A multivalued version of Krasnoselskii's theorem in generalized Banach spaces

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Abstract

The purpose of this paper is to extend Krasnoselskii's fixed point theorem to the case of generalized Banach spaces for multivalued operators. As application, we will give an existence result for a system of Fredholm-Volterra type differential inclusions.

1 Introduction, notations and auxiliary results

It is well known that A.I. Perov (see [17]) extended the classical Banach contraction principle in the setting of spaces endowed with vector-valued metrics (see also A.I. Perov, A.V. Kibenko [18]).

Also, is known that Krasnoselskii's theorem combine a metrical fixed point result (a contraction principle) and a topological one (a Schauder-type theorem). Moreover, the multivalued forms of Krasnoselskii's theorem deals with the notion of selection. Auxiliary results are needed to ensure that a multivalued operator admits a continuous selection. There is also a vast literature concerning on this aspects in nonlinear analysis, see, for example, [3], [5], [9], [12], [16], [19], [20], [21], [22], [29], [36], [39], [40].

Thus, it is obvious that Krasnoselskii's theorem represents an important abstract tool in the study of differential and integral inclusions systems. Therefore, the aim of this paper is to present a multivalued version of Krasnoselskii's theorem in vector Banach spaces and a nice existence result for a Fredholm-Volterra type integral inclusions system.

Now, we recall some basic results (see [2], [8] and [28]), which are needed for the main results of this paper. Notice that in [28] and [8], are pointed out some advantages of a vector-valued norm with respect to the usual scalar norms.

Definition 1.1. ([17]) Let X be a nonempty set and consider the space $\mathbb{R}_+^m$ endowed with the usual component-wise partial order. The mapping $d : X \times X \to \mathbb{R}_+^m$ which satisfies all the usual axioms of the metric is called a *generalized metric* in the Perov's sense and (X, d) is called a *generalized metric space*.

Let (X, d) be a generalized metric space in Perov's sense. Thus, if $v, r \in \mathbb{R}^m$, $v := (v_1, v_2, \ldots, v_m)$ and $r := (r_1, r_2, \ldots, r_m)$, then by $v \leq r$ we mean $v_i \leq r_i$, for each $i \in \{1, 2, \ldots, m\}$ and by $v < r$ we mean $v_i < r_i$, for each $i \in \{1, 2, \ldots, m\}$. Also, $|v| := (|v_1|, |v_2|, \ldots, |v_m|)$.

If $u, v \in \mathbb{R}^m$, with $u := (u_1, u_2, \ldots, u_m)$ and $v := (v_1, v_2, \ldots, v_m)$, then $\max(u, v) := (\max(u_1, v_1), \ldots, \max(u_m, v_m))$. If $c \in \mathbb{R}$, then $v \leq c$ means $v_i \leq c$, for each $i \in \{1, 2, \ldots, m\}$. Notice that, through this paper, we will make an identification between row and column vectors in $\mathbb{R}^m$. Let (X, d) be a generalized metric space in Perov's sense. For $r := (r_1, \cdots, r_m) \in \mathbb{R}^m$ with $r_i > 0$ for each $i \in \{1, 2, \cdots, m\}$, we will denote by

$$B(x_0, r) := \{x \in X : d(x_0, x) < r\}$$

the open ball centered in x_0 with radius r and by

$$\bar{B}(x_0, r) := \{x \in X : d(x_0, x) \leq r\}$$

the closed ball centered in x_0 with radius r.

We mention that for generalized metric spaces in Perov's sense, the notions of convergent sequence, Cauchy sequence, completeness, open subset and closed subset are similar to those for usual metric spaces.

Definition 1.2. A square matrix of real numbers is said to be *convergent to zero* if and only if its spectral radius $\rho(A)$ is strictly less than 1. In other words, this means that all the eigenvalues of A are in the open unit disc, i.e., $|\lambda| < 1$, for every $\lambda \in \mathbb{C}$ with $\det(A - \lambda I) = 0$, where I denotes the unit matrix of $\mathcal{M}_{m,m}(\mathbb{R})$ (see [35]).

For a matrix $A := (a_{ij})_{i,j \in \{1,\cdots,m\}} \in \mathcal{M}_{m,m}(\mathbb{R})$ we denote by

$$|A| := (|a_{ij}|)_{i,j \in \{1,\cdots,m\}} \in \mathcal{M}_{m,m}(\mathbb{R}_+).$$

In this context, we say that a non-singular matrix A has the *absolute value property* if $A^{-1}|A| \leq I$. Some examples of matrices convergent to zero $A \in \mathcal{M}_{m,m}(\mathbb{R})$, which also satisfies the property $(I - A)^{-1}|I - A| \leq I$ are:

1) $A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, where $a, b \in \mathbb{R}_+$ and $\max(a, b) < 1$;

2) $A = \begin{pmatrix} a & -c \\ 0 & b \end{pmatrix}$, where $a, b, c \in \mathbb{R}_+$ and $a + b < 1, c < 1$;

3) $A = \begin{pmatrix} a & -a \\ b & -b \end{pmatrix}$, where $a > 1, b > 0$ and $|a - b| < 1$.

In particular, if E is a linear space, then $\|\cdot\| : E \to \mathbb{R}_+^m$ is a *vector-valued norm* if (in a similar way to the vector-valued metric) it satisfies the classical axioms of a norm. In this case, the pair $(E, \|\cdot\|)$ is called a *generalized normed space*. If the generalized metric generated by the norm $\|\cdot\|$ (i.e., $d(x, y) := \|x - y\|$) is complete then the space $(E, \|\cdot\|)$ is called a *generalized Banach space*.

In the context of a generalized metric space (X, d), we will use the following notations and definitions.

$P(X)$ - the set of all nonempty subsets of X;
$\mathcal{P}(X) = P(X) \cup \{\emptyset\}$;
$P_b(X)$ - the set of all nonempty bounded subsets of X;
$P_{b,cl}(X)$ - the set of all nonempty bounded and closed subsets of X;

If $(X, \|\cdot\|)$ is a generalized normed space, then:
$P_{b,cl,cv}(X)$ - the set of all nonempty bounded, closed and convex subsets of X;
$P_{cp,cv}(X)$ - the set of all nonempty compact and convex subsets of X.

Let (X, d) be a metric space. Then we introduce the following functionals.

$D_d : P(X) \times P(X) \to \mathbb{R}_+, D_d(A, B) = \inf\{d(a, b) : a \in A, b \in B\}$ - the gap functional;
$\rho_d : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\}, \rho_d(A, B) = \sup\{D(a, B) : a \in A\}$ - the excess functional;
$H_d : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\}, H_d(A, B) = \max\{\rho(A, B), \rho(B, A)\}$ - the Pompeiu-Hausdorff functional.

If (X, d) is a generalized metric space with $d(x, y) := \begin{pmatrix} d_1(x, y) \\ \cdots \\ d_m(x, y) \end{pmatrix}$, then

we denote by

$$D(A, B) := \begin{pmatrix} D_{d_1}(A, B) \\ \cdots \\ D_{d_m}(A, B) \end{pmatrix} \text{ the vector gap functional on } P(X),$$

by

$$\rho(A, B) := \begin{pmatrix} \rho_{d_1}(A, B) \\ \cdots \\ \rho_{d_m}(A, B) \end{pmatrix} \text{ the vector excess functional,}$$

and by

$$H(A, B) := \begin{pmatrix} H_{d_1}(A, B) \\ \cdots \\ H_{d_m}(A, B) \end{pmatrix} \text{ the vector Pompeiu-Hausdorff functional.}$$

For a multivalued operator $F : X \to P(X)$, we denote by $\text{Fix}(F)$ the fixed point set of F, i.e., $\text{Fix}(F) := \{x \in X \mid x \in F(x)\}$. The symbol $\text{Graph}(F) := \{(x, y) \in X \times X : y \in F(x)\}$ denotes the graph of F.

For some topological aspects in generalized metric spaces see, for example, [32], [9], [38], [39]. We recall now the following Schauder type theorem.

Theorem 1.3. *([9]) Let $(X, \|\cdot\|)$ be a generalized Banach space, let $Y \in P_{cv}(X)$ and $g : Y \to Y$ be a continuous operator with relatively compact range. Then g has at least one fixed point in Y.*

If (X, d) is a generalized metric space, then $F : X \to P(X)$ is said to be a *multivalued weak Picard operator* if, for each $x \in X$ and $y \in F(x)$, there exists a sequence $(x_n)_{n \in \mathbb{N}}$ such that:

i) $x_0 = x$, $x_1 = y$;

ii) $x_{n+1} \in F(x_n)$;

iii) the sequence $(x_n)_{n \in \mathbb{N}}$ is convergent to a fixed point of F.

A sequence $(x_n)_{n \in \mathbb{N}}$ satisfying (i) and (ii) is said to be a *sequence of successive approximations* for F starting from $(x_0, x_1) \in \text{Graph}(F)$.

Definition 1.4. ([2]) Let (X, d) be a generalized metric space, $Y \subset X$ and $F : Y \to P(X)$ be a multivalued operator. Then, F is called a *multivalued A-contraction* if and only if, $A \in \mathcal{M}_{m,m}(\mathbb{R}_+)$ is a matrix convergent to zero and for any $x, y \in Y$ and for each $u \in F(x)$, there exists $v \in T(y)$ such that

$$d(u, v) \leq Ad(x, y).$$

Notice now that using the generalized Pompeiu-Hausdorff functional on $P_{b,cl}(X)$, the concept of multivalued contraction mapping introduced by S.B. Nadler Jr. can be extended to generalized metric spaces in the sense of Perov.

Definition 1.5. ([2]) Let (X, d) be a generalized metric space, $Y \subseteq X$ and let $F : Y \to P_{b,cl}(X)$ be a multivalued operator. Then, F is called a *multivalued A-contraction in the sense of Nadler* if and only if, $A \in \mathcal{M}_{m,m}(\mathbb{R}_+)$ is a matrix convergent to zero and

$$H(F(x), F(y)) \leq Ad(x, y), \text{ for any } x, y \in Y.$$

If X, Y are two generalized metric spaces, then a multivalued operator $F : X \to P(Y)$ is said to be:

a) *lower semi-continuous* (briefly l.s.c.) in $x_0 \in X$ if and only if, for any open set $U \subset X$ such that $F(x_0) \cap U \neq \emptyset$, there exists a neighborhood V for x_0 such that for any $x \in V$, we have that $F(x) \cap U \neq \emptyset$.

b) *Hausdorff lower semi-continuous* (briefly H-l.s.c.) in $x_0 \in X$ if and only if, for any $\varepsilon = (\varepsilon_1, \cdots, \varepsilon_m) \in \mathbb{R}_+^m$ with $\varepsilon_i > 0$ for each $i \in \{1, \cdots, m\}$, there exists $\eta = (\eta_1, \cdots, \eta_m) \in \mathbb{R}_+^m$ with $\eta_i > 0$ for each $i \in \{1, \cdots, m\}$, such that for any $x \in B(x_0, \eta)$, we have $F(x_0) \subset V(F(x); \varepsilon)$, where

$$V(F(x); \varepsilon) = \{x \in X : D(x, F(x)) \leq \varepsilon\}.$$

Now, we recall several auxiliary results proved in paper [19].

Theorem 1.6. *([19]) Let (X, d) be a complete generalized metric space and $F : X \to P_{cl}(X)$ be a multivalued A-contraction in Nadler's sense. Then, for each $x \in X$ and $y \in F(x)$, there exists a sequence $(x_n)_{n \in \mathbb{N}}$ of successive approximations for F starting from $(x, y) \in \mathrm{Graph}(F)$, which converge to a fixed point $x^* \in X$ of F and we have the following estimations:*
(a) $d(x_n, x^*) \leq A^n(I - A)^{-1}d(x_0, x_1)$, for any $n \in \mathbb{N}^*$.
(b) $d(x_0, x^*) \leq (I - A)^{-1}d(x_0, x_1)$.

Another useful result for our aim is the following data dependence lemma.

Lemma 1.7. *([19]) Let (X, d) be a complete generalized metric space and $F_1, F_2 : X \to P_{b,cl}(X)$ be two multivalued A-contractions in Nadler' sense. Then:*

$$\rho\left(\text{Fix}\left(F_1\right), \text{Fix}\left(F_2\right)\right) \leq (I - A)^{-1} \begin{pmatrix} \sup_{x \in X} \rho_{d_1}\left(F_1(x), F_2(x)\right) \\ \cdots \\ \sup_{x \in X} \rho_{d_m}\left(F_1(x), F_2(x)\right) \end{pmatrix}.$$

Also, we need an extended result of a Rybinski-type selection theorem.

Theorem 1.8. *([19]) Let (X, d) be a generalized metric space and Y be a closed subset of a generalized Banach space $(Z, \|\cdot\|)$. Assume that the multi-valued operator $F : X \times Y \to P_{cl,cv}(Y)$ satisfies the following conditions:*

i) A is a matrix convergent to zero and

$$H\left(F\left(x, y_1\right), F\left(x, y_2\right)\right) \leq A \|y_1 - y_2\|, \text{ for each } (x, y_1), (x, y_2) \in X \times Y;$$

ii) for every $y \in Y$, $F(\cdot, y)$ is H-l.s.c. on X.

Then there exists a continuous mapping $f : X \times Y \to Y$ such that:

$$f(x, y) \in F(x, f(x, y)), \text{ for each } (x, y) \in X \times Y.$$

Lemma 1.9. *([32]) Let $(X, \|\cdot\|)$ be a generalized Banach space. Then:*

$$H(Y + Z, Y + W) \leq H(Z, W), \text{ for each } Y, Z, W \in P_b(X).$$

We recall that a measurable multivalued operator $F : [a, b] \to P_{cp}(\mathbb{R}^n)$ is said to be *integrably bounded* if and only if, there exists a Lebesgue integrable function $m : [a, b] \to \mathbb{R}^n$ such that for each $v \in F(t)$, we have $\|v\| \leq m(t)$, a.e. on $[a, b]$. For a measurable and integrably bounded multivalued operator F, the set S_F^1 of all Lebesgue integrable selections for F is closed and nonempty (see [6]).

2 Main results

In this section, we prove a Krasnoselskii type fixed point theorem in generalized Banach spaces for a sum of two multivalued operators, where one of the operators satisfies a multivalued A-contraction condition in Nadler's sense and the other operator satisfies a compactness condition. Also, an application to a Fredholm-Volterra type differential inclusions system is given here. Notice that both results deals with the absolute value property of matrixes that converges to zero.

Theorem 2.1. *Let* $(X, \|\cdot\|)$ *be a generalized Banach space and* $Y \in P_{b,cl,cv}(X)$. *Assume that the operators* $F : Y \to P_{b,cl,cv}(X)$, $G : Y \to P_{cp,cv}(X)$ *satisfies the properties:*

i) $F(y_1) + G(y_2) \subset Y$, *for each* $y_1, y_2 \in Y$;

ii) F *is a multivalued* A-*contraction mapping in Nadler's sense;*

iii) G *is l.s.c and* $G(Y)$ *is relatively compact;*

iv) the matrix $I - A$ *has the absolute value property.*

Then $F + G$ *has a fixed point in* Y.

Proof. We show that for any $x \in Y$, the operator $T_x : Y \to P_{cp,cv}(Y)$, $T_x(y) = F(y) + G(x)$ is a multivalued A-contraction. We have that

$$H(T_x(y_1), T_x(y_2)) = H(F(y_1) + G(x), F(y_2) + G(x))$$
$$\leq H(F(y_1), F(y_2)) \leq A \|y_1 - y_2\|, \text{ for any } y_1, y_2 \in Y.$$

Thus, T_x is a multivalued A-contraction. By Theorem 1.6, it follows that for any $x \in Y$ the fixed point set for the multivalued operator T_x, $\text{Fix}(T_x) = \{y \in F(y) + G(x)\}$ is nonempty and closed.

Since, the multivalued operator

$$U : Y \times Y \to P_{cp,cv}(Y), U(x, y) = F(y) + G(x)$$

satisfies the hypothesis of Theorem 1.8, there exists a continuous mapping $u : Y \times Y \to Y$ such that $u(x, y) \in F(u(x, y)) + G(x)$, for each $(x, y) \in Y \times Y$.

We define $C : Y \to P_{cl}(Y)$, $C(x) = \text{Fix}(T_x)$ and we consider the singlevalued operator $c : Y \to Y$, $c(x) = u(x, x)$, for each $x \in Y$.

Now, we prove that $c(Y)$ is relatively compact. For this purpose it is sufficient to show that $C(Y)$ is relatively compact. Since $G(Y)$ is relatively compact, we have that $G(Y)$ is also totally bounded (see [38], pp. 500). Thus, for any $\varepsilon \in \mathbb{R}^m_+$ (with $\varepsilon_i > 0$ for each $i \in \{1, \ldots, m\}$), there exists $Z = \{x_1, \ldots, x_n\} \subset Y$ such that $G(Y) \subset \{z_1, \ldots, z_n\} + \bar{B}(0, |I - A|\varepsilon) \subset \{G(x_1), \ldots, G(x_n)\} + \bar{B}(0, |I - A|\varepsilon)$, where $z_i \in G(x_i)$ for any $i \in \{1, 2, \ldots, n\}$.

It follows that, for any $x \in Y$, $G(x) \subset \bigcup_{i=1}^{n} G(x_i) + \bar{B}(0, |I - A|\varepsilon)$ and thus,

there exists an element $x_k \in Z$ such that $\rho\left(G\left(x\right), G\left(x_k\right)\right) \leq |I - A|\varepsilon$. Then

$$\rho\left[C\left(x\right), C\left(x_k\right)\right] = \rho\left[\mathrm{Fix}\left(T_x\right), \mathrm{Fix}\left(T_{x_k}\right)\right]$$

$$\leq (I - A)^{-1} \begin{pmatrix} \sup_{y \in Y} \rho_{d_1}\left[T_x(y), T_{x_k}(y)\right] \\ \cdots \\ \sup_{y \in Y} \rho_{d_m}\left[T_x(y), T_{x_k}(y)\right] \end{pmatrix}$$

$$= (I - A)^{-1} \begin{pmatrix} \sup_{y \in Y} \rho_{d_1}\left[F(y) + G(x), F(y) + G(x_k)\right] \\ \cdots \\ \sup_{y \in Y} \rho_{d_m}\left[F(y) + G(x), F(y) + G(x_k)\right] \end{pmatrix}$$

$$\leq (I - A)^{-1} \begin{pmatrix} \sup_{y \in Y} \rho_{d_1}\left[G(x), G(x_k)\right] \\ \cdots \\ \sup_{y \in Y} \rho_{d_m}\left[G(x), G(x_k)\right] \end{pmatrix}$$

$$\leq (I - A)^{-1} |I - A|\varepsilon = \varepsilon.$$

It follows that for any $v \in C\left(x\right)$, there exists $w_k \in C\left(x_k\right)$ such that $\|v - w_k\| \leq \varepsilon$. Thus, for any $x \in Y$, $C\left(x\right) \subset \bigcup_{k=1}^{n} \bar{B}\left(w_k, \varepsilon\right)$. So, $C\left(Y\right)$ is relatively compact. Moreover, the operator $c : Y \to Y$ satisfies the assumptions of Theorem 1.3. Let $x^* \in Y$ be a fixed point for c. Hence, we have that $x^* = c\left(x^*\right) \in F\left(c\left(x^*\right)\right) + G\left(x^*\right) = F\left(x^*\right) + G\left(x^*\right)$. $\qquad \square$

Now, using Theorem 2.1 we can obtain a nice existence result for a system of Fredholm-Volterra type integral inclusions.

Theorem 2.2. *Let $I = [0, a]$ (with $a > 0$) be an interval of the real axis and let us consider the following inclusions system in $C\left(I, \mathbb{R}^n\right) \times C\left(I, \mathbb{R}^p\right)$:*

$$\begin{cases} x_1\left(t\right) \in \lambda_{11} \int_0^t K_1\left(t, s, x_1\left(s\right), x_2\left(s\right)\right) ds + \lambda_{12} \int_0^a L_1\left(t, s, x_1\left(s\right), x_2\left(s\right)\right) ds \\ x_2\left(t\right) \in \lambda_{21} \int_0^t K_2\left(t, s, x_1\left(s\right), x_2\left(s\right)\right) ds + \lambda_{22} \int_0^a L_2\left(t, s, x_1\left(s\right), x_2\left(s\right)\right) ds \end{cases}$$

for $t \in I$, where $\lambda_{ij} \in \mathbb{R}$, $i, j \in \{1, 2\}$. We assume that:

i) *$K_1 : I^2 \times \mathbb{R}^n \times \mathbb{R}^p \to P_{cl,cv}\left(\mathbb{R}^n\right)$, $K_2 : I^2 \times \mathbb{R}^n \times \mathbb{R}^p \to P_{cl,cv}\left(\mathbb{R}^p\right)$ are two l.s.c., measurable and integrable bounded multivalued operators;*

ii) *$L_1 : I^2 \times \mathbb{R}^n \times \mathbb{R}^p \to P_{cp,cv}\left(\mathbb{R}^n\right)$, $L_2 : I^2 \times \mathbb{R}^n \times \mathbb{R}^p \to P_{cp,cv}\left(\mathbb{R}^p\right)$ are two l.s.c., measurable and integrable bounded (by two integrable functions m_{L_1}, m_{L_2}) multivalued operators;*

iii) there exists the matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in M_{2,2}(\mathbb{R}_+)$ *such that for each* $(t, s, u_1, u_2), (t, s, v_1, v_2) \in I^2 \times \mathbb{R}^n \times \mathbb{R}^p$ *and for* $i \in \{1, 2\}$, *we have:*

$$H(K_i(t, s, u_1, u_2), K_i(t, s, v_1, v_2)) \le a_{i1}|u_1 - v_1| + a_{i2}|u_2 - v_2|;$$

iv) $\begin{pmatrix} |\lambda_{11}| \\ |\lambda_{21}| \end{pmatrix} \le \begin{pmatrix} \frac{R_1}{2a(a_{11}R_1 + a_{12}R_2)} \\ \frac{R_2}{2a(a_{21}R_1 + a_{22}R_2)} \end{pmatrix}$ *and* $\begin{pmatrix} |\lambda_{12}| \\ |\lambda_{22}| \end{pmatrix} \le \begin{pmatrix} \frac{R_1}{2M_{L_1}a} \\ \frac{R_2}{2M_{L_2}a} \end{pmatrix}$, *where*

$$M_{L_1} = \max_{t,s \in [0,a]} |m_{L_1}|_{\mathbb{R}^n}, \ M_{L_2} = \max_{t,s \in [0,a]} |m_{L_2}|_{\mathbb{R}^p}$$

and m_{L_i} *represents the set of continuous selections for the multivalued operator* $t \mapsto \lambda_{i2} \int_0^a L_i(t, s, x_1(s), x_2(s)) ds$ *for* $x_1 \in C(I, \mathbb{R}^n)$ *and* $x_2 \in C(I, \mathbb{R}^p)$.

v) the matrix $I - M$ *has the absolute value property, where*

$$M = \left(\frac{|\lambda_{i1}| a_{ij}}{\tau} \right)_{i,j=1,2}, \ \tau > 0.$$

Then, there exists $(x_1^0, x_2^0) \in C(I, \mathbb{R}^n) \times C(I, \mathbb{R}^p)$ *such that our inclusions system has at least one solution* $x^* := (x_1^*, x_2^*) \in \bar{B}(x_1^0, R_1) \times \bar{B}(x_2^0, R_2) \subset C(I, \mathbb{R}^n) \times C(I, \mathbb{R}^p)$.

Proof. For the sake of simplicity let us denote $X_1 := \mathbb{R}^n$, $X_2 := \mathbb{R}^p$ and $X := C(I, X_1) \times C(I, X_2)$. Let $F_1, G_1 : X \to \mathcal{P}(C(I, X_1))$ be two multivalued operators given by:

$$F_1(x_1, x_2) = \left\{ u \in X : u(t) \in \lambda_{11} \int_0^t K_1(t, s, x_1(s), x_2(s)) ds \text{ a.e. on } I \right\},$$

$$G_1(x_1, x_2) = \left\{ v \in X : v(t) \in \lambda_{12} \int_0^a L_1(t, s, x_1(s), x_2(s)) ds \text{ a.e. on } I \right\}.$$

Let $F_2, G_2 : X \to \mathcal{P}(C(I, X_2))$ be two multivalued operators given by:

$$F_2(x_1, x_2) = \left\{ u \in X : u(t) \in \lambda_{21} \int_0^t K_2(t, s, x_1(s), x_2(s)) ds \text{ a.e. on } I \right\},$$

$$G_2(x_1, x_2) = \left\{ v \in X : v(t) \in \lambda_{22} \int_0^a L_2(t, s, x_1(s), x_2(s)) ds \text{ a.e. on } I \right\}.$$

Then, $F = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}, G = \begin{pmatrix} G_1 \\ G_2 \end{pmatrix} : X \to \mathcal{P}(X)$.

For $x = (x_1, x_2) \longrightarrow F(x) = (F_1(x), F_2(x))$, $G(x) = (G_1(x), G_2(x))$. Obviously, $x^* = (x_1^*, x_2^*)$ is a solution for our inclusions system if and only if x^* is a fixed point for $F(x) + G(x)$. We need to show that the multivalued operators F and G satisfies the assumptions of Theorem 2.1. By Ascoli-Arzelà theorem, we have that $F : X \to P_{cp,cv}(X)$.

Let $x := (x_1, x_2)$, $y := (y_1, y_2) \in X$ and let $u = (u_1, u_2) \in F(x)$. That is $u_1 \in F_1(x)$, $u_2 \in F_2(x)$. Then $u_1, u_2 \in X$ and

$$u_1(t) \in \lambda_{11} \int_0^t K_1(t, s, x_1(s), x_2(s)) \, ds \text{ a.e. on } I, \text{ respectively}$$

$$u_2(t) \in \lambda_{21} \int_0^t K_2(t, s, x_1(s), x_2(s)) \, ds \text{ a.e. on } I.$$

It follows that there exists a mapping $k_x^1 \in S_{K_1(\cdot, \cdot, x_1(\cdot), x_2(\cdot))}^1$, respectively $k_x^2 \in S_{K_2(\cdot, \cdot, x_1(\cdot), x_2(\cdot))}^1$ such that

$$u_1(t) \in \lambda_{11} \int_0^t k_x^1(t, s) \, ds \text{ a.e. on } I, \text{ respectively}$$

$$u_2(t) \in \lambda_{21} \int_0^t k_x^2(t, s) \, ds \text{ a.e. on } I.$$

For $i \in \{1, 2\}$, from the relation

$$H_{X_i}\left(K_i(t, s, x_1(t), x_2(t)), K_i(t, s, y_1(t), y_2(t))\right)$$
$$\leq a_{i1} |x_1(t) - y_1(t)|_{X_1} + a_{i2} |x_2(t) - y_2(t)|_{X_2},$$

we get that there exists $w_1 \in K_1(t, s, y_1(t), y_2(t))$, respectively $w_2 \in K_2(t, s, y_1(t), y_2(t))$ such that

$$\left|k_x^1(t, s) - w_1\right|_{X_1} \leq a_{11} |x_1(t) - y_1(t)|_{X_1} + a_{12} |x_2(t) - y_2(t)|_{X_2}, \text{ respectively}$$
$$\left|k_x^2(t, s) - w_2\right|_{X_2} \leq a_{21} |x_1(t) - y_1(t)|_{X_1} + a_{22} |x_2(t) - y_2(t)|_{X_2}.$$

Thus, the multivalued operator $T = (T_1, T_2)$, defined by

$$T_1(t, s) = K_1(t, s, y_1(t), y_2(t)) \cap K_{T_1}(t, s),$$
$$T_2(t, s) = K_2(t, s, y_1(t), y_2(t)) \cap K_{T_2}(t, s)$$

has nonempty values and is measurable, where

$$K_{T_i}(t, s) = \left\{ w_i : \left|k_x^i(t, s) - w_i\right| \leq a_{i1} |x_1(t) - y_1(t)|_{X_1} + a_{i2} |x_2(t) - y_2(t)|_{X_2} \right\},$$

for $i \in \{1, 2\}$.

Let k_y^1 be a measurable selection for T_1, respectively k_y^2 be a measurable selection for T_2 (which exists by Kuratowski-Ryll-Nardzewski's selection theorem, see [7]), then $k_y^1(t,s) \in K_1(t,s,y_1(t),y_2(t))$, respectively $k_y^2(t,s) \in K_2(t,s,y_1(t),y_2(t))$ and

$$\left|k_x^1(t,s) - k_y^1(t,s)\right|_{X_1} \leq a_{11}\left|x_1(t) - y_1(t)\right|_{X_1} + a_{12}\left|x_2(t) - y_2(t)\right|_{X_2},$$
$$\left|k_x^2(t,s) - k_y^2(t,s)\right|_{X_2} \leq a_{21}\left|x_1(t) - y_1(t)\right|_{X_1} + a_{22}\left|x_2(t) - y_2(t)\right|_{X_2}.$$

We define $v_1(t) = \lambda_{11}\int_0^t k_y^1(t,s)\,ds$, respectively $v_2(t) = \lambda_{21}\int_0^t k_y^2(t,s)\,ds$. It follows that $v_1(t) \in F_1(y)$, respectively $v_2(t) \in F_2(y)$ and for $i \in \{1,2\}$, we have

$$|u_i(t) - v_i(t)|_{X_i}$$
$$\leq |\lambda_{i1}|\int_0^t \left|k_x^i(t,s) - k_y^i(t,s)\right|_{X_i}\,ds$$
$$\leq |\lambda_{i1}|\left(a_{i1}\int_0^t |x_1(s) - y_1(s)|_{X_1}\,ds + a_{i2}\int_0^t |x_2(s) - y_2(s)|_{X_2}\,ds\right)$$
$$= |\lambda_{i1}|\left(a_{i1}\|x_1 - y_1\|_{B_1}\int_0^t e^{\tau s}\,ds + a_{i2}\|x_2 - y_2\|_{B_2}\int_0^t e^{\tau s}\,ds\right)$$
$$\leq \frac{|\lambda_{i1}|}{\tau}e^{\tau t}\left(a_{i1}\|x_1 - y_1\|_{B_1} + a_{i2}\|x_2 - y_2\|_{B_2}\right),$$

where $\|u\|_B := \begin{pmatrix} \|u_1\|_{B_1} \\ \|u_2\|_{B_2} \end{pmatrix} = \begin{pmatrix} \sup\limits_{t\in[0,a]} e^{-\tau t}|u_1(t)|_{X_1} \\ \sup\limits_{t\in[0,a]} e^{-\tau t}|u_2(t)|_{X_2} \end{pmatrix}$, $\tau > 0$ denotes the Bielecki-type norm on the generalized Banach space X. Thus, for $i \in \{1,2\}$, we obtain that

$$\|u_i - v_i\|_{B_i} \leq \frac{|\lambda_{i1}|}{\tau}\left(a_{i1}\|x_1 - y_1\|_{B_1} + a_{i2}\|x_2 - y_2\|_{B_2}\right)$$

and similarly, interchanging the roles between x and y, for $i \in \{1,2\}$, we get that

$$H_{B_i}(F_i(x), F_i(y)) \leq \frac{|\lambda_{i1}|}{\tau}\left(a_{i1}\|x_1 - y_1\|_{B_1} + a_{i2}\|x_2 - y_2\|_{B_2}\right),$$
for each $x,y \in X$.

These inequalities can be written in the matrix form

$$H_B(F(x), F(y)) \leq M\|x - y\|_B,$$

where

$$M = \left(\frac{|\lambda_{i1}| \, a_{ij}}{\tau} \right)_{i,j=1,2}.$$

Taking τ large enough it follows that the matrix M is convergent to zero and thus, F is a multivalued M-contraction. By Theorem 1.6, we have that there exists a fixed point $x^0 = \left(x_1^0, x_2^0 \right) \in X$ for F, i.e. $x^0 \in F\left(x^0 \right)$. That is $x_1^0 \in F_1\left(x^0 \right)$ and $x_2^0 \in F_2\left(x^0 \right)$.

Let us consider $Y = \bar{B}\left(x_1^0, R_1 \right) \times \bar{B}\left(x_2^0, R_2 \right) \subset C\left(I, X_1 \right) \times C\left(I, X_2 \right)$. We show that $F\left(Y \right) \subset \bar{B}\left(x_1^0, \frac{R_1}{2} \right) \times \bar{B}\left(x_2^0, \frac{R_2}{2} \right)$. Let $x \in Y$, then $(x_1, x_2) \in \bar{B}\left(x_1^0, R_1 \right) \times \bar{B}\left(x_2^0, R_2 \right)$. Thus,

$$\left\| x_1 - x_1^0 \right\|_{C_1} \leq R_1 \text{ and } \left\| x_2 - x_2^0 \right\|_{C_2} \leq R_2.$$

Let $u = (u_1, u_2) \in F\left(x \right)$ be arbitrarily chosen. That is $u_1 \in F_1\left(x \right)$, $u_2 \in F_2\left(x \right)$. Then $u_1\left(t \right) \in \lambda_{11} \int_0^t K_1\left(t, s, x_1\left(s \right), x_2\left(s \right) \right) ds$, respectively $u_2\left(t \right) \in \lambda_{21} \int_0^t K_2\left(t, s, x_1\left(s \right), x_2\left(s \right) \right) ds$ a.e. on I. It follows that $u_i\left(t \right) = \lambda_{i1} \int_0^t k_x^i\left(t, s \right) ds$, where $k_x^i\left(t, s \right) \in K_i\left(t, s, x_1\left(s \right), x_2\left(s \right) \right)$ a.e. on I^2, for $i \in \{1, 2\}$. We have

$$\left| u_i\left(t \right) - x_i^0\left(t \right) \right|_{X_i}$$
$$\leq |\lambda_{i1}| \int_0^t \left| k_x^i\left(t, s \right) - k_{x^0}^i\left(t, s \right) \right|_{X_i} ds$$
$$\leq |\lambda_{i1}| \left(a_{i1} \int_0^t \left| x_1\left(s \right) - x_1^0\left(s \right) \right|_{X_1} ds + a_{i2} \int_0^t \left| x_2\left(s \right) - x_2^0\left(s \right) \right|_{X_2} ds \right)$$
$$\leq |\lambda_{i1}| \int_0^t \left(a_{i1} \left\| x_1 - x_1^0 \right\|_{C_1} + a_{i2} \left\| x_2 - x_2^0 \right\|_{C_2} \right)$$
$$\leq |\lambda_{i1}| \, a \left(a_{i1} R_1 + a_{i2} R_2 \right), \text{ for } i \in \{1, 2\}.$$

Taking $\max_{t \in [0,a]} |u_i\left(t \right)|_{X_i}$, we get that

$$\left\| u_i - x_i^0 \right\|_{C_i} \leq |\lambda_{i1}| \, a \left(a_{i1} R_1 + a_{i2} R_2 \right) \leq \frac{R_i}{2}, \text{ for } i \in \{1, 2\}.$$

The multivalued operator G is l.s.c. and compact.

We show that $G\left(Y \right)$ is relatively compact.

Let $x := (x_1, x_2) \in Y$ and we prove that $G\left(Y \right) \subset \bar{B}\left(0, \frac{R_1}{2} \right) \times \bar{B}\left(0, \frac{R_2}{2} \right)$. Let $v = (v_1, v_2) \in G\left(x \right)$ be arbitrarily chosen. That is $v_1 \in G_1\left(x \right)$, respectively $v_2 \in G_2\left(x \right)$. Then $v_1\left(t \right) \in \lambda_{12} \int_0^a L_1\left(t, s, x_1\left(s \right), x_2\left(s \right) \right) ds$, respectively $v_2\left(t \right) \in \lambda_{22} \int_0^a L_2\left(t, s, x_1\left(s \right), x_2\left(s \right) \right) ds$ a.e. on I. It follows that

$v_i(t) = \lambda_{i2} \int_0^a l_x^i(t,s)\,ds$, where $l_x^i(t,s) \in L_i(t,s,x_1(s),x_2(s))$ a.e. on I^2, for $i \in \{1,2\}$. Clearly,

$$|v_i(t)|_{X_i} \leq |\lambda_{i2}| \int_0^a \left| l_x^i(t,s) \right|_{X_i} ds \leq |\lambda_{i2}|\, a M_{L_i} \leq \frac{R_i}{2}, \text{ for } i \in \{1,2\}.$$

Thus, $v_1 \in \bar{B}\left(0, \frac{R_1}{2}\right)$, $v_2 \in \bar{B}\left(0, \frac{R_2}{2}\right)$. Then, the multivalued operator $U = F + G$ has the property $U : Y \to P_{cp,cv}(Y)$, i.e. $F(x) + G(x) \subset Y$, for each $x \in Y$. Hence, the conclusion follows by Theorem 2.1. □

Remark 2.3. The above Theorem can be improved by supposing instead of the existence of a real positive number square matrix A, another square matrix $A = (a_{ij})_{i,j=1,2}$, where $a_{ij} \in L^p([0,a], \mathbb{R}_+)$, $i,j \in \{1,2\}$ and using the Hölder's inequality, we can obtain too, another similar result.

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