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Extremal problems on the generalized (n, d)-equiangular system of points

Abstract

The paper of Lavrent'ev [1] was the beginning of geometrical theory of functions of the complex variable. He solved a problem on the product of conformal radiuses of two non-overlapping domains. In many papers (see [2] – [13]) the Lavrent'ev's result are generalized. In this paper are obtained the new results of this direction.

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Let $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ be the sets of natural, real and complex numbers respectively. We define $\overline{\mathbb{C}} := \mathbb{C} \bigcup \{\infty\}$ and $\mathbb{R}^+ := (0, \infty)$.

Let $n, m, d \in \mathbb{N}$ such that $m = nd$. Consider the set of natural numbers $\{m_k\}_{k=1}^{n}$ such that

$$\sum_{k=1}^{n} m_k = m. \tag{1}$$

The following system of points

$$A_{n,d} := \left\{ a_{k,p} \in \mathbb{C} : \; k = \overline{1, n}, \; p = \overline{1, m_k} \right\},$$

are called the generalized (n, d)-equiangular system of points on the rays, if the condition (1) is fulfilled and if for all $k = \overline{1, n}$, $p = \overline{1, m_k}$ the following relations are true:

$$\begin{array}{l} 0 < |a_{k,1}| < \ldots < |a_{k,m_k}| < \infty; \\ \arg a_{k,1} = \arg a_{k,2} = \ldots = \arg a_{k,m_k} = \frac{2\pi}{n}(k-1). \end{array} \tag{2}$$

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An arbitrary generalized (n,d)-equiangular system of points (with variable number of points on the rays $A_{n,d}$) formed the set of domains $\{P_k\}_{k=1}^{n}$, where

$$P_k := \left\{ w \in \mathbb{C}\backslash\{0\} : \frac{2\pi}{n}(k-1) < \arg w < \frac{2\pi}{n}k \right\},\ k = \overline{1,n}.$$

For an arbitrary generalized (n,d)-equiangular system of points (with the variable amount of points on the rays $A_{n,d}$) we consider the following "managing" functional

$$\mu\left(A_{n,d}\right) := \prod_{k=1}^{n}\prod_{p=1}^{m_k}\left[\chi\left(\left|a_{k,p}\right|^{\frac{n}{2}}\right)\left|a_{k,p}\right|\right],$$

where $\chi(t) = \frac{1}{2}\left(t + t^{-1}\right)$.

Let $\{B_0, B_{k,p}\}$, $\{B_{k,p}, B_\infty\}$ be the arbitrary non-overlapping domains such that

$$0 \in B_0,\ a_{k,p} \in B_{k,p},\ \infty \in B_\infty,\ B_0, B_{k,p}, B_\infty \subset \overline{\mathbb{C}}, \quad k = \overline{1,n},\ p = \overline{1,m_k}.$$

Let $D \subset \overline{\mathbb{C}}$ be an arbitrary open set and $w = a \in D$. Then $D(a)$ is a connected component of D which contain the point a. For an arbitrary system of points $A_{n,d}$ and for open set D, $A_{n,d} \subset D$ we define $D_k(a_{l,p})$ as the connected component of the set for which $D(a_{l,p})\bigcap\overline{P_k}$ contain the point $a_{l,p}$ for $k = \overline{1,n}$, $l = k, k+1$, $p = \overline{1,m_l}$, where $m_{n+1} = m_1$, $a_{n+1,p} := a_{1,p}$. We have that $D_k(0)$ (respectively $D_k(\infty)$) define the connected component of the set $D(0)\bigcap\overline{P_k}$ (respectively $D(\infty)\bigcap\overline{P_k}$) which contain the point $w = 0$ (respectively $w = \infty$).

An open set D with $\{0\}\cup A_{n,d} \subset D$ satisfies the non-overlapping conditions with respect to the system of points $\{0\}\cup A_{n,d}$ if satisfies the condition:

$$\left[D_k(a_{s,p})\bigcap D_k(a_{l,q})\right]\bigcup\left[D_k(0)\bigcap D_k(a_{l,q})\right] = \varnothing, \tag{3}$$

$$k = \overline{1,n}, \quad l,s = k,k+1, \quad p = \overline{1,m_s}, \quad q = \overline{1,m_l}$$

for all corners $\overline{P_k}$.

An open set D with $\{\infty\}\cup A_{n,d} \subset D$ satisfies the non-overlapping conditions with respect to the system of points $\{\infty\}\cup A_{n,d}$ if satisfies the condition:

$$\left[D_k(a_{s,p})\bigcap D_k(a_{l,q})\right]\bigcup\left[D_k(\infty)\bigcap D_k(a_{l,q})\right] = \varnothing, \tag{4}$$

$$k = \overline{1,n}, \quad l,s = k,k+1, \quad p = \overline{1,m_s}, \quad q = \overline{1,m_l}$$

for all corners $\overline{P_k}$.

The system of domains $\{B_0\cup B_{k,p}\}$ ($\{B_\infty\cup B_{k,p}\}$), $k=\overline{1,n}$, $p=\overline{1,m_k}$, are defined as system of partially non-overlapping domains if $D:=\bigcup\limits_{k=1}^{n}\bigcup\limits_{p=1}^{m_k}B_{k,p}\cup B_0$ ($D:=\bigcup\limits_{k=1}^{n}\bigcup\limits_{p=1}^{m_k}B_{k,p}\cup B_\infty$) is an open set and if it satisfies the condition (3) (condition (4)).

The definition of inner radius $r(B;a)$ of domain $B\subset\overline{\mathbb{C}}$ with respect to a point $a\in B$ can be found in the papers [4 – 6].

For an arbitrary $n,d\in\mathbb{N}$, $n\geq 2$ we denote by $A^{(1)}_{n,d}$ the generalized (n,d)-equiangular system of points which is formed by poles of the quadratic differential $Q_1(w)dw^2$, where

$$Q_1(w)dw^2:=-\frac{w^{n-2}(1+w^n)^{2d-1}}{\left[(1-iw^{\frac{n}{2}})^{2d+1}-(1+iw^{\frac{n}{2}})^{2d+1}\right]^2}\,dw^2. \tag{5}$$

We denote by $A^{(2)}_{n,d}$ the generalized (n,d)-equiangular system of points which is formed by poles of the quadratic differential $Q_2(w)dw^2$, where

$$Q_2(w)dw^2:=\frac{w^{n-2}(1+w^n)^{2d-1}}{\left[(1-iw^{\frac{n}{2}})^{2d+1}+(1+iw^{\frac{n}{2}})^{2d+1}\right]^2}\,dw^2. \tag{6}$$

We remark that the condition (2) is satisfied for the system of points $A^{(1)}_{n,d}$, $A^{(2)}_{n,d}$ when $m_k=d$, $k=\overline{1,n}$. This statement easy follows from the general theory of quadratic differentials [16].

In this paper we investigate the following problem.

Problem. Let $n,m,d\in\mathbb{N}$, $m=nd$, $n\geq 2$. We intend to find a maximum of the functional I_n and to describe all its extremals, if

$$I_n:=r^{\frac{n^2}{4}}(D,0)\cdot\prod_{k=1}^{n}\prod_{p=1}^{m_k}r(D,a_{k,p}),$$

where $A_{n,d}=\{a_{k,p}\}$ is an arbitrary generalized (n,d)-equiangular system of points satisfying relation (2) and D is an arbitrary open set satisfying condition (3).

We remark that this problem is more general with respect to the conditions which are considered in [8 – 13].

Theorem 1. *Let $n,m,d\in\mathbb{N}$, $m=nd$, $n\geq 2$ and let $A_{n,d}=\{a_{k,p}\}$, $\mu(A_{n,d})=\mu(A^{(1)}_{n,d})$ be an arbitrary generalized (n,d)-equiangular system of points; the set of numbers $\{m_k\}_{k=1}^{n}$ satisfies the condition (1); an arbitrary* *open set D, $A_{n,d} \subset D \subset \overline{\mathbb{C}}$ satisfies the non-overlapping conditions with respect to the system of points $A_{n,d}$. Then we have the inequality*

$$r^{\frac{n^2}{4}}(D,0) \cdot \prod_{k=1}^{n} \prod_{p=1}^{m_k} r(D, a_{k,p}) \leq \left(\frac{8}{2m+n}\right)^m \cdot \left(\frac{2n}{2m+n}\right)^{\frac{n}{2}} \cdot \mu.$$

The equality sign holds, if the open set $D = \bigcup_{k=1}^{n} \bigcup_{s=1}^{m_k} B_{k,s}$, where $B_{k,s}$ is the system of circular domains of the quadratic differential (5).

Corollary 1. *Let $n, m, d \in \mathbb{N}$, $m = nd$, $n \geq 2$ and let $A_{n,d} = \{a_{k,p}\}$, $\mu(A_{n,d}) = \mu(A^{(1)}_{n,d})$ be an arbitrary generalized (n,d)-equiangular system of points; the set of numbers $\{m_k\}_{k=1}^{n}$ satisfies the condition (1). Let also $\{B_{k,p}\}$, $a_{k,p} \in B_{k,p} \subset \overline{\mathbb{C}}$ be an arbitrary set of non-overlapping domains. Then we have the inequality*

$$r^{\frac{n^2}{4}}(B_0,0) \cdot \prod_{k=1}^{n} \prod_{p=1}^{m_k} r(B_{k,p}, a_{k,p}) \leq \left(\frac{8}{2m+n}\right)^m \cdot \left(\frac{2n}{2m+n}\right)^{\frac{n}{2}} \cdot \mu.$$

The equality sign holds, if the points $a_{k,p}$ and domains $B_{k,p}$ are the poles and the circular domains of the quadratic differential (5).

Corollary 2. *Let $n, m, d \in \mathbb{N}$, $m = nd$, $n \geq 2$ and let $A_{n,d} = \{a_{k,p}\}$, $\mu(A_{n,d}) = \mu(A^{(1)}_{n,d})$ be an arbitrary generalized (n,d)-equiangular system of points; the set of numbers $\{m_k\}_{k=1}^{n}$ satisfies the condition (1). Let also $\{B_{k,p}\}$, $a_{k,p} \in B_{k,p} \subset \overline{\mathbb{C}}$ be an arbitrary set of partially non-overlapping domains. Then the inequality of Corollary 1 is true.*

Theorem 2. *Let $n, m, d \in \mathbb{N}$, $m = nd$, $n \geq 2$ and let $A_{n,d} = \{a_{k,p}\}$, $\mu(A_{n,d}) = \mu(A^{(2)}_{n,d})$ be an arbitrary generalized (n,d)-equiangular system of points; the set of numbers $\{m_k\}_{k=1}^{n}$ satisfies condition (1); an arbitrary open set D, $A_{n,d} \subset D \subset \overline{\mathbb{C}}$ satisfies the non-overlapping conditions with respect to the system of points $A_{n,d}$. Then we have the inequality*

$$r^{\frac{n^2}{4}}(D,\infty) \cdot \prod_{k=1}^{n} \prod_{p=1}^{m_k} r(D, a_{k,p}) \leq \left(\frac{8}{2m+n}\right)^m \cdot \left(\frac{2n}{2m+n}\right)^{\frac{n}{2}} \cdot \mu.$$

The equality sign holds, if the open set $D = \bigcup_{k=1}^{n} \bigcup_{s=1}^{m_k} B_{k,s}$, where $B_{k,s}$ is the system of circular domains of the quadratic differential (6).

Corollary 3. *Let $n, m, d \in \mathbb{N}$, $m = nd$, $n \geq 2$ and let $A_{n,d} = \{a_{k,p}\}$, $\mu(A_{n,d}) = \mu(A^{(2)}_{n,d})$ be an arbitrary generalized (n,d)-equiangular system of points; the set of numbers $\{m_k\}_{k=1}^{n}$ satisfies condition (1). Let also $\{B_{k,p}\}$,*

$a_{k,p} \in B_{k,p} \subset \overline{\mathbb{C}}$ be an arbitrary set of non-overlapping domains. Then we have the inequality

$$r^{\frac{n^2}{4}}(B_\infty,\infty)\cdot\prod_{k=1}^{n}\prod_{p=1}^{m_k} r(B_{k,p},a_{k,p}) \le \left(\frac{8}{2m+n}\right)^m\cdot\left(\frac{2n}{2m+n}\right)^{\frac{n}{2}}\cdot\mu.$$

The equality sign is holds, if the points $a_{k,p}$ and domains $B_{k,p}$ are the poles and the circular domains of the quadratic differential (6).

Corollary 4. *Let $n,m,d\in\mathbb{N}$, $m=nd$, $n\ge 2$ and let $A_{n,d}=\{a_{k,p}\}$, $\mu(A_{n,d})=\mu(A^{(2)}_{n,d})$ be an arbitrary generalized (n,d)-equiangular system of points; the set of numbers $\{m_k\}_{k=1}^n$ satisfies the condition (1). Let also $\{B_{k,p}\}$, $a_{k,p}\in B_{k,p}\subset\overline{\mathbb{C}}$ be an arbitrary set of partially non-overlapping domains. Then the inequality of Corollary 3 is true.*

Proof of Theorem 1. We note that from the non-overlapping condition follows that $\operatorname{cap}\overline{\mathbb{C}}\backslash D>0$ and the set D with respect to a point $a\in D$ possesses the Green generalized function $g_D(z,a)$, which has the form

$$g_D(z,a):=\begin{cases} g_{D(a)}(z,a),\ z\in D(a),\\ 0,\ z\in\overline{\mathbb{C}}\backslash\overline{D(a)},\\ \lim\limits_{\zeta\to z} g_{D(a)}(\zeta,a),\ \zeta\in D(a), z\in\partial D(a),\end{cases}$$

where $g_{D(a)}(z,a)$ is the Green function of the domain $D(a)$ with respect to a point $a\in D(a)$.

Further, we will use the methods of the paper [8]. Consider the sets $E_0=\overline{\mathbb{C}}\backslash D$; $\overline{U}_t=\{w\in\mathbb{C}: |w|\leqslant t\}$, $E(a_{k,p},t)=\{w\in\mathbb{C}: |w-a_{k,p}|\leqslant t\}$, $k=\overline{1,n}$, $p=\overline{1,m_k}$, $n\geqslant 2$, $n,m_k\in\mathbb{N}$, $t\in\mathbb{R}^+$. For a rather small $t>0$, we consider the condenser

$$C\,(t,\,D,\,A_{n,d})=\left\{E_0,\,\overline{U}_t,\,E_1\right\},$$

where $E_1=\bigcup\limits_{k=1}^{n}\bigcup\limits_{p=1}^{m_k}E(a_{k,p},t)$. The capacity of the condenser $C\,(t,\,D,\,A_{n,d})$ is defined as

$$\operatorname{cap}C\,(t,\,D,\,A_{n,d})=\inf\int\int\left[(G'_x)^2+(G'_y)^2\right]\,dxdy$$

(see [5]), where an infimum takes in $\overline{\mathbb{C}}$ over all Lipschitzian functions $G=G(z)$, such that $G\Big|_{E_0}=0$, $G\Big|_{E_1}=1$, $G\Big|_{\overline{U}_t}=\frac{n}{2}$.

The module of the condenser C is defined as

$$|C|=[\operatorname{cap}C]^{-1}.$$

From the Theorem 1 from [8] we get

$$|C\left(t,D,A_{n,d}\right)|=\frac{1}{2\pi}\cdot\frac{1}{\frac{n^2}{4}+m}\cdot\log\frac{1}{t}+M(D,A_{n,d})+o(1),\qquad t\to 0,\quad (7)$$

where

$$M(D,A_{n,d})=\frac{1}{2\pi}\cdot\frac{1}{\left(\frac{n^2}{4}+m\right)^2}\cdot\Big[\frac{n^2}{4}\log r(D,0)+\sum_{k=1}^{n}\sum_{p=1}^{m_k}g_D(0,a_{k,p})+$$
$$+\sum_{k=1}^{n}\sum_{p=1}^{m_k}\log r(D,a_{k,p})+\sum_{(k,p)\neq(q,s)}g_D(a_{k,p},a_{q,s})\Big].\qquad (8)$$

The function

$$z_k(w)=(-1)^k\, i\cdot w^{\frac{n}{2}},$$

$k=\overline{1,n}$ realizes univalent and conformal transformations of domain P_k on the right half-plane $\mathrm{Re}z>0$.

Therefore function

$$\zeta_k(w):=\frac{1-z_k(w)}{1+z_k(w)}\qquad (9)$$

is a univalent and conformal mapping of the domain P_k on the unit circle $U=\{z:\,|z|\le 1\}$, $k=\overline{1,n}$.

Obviously, we have $\zeta_k\left(0\right)=1$, $k=\overline{1,n}$.

Let $\omega^{(1)}_{k,p}:=\zeta_k\left(a_{k,p}\right)$, $\omega^{(2)}_{k-1,p}:=\zeta_{k-1}\left(a_{k,p}\right)$, $a_{n+1,p}:=a_{1,p}$, $\omega^{(2)}_{0,p}:=\omega^{(2)}_{n,p}$, $\zeta_0:=\zeta_n$ $(k=\overline{1,n},\ p=\overline{1,m_k})$. For any domain $\Delta\in\mathbb{C}$, we define $(\Delta)^*:=\left\{w\in\overline{\mathbb{C}}:\,\frac{1}{\overline{w}}\in\Delta\right\}$.

From the formula (9) from [7], we obtain the following asymptotic expressions

$$\Big|\zeta_k(w)-\zeta_k(a_{k,p})\Big|\sim\left[\frac{2}{n}\cdot\chi\left(\Big|a_{k,p}\Big|^{\frac{n}{2}}\right)|a_{k,p}|\right]^{-1}\cdot|w-a_{k,p}|,$$
$$w\to a_{k,p},\quad w\in\overline{P}_k.$$
$$\Big|\zeta_{k-1}(w)-\zeta_{k-1}(a_{k,p})\Big|\sim\left[\frac{2}{n}\cdot\chi\left(\Big|a_{k,p}\Big|^{\frac{n}{2}}\right)|a_{k,p}|\right]^{-1}\cdot|w-a_{k,p}|,$$
$$w\to a_{k,p},\quad w\in\overline{P}_{k-1},\quad k=\overline{1,n},p=\overline{1,m_k}.\qquad (10)$$

The coefficients of piece-dividing transformation at the point $w=0$ are defined by the following asymptotic equalities

$$\Big|\zeta_k(w)-1\Big|\sim 2|w|^{\frac{n}{2}},\quad w\to 0,\,w\in\overline{P}^0_k,\,k=\overline{1,n}.\qquad (11)$$

Let $\mathbf{\Omega}_{k,p}^{(1)}$ be a connected component $\zeta_k\left(D\bigcap\overline{P}_k\right)\bigcup\left(\zeta_k\left(D\bigcap\overline{P}_k\right)\right)^*$ containing the point $\omega_{k,p}^{(1)}$ and let $\mathbf{\Omega}_{k-1,p}^{(2)}$ be a connected component $\zeta_{k-1}\left(D\bigcap\overline{P}_{k-1}\right)\bigcup\left(\zeta_{k-1}\left(D\bigcap\overline{P}_{k-1}\right)\right)^*$ containing the point $\omega_{k-1,p}^{(2)}$, $k=\overline{1,n}$, $p=\overline{1,m_k}$, $\overline{P}_0:=\overline{P}_n$, $\mathbf{\Omega}_{0,p}^{(2)}:=\mathbf{\Omega}_{n,p}^{(2)}$. It is clear that in generally $\mathbf{\Omega}_{k,p}^{(s)}$ are multiconnected domains, $k=\overline{1,n}$, $p=\overline{1,m_k}$, $s=1,2$. Pair of the domains $\mathbf{\Omega}_{k-1,p}^{(2)}$ and $\mathbf{\Omega}_{k,p}^{(1)}$ is the result of piece-dividing transformation of the open set D with respect to the family $\{P_{k-1},P_k\}$, $\{\zeta_{k-1},\zeta_k\}$ at the point $a_{k,p}$, $k=\overline{1,n}$, $p=\overline{1,m_k}$. Let $\mathbf{\Omega}_k^{(0)}$ be a connected component $\zeta_k\left(D\bigcap\overline{P}_k\right)\bigcup\left(\zeta_k\left(D\bigcap\overline{P}_k\right)\right)^*$ containing the point 1, $k=\overline{1,n}$. The family of the domains $\left\{\mathbf{\Omega}_k^{(0)}\right\}_{k=1}^n$ is the result of piece-dividing transformation of the open set D with respect to the family $\{P_k\}_{k=1}^n$ and the functions $\{\zeta_k\}_{k=1}^n$ at the point $w=0$, $k=\overline{1,n}$.

In the following, we consider the condensers

$$C_k\left(t,\,D,\,A_{n,d}\right)=\left(E_0^{(k)},\,\overline{U}_t^{(k)},\,E_1^{(k)}\right),$$

where

$$E_s^{(k)}=\zeta_k\left(E_s\bigcap\overline{P}_k\right)\bigcup\left[\zeta_k\left(E_s\bigcap\overline{P}_k\right)\right]^*,$$

$$\overline{U}_t^{(k)}=z_k\left(\overline{U}_t\bigcap\overline{P_k}\right)\bigcup\left\{z_k\left(\overline{U}_t\bigcap\overline{P_k}\right)\right\}^*,$$

$k=\overline{1,n}$, $s=0,1$, $\{P_k\}_{k=1}^n$ is a system of corners corresponding to a system of points $A_{n,d}$; the set $[A]^*$ is a set which is symmetrical to the set A with respect a unit circle $|w|=1$. From this, it follows that for dividing transformation with respect to $\{P_k\}_{k=1}^n$ and $\{\zeta_k\}_{k=1}^n$ for the condenser $C\left(t,\,D,\,A_{n,d}\right)$ corresponds the set of condensers $\{C_k\left(t,\,D,\,A_{n,d}\right)\}_{k=1}^n$. The last condensers are symmetrical with respect to $\{z:\,|z|=1\}$. According to the paper [8], we obtain

$$\mathrm{cap}C\left(t,D,A_{n,d}\right)\geqslant\frac{1}{2}\sum_{k=1}^n\mathrm{cap}C_k\left(t,D,A_{n,d}\right). \tag{12}$$

Therefore we obtain

$$|C\left(t,D,A_{n,d}\right)|\leqslant 2\left(\sum_{k=1}^n|C_k\left(t,D,A_{n,d}\right)|^{-1}\right)^{-1}. \tag{13}$$

The formula (7) gives a module of asymptotic $C\left(t,\,D,\,A_{n,d}\right)$ when $t\to 0$ and $M\left(D,A_{n,d}\right)$ is a module of the set D with respect to $A_{n,d}$. Using the formulae (10), (11), and the fact that the set D satisfies a non-overlapping conditions with respect to the system of points $0\cup A_{n,d}$, we have the following asymptotic representations for the condensers $C_k\left(t,D,A_{n,d}\right)$, $k=\overline{1,n}$:

$$|C_k\left(t,D,A_{n,d}\right)| =$$
$$= \frac{1}{2\pi\left(\frac{n}{2}+m_k+m_{k+1}\right)}\log\frac{1}{t}+M_k\left(D,A_{n,d}\right)+o(1), \quad t\to 0, \quad m_{n+1}:=m_1, \tag{14}$$

where

$$M_k\left(D,A_{n,d}\right)=\frac{1}{2\pi\left(\frac{n}{2}+m_k+m_{k+1}\right)^2}\cdot\left[\log\frac{r\left(\mathbf{\Omega}_k^{(0)},1\right)}{2}+\right.$$
$$\left.+\sum_{p=1}^{m_k}\log\frac{r\left(\mathbf{\Omega}_{k,p}^{(1)},\omega_{k,p}^{(1)}\right)}{\left[\frac{2}{n}\cdot\chi\left(|a_{k,p}|^{\frac{n}{2}}\right)|a_{k,p}|\right]^{-1}}+\sum_{t=1}^{m_{k+1}}\log\frac{r\left(\mathbf{\Omega}_{k,t}^{(2)},\omega_{k,t}^{(2)}\right)}{\left[\frac{2}{n}\cdot\chi\left(|a_{k+1,t}|^{\frac{n}{2}}\right)|a_{k+1,t}|\right]^{-1}}\right],$$

and $k=\overline{1,n}$.

Using (14), we get

$$|C_k\left(t,D,A_{n,d}\right)|^{-1}=\frac{2\pi\left(\frac{n}{2}+m_k+m_{k+1}\right)}{\log\frac{1}{t}}\times$$
$$\times\left(1+\frac{2\pi\left(\frac{n}{2}+m_k+m_{k+1}\right)}{\log\frac{1}{t}}M_k\left(D,A_{n,d}\right)+o\left(\frac{1}{\log\frac{1}{t}}\right)\right)^{-1}=$$
$$=\frac{2\pi\left(\frac{n}{2}+m_k+m_{k+1}\right)}{\log\frac{1}{t}}-$$
$$-\left(\frac{2\pi\left(\frac{n}{2}+m_k+m_{k+1}\right)}{\log\frac{1}{t}}\right)^2 M_k\left(D,A_{n,d}\right)+o\left(\left(\frac{1}{\log\frac{1}{t}}\right)^2\right), t\to 0. \tag{15}$$

Using the equality $\sum\limits_{k=1}^{n}m_k=m$ and the condition (15), we have

$$\sum_{k=1}^{n}|C_k\left(t,D,A_{n,d}\right)|^{-1}=\frac{2\pi\left(\frac{n^2}{2}+2m\right)}{\log\frac{1}{t}}-$$
$$-\left(\frac{2\pi}{\log\frac{1}{t}}\right)^2\cdot\sum_{k=1}^{n}\left(\frac{n}{2}+m_k+m_{k+1}\right)^2 M_k\left(D,A_{n,d}\right)+o\left(\left(\frac{1}{\log\frac{1}{t}}\right)^2\right), t\to 0. \tag{16}$$

In turn, from the relation (16), we obtain the following asymptotic representation

$$\left(\sum_{k=1}^{n}|C_k\left(t,D,A_{n,d}\right)|^{-1}\right)^{-1}=\frac{\log\frac{1}{t}}{2\pi\left(\frac{n^2}{2}+2m\right)}\left(1-\frac{2\pi}{\left(\frac{n^2}{2}+2m\right)}\cdot\frac{1}{\log\frac{1}{t}}\times\right.$$

$$\left.\times\sum_{k=1}^{n}\left(\frac{n}{2}+m_k+m_{k+1}\right)^2M_k\left(D,A_{n,d}\right)+o\left(\frac{1}{\log\frac{1}{t}}\right)\right)^{-1}=\frac{\log\frac{1}{t}}{2\pi\left(\frac{n^2}{2}+2m\right)}+$$

$$+\frac{1}{\left(\frac{n^2}{2}+2m\right)^2}\cdot\sum_{k=1}^{n}\left(\frac{n}{2}+m_k+m_{k+1}\right)^2M_k\left(D,A_{n,d}\right)+o(1),\qquad t\to 0.\quad(17)$$

From the inequalities (12) and (13), using (7) and (17), we obtain

$$\frac{1}{2\pi\left(\frac{n^2}{4}+m\right)}\log\frac{1}{t}+M\left(D,A_{n,d}\right)+o(1)\leqslant$$

$$\leqslant\frac{1}{2\pi\left(\frac{n^2}{4}+m\right)}\log\frac{1}{t}+\frac{2}{\left(\frac{n^2}{2}+2m\right)^2}\cdot\sum_{k=1}^{n}\left(\frac{n}{2}+m_k+m_{k+1}\right)^2M_k\left(D,A_{n,d}\right)+o(1).\quad(18)$$

From (18) when $t\to 0$, we get

$$M(D,A_{n,d})\leqslant\frac{2}{\left(\frac{n^2}{2}+2m\right)^2}\cdot\sum_{k=1}^{n}\left(\frac{n}{2}+m_k+m_{k+1}\right)^2M_k\left(D,A_{n,d}\right).\quad(19)$$

The formulae (8), (14) and (19) imply the following expression

$$\frac{1}{2\pi}\cdot\frac{1}{\left(\frac{n^2}{4}+m\right)^2}\cdot\left[\frac{n^2}{4}\log r(D,0)+\sum_{k=1}^{n}\sum_{p=1}^{m_k}g_D(0,a_{k,p})+\right.$$

$$\left.+\sum_{k=1}^{n}\sum_{p=1}^{m_k}\log r(D,a_{k,p})+\sum_{(k,p)\neq(q,s)}g_D(a_{k,p},a_{q,s})\right]\le\frac{1}{4\pi}\cdot\frac{1}{\left(\frac{n^2}{2}+m\right)^2}\times$$

$$\times\sum_{k=1}^{n}\left[\log\frac{r\left(\mathbf{\Omega}_k^{(0)},1\right)}{2}+\sum_{p=1}^{m_k}\log\frac{r\left(\mathbf{\Omega}_{k,p}^{(1)},\omega_{k,p}^{(1)}\right)}{\left[\frac{2}{n}\cdot\chi\left(|a_{k,p}|^{\frac{n}{2}}\right)|a_{k,p}|\right]^{-1}}+\right.$$

$$\left.+\sum_{t=1}^{m_{k+1}}\log\frac{r\left(\mathbf{\Omega}_{k,t}^{(2)},\omega_{k,t}^{(2)}\right)}{\left[\frac{2}{n}\cdot\chi\left(|a_{k+1,t}|^{\frac{n}{2}}\right)|a_{k+1,t}|\right]^{-1}}\right],\qquad k=\overline{1,n}.$$

Therefore, we have

$$r^{\frac{n^2}{4}}(D,0)\cdot\prod_{k=1}^{n}\prod_{p=1}^{m_k} r(D,a_{k,p})\leq 2^{-\frac{n}{2}}\cdot\left(\frac{2}{n}\right)^m\cdot\mu\left(A_{n,d}\right)\times$$

$$\times\prod_{k=1}^{n}\left\{r\left(\mathbf{\Omega}_k^{(0)},1\right)\cdot\prod_{p=1}^{m_k} r\left(\mathbf{\Omega}_{k,p}^{(1)},\omega_{k,p}^{(1)}\right)\cdot\prod_{t=1}^{m_{k+1}} r\left(\mathbf{\Omega}_{k,t}^{(2)},\omega_{k,t}^{(2)}\right)\right\}^{\frac{1}{2}}. \qquad (20)$$

From results of the paper [6, 8, 9], we have the following inequalities

$$r\left(\mathbf{\Omega}_k^{(0)},1\right)\cdot\prod_{p=1}^{m_k} r\left(\mathbf{\Omega}_{k,p}^{(1)},\omega_{k,p}^{(1)}\right)\cdot\prod_{t=1}^{m_{k+1}} r\left(\mathbf{\Omega}_{k,t}^{(2)},\omega_{k,t}^{(2)}\right)\leq$$

$$\leq\prod_{s=1}^{m_k+m_{k+1}+1} r\left(G_s^{(k)},e^{i\frac{2\pi}{m_k+m_{k+1}+1}(s-1)}\right), \qquad (21)$$

where $G_s^{(k)}$ is a system of circular domains of the quadratic differential

$$Q\left(\zeta_k\right)d\zeta_k^2=-\frac{\zeta_k^{m_k+m_{k+1}-1}}{\left(\zeta_k^{m_k+m_{k+1}+1}-1\right)^2}\cdot d\zeta_k^2.$$

Using the inequalities (20), (21), we obtain

$$r^{\frac{n^2}{4}}(D,0)\cdot\prod_{k=1}^{n}\prod_{p=1}^{m_k} r(D,a_{k,p})\leq 2^{-\frac{n}{2}}\cdot\left(\frac{2}{n}\right)^m\cdot\mu\left(A_{n,d}\right)\times$$

$$\times\prod_{k=1}^{n}\left\{\prod_{s=1}^{m_k+m_{k+1}+1} r\left(G_s^{(k)},e^{i\frac{2\pi}{m_k+m_{k+1}+1}(s-1)}\right)\right\}^{\frac{1}{2}}. \qquad (22)$$

Now consider the family of functions

$$\xi_k=\sqrt[n]{\zeta_k}\cdot e^{i\frac{2\pi}{n}(k-1)},\quad k=\overline{1,n},$$

which transform the unit circle to a sector with size $\frac{2\pi}{n}$. Then the domains $G_s^{(k)}$, $k=\overline{1,n}$, $s=\overline{1,m_k+m_{k+1}+1}$ will be transformed to the domain $\Sigma_s^{(k)}$ and the points $e^{i\frac{2\pi}{m_k+m_{k+1}+1}(s-1)}$ will be transformed into $e^{i\frac{2\pi}{n}\left(\frac{s-1}{m_k+m_{k+1}+1}+k-1\right)}$. By union all sectors we obtain the unit circle containing $(2m+n)$ non-overlapping domains $\Sigma_s^{(k)}$, $k=\overline{1,n}$, $s=\overline{1,m_k+m_{k+1}+1}$. Then

$$r\left(G_s^{(k)},e^{i\frac{2\pi}{m_k+m_{k+1}+1}(s-1)}\right)\leq n\cdot r\left(\Sigma_s^{(k)},e^{i\frac{2\pi}{n}\left(\frac{s-1}{m_k+m_{k+1}+1}+k-1\right)}\right). \qquad (23)$$

Using the inequalities (22), (23), we have

$$r^{\frac{n^2}{4}}(D,0)\cdot\prod_{k=1}^{n}\prod_{p=1}^{m_k} r(D,a_{k,p})\leq 2^m\cdot\left(\frac{n}{2}\right)^{\frac{n}{2}}\cdot\mu\left(A_{n,d}\right)\times$$

$$\times\left\{\prod_{k=1}^{n}\prod_{s=1}^{m_k+m_{k+1}+1} r\left(\Sigma_s^{(k)},e^{i\frac{2\pi}{n}\left(\frac{s-1}{m_k+m_{k+1}+1}+k-1\right)}\right)\right\}^{\frac{1}{2}}. \quad (24)$$

Using the results of the paper [6, 8, 9], we obtain the following inequality

$$\prod_{k=1}^{n}\prod_{s=1}^{m_k+m_{k+1}+1} r\left(\Sigma_s^{(k)},e^{i\frac{2\pi}{n}\left(\frac{s-1}{m_k+m_{k+1}+1}+k-1\right)}\right)\leq\prod_{t=1}^{2m+n} r\left(B_t,b_t\right) \quad (25)$$

$$=\left(\frac{4}{2m+n}\right)^{2m+n}.$$

The sign of equality is obtained when the domains B_t and the points b_t are the circular domains and the poles of the quadratic differential

$$Q\left(\xi\right)d\xi^2=-\frac{\xi^{2m+n-2}}{\left(\xi^{2m+n}-1\right)^2}\cdot d\xi^2. \quad (26)$$

Finally, from the inequalities (25), (24), we obtain

$$r^{\frac{n^2}{4}}(D,0)\cdot\prod_{k=1}^{n}\prod_{p=1}^{m_k} r(D,a_{k,p})\leq\left(\frac{8}{2m+n}\right)^m\cdot\left(\frac{2n}{2m+n}\right)^{\frac{n}{2}}\cdot\mu\left(A_{n,d}\right). \quad (27)$$

The statement of the theorem follows directly from the inequality (27) and from the quadratic differential (26), in which we must make a necessary exchange of variables. **The theorem is proved**.

Proof of the Theorem 2 is similar to the proof of the Theorem 1.

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