity of $(S, \cdot)$. And we get the statement.

As far as part 2 is concerned, suppose that $e \leq u$, where u is the identity of $(S, \cdot)$. Since $(S, \cdot, \leq)$ is a quasi-ordered monoid, we have that also $e \cdot x \leq u \cdot x = x$ and $e \cdot e \cdot x \leq e \cdot x$ for an arbitrary $x \in S$. From transitivity of the relation $\leq$ we get that $e \cdot e \cdot x \leq x$, i.e. $x \in [e \cdot e \cdot x)_{\leq} = f(e, e, x)$. But we could have also multiplied by x from the left and get $x \cdot e \leq x \cdot u = x$. Then from $e \cdot x \leq x$ we get that $e \cdot x \cdot e \leq x \cdot e$ and from transitivity we get that $e \cdot x \cdot e \leq x$, i.e. $x \in [e \cdot x \cdot e)_{\leq}$, i.e. $x \in f(e, x, e)$. Finally, from $x \cdot e \leq x$ and $x \cdot e \cdot e \leq x \cdot e$ we get that $x \in f(x, e, e)$, which completes the proof for arity $n = 3$. In order to prove the statement for higher arities we may obviously use the same strategies. □

Remark 5.3. *Notice that for arity $n = 2$ Theorem 5.2 turns into equivalence stating that $e \in S$ is an identity of (S, f) if and only if $e \leq u$, which has already been included in [19] as Theorem 3.4. Further notice that we obtain the same result for idempotent $\cdot$ and $n > 2$.*

Corollary 5.4. *If in Theorem 5.2 $(S, \cdot, \leq)$ is a quasi-ordered group, then if $e \in S$ is an identity of (S, f), then also* $\underbrace{e \cdot \ldots \cdot e}_{n-1} \leq \underbrace{e^{-1} \cdot \ldots \cdot e^{-1}}_{n-1}$.

□

Proof. We continue the proof of part 1 of Theorem 5.2. By $n-1$ times repeated multiplication by e^{-1} we get that $u \leq \underbrace{e^{-1} \cdot \ldots \cdot e^{-1}}_{n-1}$ and thanks to transitivity of the relation $\leq$ we get the statement. □

Corollary 5.5. *The identity u of $(S, \cdot)$ is an identity of its associated n–ary EL–semihypergroup (S, f).*

Proof. Obvious. □

Example 5.6. *If we regard the hypergroup $(\mathbb{R}, f)$, where*

$$f(a_1^n) = [\underbrace{a_1 + \ldots + a_n}_{n})_{\leq} = \{x \in \mathbb{R}; \underbrace{a_1 + \ldots + a_n}_{n} \leq x\}$$

for arbitrary real numbers $a_1, \ldots, a_n$, we get that 0 and all negative numbers are all identities of this hypergroup. Also, obviously, $\underbrace{x + \ldots + x}_{n-1} \leq 0$ for both 0 and an arbitrary negative x.

Example 5.7. *If we regard the set $(\mathcal{P}(S), f)$ (with $\emptyset$ included), where*

$$f(A_1^n) = [\underbrace{A_1 \cup \ldots \cup A_n}_{n})_{\subseteq} = \{X \in \mathcal{P}^*(S); \underbrace{A_1 \cup \ldots \cup A_n}_{n} \subseteq X\}$$

we get that this hypergroup has the only identity $\emptyset$.

Scalar neutral elements (or *scalar identities*) are such elements, where the inclusion in Definition 5.1 is substituted by equality.

Definition 5.8. *([2], p. 380) Element e of an n–ary hypergroup (H, f) is called a scalar neutral element if*

$$\{x\} = f(\underbrace{e, \ldots, e}_{i-1}, x, \underbrace{e, \ldots, e}_{n-i}) \tag{12}$$

for every $1 \leq i \leq n$ and for every $x \in H$.

Remark 5.9. *Notice that in [2] a slightly different notation is used: instead of* $f(\underbrace{e,\dots,e}_{i-1},x,\underbrace{e,\dots,e}_{n-i})$ *the authors write* $f(e^{(i-1)},x,e^{(n-i)})$. *Also notice that sometimes, e.g. [13], p. 168, the concept of a more general term* scalar *is used when defining that* the element $a\in H$ is called a scalar if $|f(x_1^i,a,x_{i+2}^n)|=1$ for all $x_1,\dots,x_i,x_{i+2},\dots,x_n\in H$, *i.e. defining that* $f(\underbrace{e,\dots,e}_{i-1},x,\underbrace{e,\dots,e}_{n-i})$ *must be a one-element set, not neccessarily the set* $\{x\}$ *as in the case of scalar neutral element.*

As has been done with Theorem 5.2, let us now permit a more general case of scalar neutral elements in semihypergroups. To be consistent in naming concepts we prefer the name *scalar identity* to *scalar neutral element* further on.

Theorem 5.10. *Let* $(S,\cdot,\le)$ *be a non-trivial quasi-ordered semigroup and* (S,f) *an* n*–ary EL–semihypergroup associated to it. If* $e\in S$ *is a scalar identity of* (S,f)*, then*

$$x=\underbrace{e\cdot\ldots\cdot e}_{i-1}\cdot x\cdot\underbrace{e\cdot\ldots\cdot e}_{n-i} \tag{13}$$

for all $x\in S$ *and all* $1\le i\le n$.

□

Proof. Suppose that in (S,f) there exists a scalar neutral identity e. This means that for every $x\in S$ and every i such that $1\le i\le n$ there is

$$\{x\}=f(\underbrace{e,\dots,e}_{i-1},x,\underbrace{e,\dots,e}_{n-i}).$$

Yet thanks to the definition of the hyperoperation f this means that

$$\{x\}=[\underbrace{e\cdot\ldots\cdot e}_{i-1}\cdot x\cdot\underbrace{e\cdot\ldots\cdot e}_{n-i})_\le.$$

Since $\le$ is reflexive, there is

$$\underbrace{e\cdot\ldots\cdot e}_{i-1}\cdot x\cdot\underbrace{e\cdot\ldots\cdot e}_{n-i}\in[\underbrace{e\cdot\ldots\cdot e}_{i-1}\cdot x\cdot\underbrace{e\cdot\ldots\cdot e}_{n-i})_\le,$$

which means that $x=\underbrace{e\cdot\ldots\cdot e}_{i-1}\cdot x\cdot\underbrace{e\cdot\ldots\cdot e}_{n-i}$ for all $x\in S$ and all i such that $1\le i\le n$. □

Remark 5.11. *Obviously, if for some $x \in S$ or some $i \in \{1, \dots, n\}$ condition (13) does not hold, then $e \in S$ is not a scalar identity of (S, f). This equivalent condition might be a better tool for finding scalar identities than the Theorem itself.*

Corollary 5.12. *The identity u of a quasi-ordered semigroup $(S, \cdot, \leq)$ is a scalar identity of (S, f) associated to $(S, \cdot, \leq)$ if and only if $\leq$ is the identity relation.*

□

Proof. By definition

$$f(\underbrace{u, \dots, u}_{i-1}, x, \underbrace{u, \dots, u}_{n-i}) = [\underbrace{u \cdot \dots \cdot u}_{i-1} \cdot x \cdot \underbrace{u \cdot \dots \cdot u}_{n-i})_{\leq} = [x)_{\leq}.$$

This is equal to $\{x\}$ for reflexive $\leq$ and all $x \in S$ if and only if $\leq$ is the identity relation. □

Remark 5.13. *Notice that for arity $n = 2$ condition (13) turns into $x = e \cdot x = x \cdot e$ for all $x \in S$ which is possible only for $e = u$, where u is the identity of $(S, \cdot)$. And we immediately conclude that $\leq$ must be the identity relation. As a result, there do not exist any non-trivial canonical hyperstructures constructed using Lemma 3.1.*

Example 5.14. *If we regard the hypergroup $(\mathbb{R}, f)$ from Example 5.6, we see that condition (13) can hold for $e = 0$ only, which means that $(\mathbb{R}, *)$ does not have a scalar identity.*

Apart from identities and scalar identities we might consider *zero elements* (or *absorbing elements*) of n–ary hyperstructures.

Definition 5.15. *([2], p. 380) Element 0 of an n–ary hypergroup (H, f) is called a zero element if*

$$\{0\} = f(\underbrace{x_1, \dots, x_{i-1}}_{i-1}, 0, \underbrace{x_{i+1}, \dots, x_n}_{n-i}) \tag{14}$$

for every $1 \leq i \leq n$ and for every $(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \in H^{n-1}$.

Obviously, the zero element is unique. The following Theorem might be used to detect it. We see that only maximal elements of $(S, \leq)$ can be zero elements. As in the case of identities and scalar identities of (S, f) we might again expand the definition onto semihypergroups.

Theorem 5.16. *Let $(S, \cdot, \leq)$ be a non-trivial quasi-ordered semigroup and (S, f) an n–ary EL–semihypergroup associated to it. If 0 is the zero element of (S, f), then 0 is the maximal element of $(S, \leq)$.*

□

Proof. From (14) in the definition of the zero element and from the definition of the hyperoperation f we get that

$$[\underbrace{x_1 \cdot \ldots \cdot x_{i-1}}_{i-1} \cdot 0 \cdot \underbrace{x_{i+1} \cdot \ldots \cdot x_n}_{n-i})_{\leq} = \{0\} \tag{15}$$

for every i such that $1 \leq i \leq n$ and for every $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \in S^{n-1}$. Since the relation $\leq$ is reflexive, there is

$$\underbrace{x_1 \cdot \ldots \cdot x_{i-1}}_{i-1} \cdot 0 \cdot \underbrace{x_{i+1} \cdot \ldots \cdot x_n}_{n-i} \in [\underbrace{x_1 \cdot \ldots \cdot x_{i-1}}_{i-1} \cdot 0 \cdot \underbrace{x_{i+1} \cdot \ldots \cdot x_n}_{n-i})_{\leq},$$

which combined with (15) means that for a zero element 0 there must be $\underbrace{x_1 \cdot \ldots \cdot x_{i-1}}_{i-1} \cdot 0 \cdot \underbrace{x_{i+1} \cdot \ldots \cdot x_n}_{n-i} = 0$ for every i such that $1 \leq i \leq n$ and for every $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \in S^{n-1}$. Yet if this holds, (15) reduces to $[0)_{\leq} = \{0\}$, which means that 0 is the maximal element of the relation $\leq$. □

Example 5.17. *Since there are no maximal elements in $(\mathbb{R}, +, \leq)$ there are no zero elements in $(\mathbb{R}, f)$ from Example 5.6.*

Example 5.18. *If we want to describe zero elements in $(\mathcal{P}(S), f)$ from Example 5.7, we must concentrate on the only maximal element of $(\mathcal{P}(S), \cup, \subseteq)$, i.e. on $\mathcal{P}(S)$ itself. We easily verify that it is a zero element of $(\mathcal{P}(S), f)$.*

Inverse elements in n–ary hyperstructures are studied e.g. in [2]. The property of *having a unique inverse element* required in [2] is taken over from the definition of *canonical n–ary* hypergroup included in [15]. Notice that canonical n–ary hypergroups are a special class of *commutative* n–ary hyperstructures (moreover, with the unique identity e having a certain further property), i.e. the definition of inverse elements included in [2], which has been taken over from [15], must be adjusted to a more general case.

In the following text the notation $\text{perm}\{a_1, \ldots, a_n\}$ stands for the set of all permutations of elements $a_1, \ldots, a_n$.

Definition 5.19. *Element x' of an n–ary hypergroup (H, f) is called an inverse element to $x \in H$ if there exists an identity $e \in H$ such that*

$$e \in f(\text{perm}\{x, x', \underbrace{e, \ldots, e}_{n-2}\}) \tag{16}$$

for every $1 \leq i \leq n$.

Theorem 5.20. *Let* (H, f) *be an* n*–ary* EL*–hypergroup associated to a quasi-ordered group* $(H, \cdot, \leq)$*. For an arbitrary* $x \in H$ *there holds*

1. *if* $x' \leq x^{-1}$*, then* x' *is an inverse of* x *in* (H, f)*,*
2. *if* x' *is an inverse of* x *in* (H, f)*, then* $a \leq x^{-1}$ *for all* $a \in \text{perm}\{x' \cdot \underbrace{e \cdot \ldots \cdot e}_{2(n-2)}\}$*,*

where x^{-1} *denotes the inverse of* $x \in H$ *in* $(H, \cdot)$ *and* e *is some (unspecified) identity of* (H, f)*.*

□

Proof. Suppose that $x \in H$, $x' \in H$ are arbitrary and denote by the upper index $^{-1}$ the inverse in $(H, \cdot)$. Finally, denote by u the identity of $(H, \cdot)$. Throughout the proof recall (5) on page 151 for the definition of the hyperoperation f using the single-valued operation $\cdot$ and the relation $\leq$.

ad 1: If $x' \leq x^{-1}$, then also $x' \cdot x \leq x^{-1} \cdot x = u$ and $x \cdot x' \leq x \cdot x^{-1} = u$. Moreover, we can multiply by the element u any number of times, or "insert" it anywhere "in between" x and x' or x' and x on the left side. Since according to Corollary 5.5 u is an identity of (H, f), we have that x' is an inverse of x.

ad 2: Suppose that x' is an inverse of x in (H, f). This means that there exists an identity $e \in H$ such that (16) holds. This means that

$$\underbrace{x \cdot x' \cdot e \cdot \ldots \cdot e}_{\text{arbitrary permutation of } n \text{ elements}} \leq e$$

When we multiply this by $\underbrace{e \cdot \ldots \cdot e}_{n-2}$, we get

$$\underbrace{x \cdot x' \cdot e \cdot \ldots \cdot e}_{\text{arbitrary permutation of } x, x' \text{ and } 2(n-2) \text{ instances of } e} \leq \underbrace{e \cdot \ldots \cdot e}_{n-1}.$$

However, from Theorem 5.2 and transitivity of the relation $\leq$ we get that

$$\underbrace{x \cdot x' \cdot e \cdot \ldots \cdot e}_{\text{arbitrary permutation of } x, x' \text{ and } 2(n-2) \text{ instances of } e} \leq u$$

which is equivalent to

$$\underbrace{x' \cdot e \cdot \ldots \cdot e}_{\text{arbitrary permutation of } x' \text{ and } 2(n-2) \text{ instances of } e} \leq x^{-1}.$$

It can be easily verified that commutativity / non-commutativity of the single-valued operation $\cdot$ is not relevant in the last step.

□

Remark 5.21. *Notice that for arity $n = 2$ there is $2(n-2) = 0$, i.e. Theorem 5.20 turns into an equivalence which enables us to describe the set of all inverses of an arbitrary $x \in H$ (denoted as $i(x)$) in a far more elegant way by*

$$i(x) = (x^{-1}]_{\leq} = \{x' \in G; x' \leq x^{-1}\}, \tag{17}$$

which has already been shown as [19], Theorem 3.9.

Example 5.22. *If we regard the hypergroup $(\mathbb{R}, f)$ from Example 5.6, we see that all $a \in \mathbb{R}$ such that $a \leq -x$ are inverses of an arbitrary real number x in (R, f). We also see that we might set $e = 0$ and Theorem 5.20 turns into equivalence.*

6 A more complex example

The hyperstructures $(\mathbb{R}, f)$ and $(\mathcal{P}(S), f)$ used to demonstrate the use of the above obtained results are quite simple and straightforward ones. Let us therefore conclude with a more complex example.

Example 6.1. *In paper [6] the authors deal with the relation of hyperstructures and homogeneous second order linear differential equations*

$$y'' + p(x)y' + q(x)y = 0, \tag{18}$$

such that $p \in C_+(I)$, $q \in C(I)$, where $C^k(I)$ denotes the commutative ring of all continuous real functions of one variable defined on an open interval I of reals with continuous derivatives up to order $k \geq 0$ (instead of $C^0(I)$ the authors write only $C(I)$), and $C_+(I)$ denotes its subsemiring of all positive continuous functions. They denote the set of nonsingular ordinary differential equations (18) by $\mathbb{A}_2$, the pair of functions p, q by $[p, q]$, $D = \frac{\mathrm{d}}{\mathrm{d}x}$ and the identity operator by Id. The notation $L(p, q)$ is reserved for the differential operator $L(p, q) = D^2 + p(x)D + q(x)Id$, i.e. the notation $L(p, q)(y) = 0$ is the equation (18). The set

$$\mathbb{L}\mathbb{A}_2(I) = \{L(p, q) : C^2(I) \to C(I); [p, q] \in C_+(I) \times C(I)\} \tag{19}$$

is the set of all such operators. Finally for an arbitrary $r \in \mathbb{R}$ the notation $\chi_r : I \to \mathbb{R}$ stands for the constant function with value r.

Proposition 1 of [6] states that if we define multiplication of operators by

$$L(p_1, q_1) \cdot L(p_2, q_2) = L(p_1 p_2, p_1 q_2 + q_1) \tag{20}$$

and if we define that $L(p_1, q_1) \leq L(p_2, q_2)$ if

$$p_1(x) = p_2(x),\ q_1(x) \leq q_2(x) \text{ for all } x \in I, \tag{21}$$

then $(\mathbb{LA}_2(I), \cdot, \leq)$ is a noncommutative partially ordered group with the unit element (identity) $L(\chi_1, \chi_0)$. Using Lemma 3.1 and a further proof included in [6] we get that if we put

$$\begin{array}{c} L(p_1, q_1) * L(p_2, q_2) = \\ = \{L(p, q) \in \mathbb{LA}_2(I); L(p_1, q_1) \cdot L(p_2, q_2) \leq L(p, q)\} = \\ = \{L(p_1 p_2, q); q \in C(I), p_1 q_2 + q_1 \leq q\} \end{array}, \tag{22}$$

then $(\mathbb{LA}_2(I), *)$ is a (transposition) hypergroup ([6], Theorem 3).[†]

Expand now the binary hyperoperation $*$ defined in (22) for arity $n = 3$ and suppose the 3–ary hypergroupoid $(\mathbb{LA}_2(I), f)$, where

$$f\left(L(p_1, q_1), L(p_2, q_2), L(p_3, q_3)\right) = [L(p_1, q_1) \cdot L(p_2, q_2) \cdot L(p_3, q_3))_{\leq}, \tag{23}$$

for arbitrary operators, where $\cdot$ is defined as (20) and $\leq$ is defined as (21).

According to Theorem 4.1 and Theorem 4.3, $(\mathbb{LA}_2(I), f)$ is a noncommutative 3–ary hypergroup. According to Theorem 5.2, all operators $L(p, q)$ such that $p \equiv 1$, $q(x) \leq 0$ for all $x \in I$, are identities of $(\mathbb{LA}_2(I), f)$ and one can easily verify that also part 1 of the Theorem holds.

In order to describe scalar identities of $(\mathbb{LA}_2(I), f)$, Theorem 5.10 states that we have to examine operators $L(a, b)$ such that for an arbitrary operator $L(r, s) \in \mathbb{LA}_2(I)$ there simultaneously holds

$$\begin{aligned} L(r, s) &= L(a, b) \cdot L(r, s) \cdot L(a, b) \\ L(r, s) &= L(r, s) \cdot L(a, b) \cdot L(a, b) \\ L(r, s) &= L(a, b) \cdot L(a, b) \cdot L(r, s) \end{aligned}$$

If the operator $L(a, b)$ does not have this property, then it is not a scalar identity. Yet since the result of the twice repeated multiplication in (23) is

[†]Notice that if we do not restrict our considerations to positive continuous functions p and suppose that $p(x) \neq 0$ for all $x \in I$, then we for sure know only that $(\mathbb{LA}_2(I), *)$ is a semihypergroup. However, it can be shown that even in this case it is a hypergroup.

$L(p_1, p_2p_3, p_1p_2q_3 + p_1q_2 + q_1)$, we have that the above conditions in fact mean that

$$\begin{aligned} L(r,s) &= L(a^2r, arb + as + b) \\ L(r,s) &= L(a^2r, rab + rb + s) \\ L(r,s) &= L(a^2r, a^2s + ab + b) \end{aligned}$$

which obviously holds for $a \equiv 1$, $b \equiv 0$ only. Thus by Corollary 5.12 we get that there are no scalar identities in $(\mathbb{LA}_2(I), f)$.

Theorem 5.16 states that maximal elements of $(\mathbb{LA}_2(I), \leq)$ are the only potential zero elements of $(\mathbb{LA}_2(I), f)$. However, no such elements exist in $(\mathbb{LA}_2(I), \leq)$, i.e. there are no zero elements in $(\mathbb{LA}_2(I), f)$.

As far as inverse elements of $(\mathbb{LA}_2(I), f)$ are concerned, the operator $L(\frac{1}{p}, -\frac{q}{p})$ is the single-valued inverse of $L(p,q) \in \mathbb{LA}_2(I)$. Thus according to Theorem 5.20 all operators $L(r,s) \in \mathbb{LA}_2(I)$, where $r(x) = \frac{1}{p(x)}$, $s(x) \leq -\frac{q(x)}{p(x)}$ for all $x \in I$ are inverses of an arbitrary operator $L(p,q)$ in $(\mathbb{LA}_2(I), f)$.

7 Conclusion

This paper has contributed to the study of n–ary hyperstructures started only recently by [13, 17] and especially to the development of the theoretical background of hyperstructures constructed from quasi– or partially ordered semigroups, i.e. to one of classical areas in the hyperstructure theory. Some particular results obtained earlier in papers such as e.g. [19, 21, 22] can now be regarded as special cases of results obtained for n–ary hyperstructures in this paper. Thanks to this, some results included in e.g. [3, 4, 5, 14] may be studied or described more easily or from a different perspective.

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Michal NOVÁK,
Department of Mathematics,
Faculty of Electrical Engineering and Communication,
Brno University of Technology,
Technická 8, 616 00 Brno, Czech Republic.
Email: novakm@feec.vutbr.cz