



A new proof of the bound for the first Dirichlet eigenvalue of the Laplacian operator

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Abstract

In this paper, we present a new proof of the upper and lower bound estimates for the first Dirichlet eigenvalue $\lambda_1^P(B(p, r))$ of Laplacian operator for the manifold with Ricci curvature $Rc \geq -K$, by using Li-Yau's gradient estimate for the heat equation.

1 Introduction and Main Results

Let (M^n, g) be an n -dim C^∞ complete Riemannian manifold, and Δ denote the Laplacian operator. For the compact manifold, it is well known that the eigenvalue problem $-\Delta\varphi = \lambda\varphi$ has discrete eigenvalues, which are listed as

$$0 = \lambda_0(M^n) < \lambda_1(M^n) \leq \lambda_2(M^n) \leq \dots$$

Moreover we call $\lambda_i(M^n)$ the i th eigenvalue and call a function φ_i satisfying $\Delta\varphi_i = -\lambda_i\varphi_i$ the i th eigenfunction.

Recall that the first eigenvalue $\lambda_1(M^n)$ for the closed Riemannian manifold M^n is defined by

$$\lambda_1(M^n) = \inf_{\phi \in \Omega} \frac{\int_{M^n} |\nabla\phi|^2 d\mu}{\int_{M^n} \phi^2 d\mu}, \quad (1)$$

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where Ω is the completing Hilbert space of

$$\Omega_0 = \left\{ \phi \in C^\infty(M^n) \mid \int_{M^n} \phi d\mu = 0 \right\}$$

under the norm

$$\|\phi\|_1^2 = \int_{M^n} \phi^2 d\mu + \int_{M^n} |\nabla\phi|^2 d\mu.$$

We denote the geodesic ball with center p and radius r by $B(p, r)$ in the n -dim manifold M^n . Then the first Dirichlet eigenvalue $\lambda_1^D(B(p, r))$ of Laplacian operator can be denoted as

$$\lambda_1^D(B(p, r)) = \inf_{\phi \in H_0^2(B(p, r))} \frac{\int_{B(p, r)} |\nabla\phi|^2 d\mu}{\int_{B(p, r)} \phi^2 d\mu}, \quad (2)$$

where $H_0^2(B(p, r))$ is the completing Hilbert space of $C_0^\infty(B(p, r))$ under the norm

$$\|\phi\|_1^2 = \int_{B(p, r)} \phi^2 d\mu + \int_{B(p, r)} |\nabla\phi|^2 d\mu.$$

The upper and lower bound for the first eigenvalue of Laplacian operator are very useful in geometry analysis and PDE. In [1], S. Y. Cheng used the approach of Jacobi fields to obtain an upper bound for the first Dirichlet eigenvalue $\lambda_1^D(B(p, r))$ of Laplacian operator, which are the following comparison theorems.

Theorem 1.1 (Cheng). *Let (M^n, g) be a complete Riemannian manifold satisfying $Rc \geq (n-1)K$, then*

$$\lambda_1^D(B(p, r)) \leq \lambda_1^D(B_K(p, r)),$$

where $B_K(p, r)$ is the geodesic ball in the space form M_K^n , and equality holds iff $B(p, r)$ is isometric to $B_K(p, r)$.

Corollary 1.2 (Cheng). *Let (M^n, g) be a compact Riemannian manifold satisfying $Rc \geq 0$, then*

$$\lambda_1(M^n) \leq \lambda_1^D\left(B_K\left(p, \frac{d_{M^n}}{2}\right)\right) \leq \frac{C_n}{d_{M^n}^2},$$

where $C_n = 2n(n+4)$ and d_{M^n} is the diameter of M^n .

For the lower bound of the first eigenvalue $\lambda_1(M^n)$ for closed Riemannian manifold M^n , in 1980, P. Li and S. T. Yau [3] used the approach of gradient estimates to derive a lower bound as follows:

Theorem 1.3 (Li-Yau). *Let (M^n, g) be a closed Riemannian manifold satisfying $Rc \geq 0$, then*

$$\lambda_1(M^n) \geq \frac{\pi^2}{2d_{M^n}},$$

where d_{M^n} denotes the diameter of M^n .

Moreover, their result was improved to a sharp lower bound by J. Q. Zhong and H. C. Yang [5] as follows. Their proofs both rely on the gradient estimate and the maximum principle.

Theorem 1.4 (Zhong-Yang). *Let (M^n, g) be a closed Riemannian manifold satisfying $Rc \geq 0$, then*

$$\lambda_1(M^n) \geq \frac{\pi^2}{d_{M^n}}.$$

In this paper, we will present a new and simpler proof of upper and lower bound estimates for the first Dirichlet eigenvalue $\lambda_1^D(B(p, r))$ of the Laplacian operator for the manifold with Ricci curvature $Rc \geq -K$, where $K \geq 0$. Our approach is applying the following gradient estimate which leads to the famous Li-Yau Harnack estimate for the heat equation [4].

Theorem 1.5 (Li-Yau). *Let (M^n, g) be a complete Riemannian manifold with Ricci curvature $Rc(g) \geq -K$, and let $B(p, 2r)$ denote the geodesic ball with center p and radius $2r$ in M^n . If $u : B(p, 2r) \times [0, \infty) \rightarrow \mathbb{R}$ is a positive solution to the heat equation*

$$\frac{\partial}{\partial t} u(x, t) = \Delta u(x, t),$$

then in $B(p, r)$ we have

$$\sup_{B(p, r)} \left(\frac{|\nabla u|^2}{u^2} - \alpha \frac{u_t}{u} \right) \leq \frac{C(n) \alpha^2}{r^2} \left(\frac{\alpha^2}{\alpha^2 - 1} + \sqrt{K} r \right) + \frac{n \alpha^2 K}{2(\alpha - 1)} + \frac{n \alpha^2}{2t}, \quad (3)$$

where $\alpha > 1$ and $C(n)$ is a constant depended on n .

Remark 1. To use Theorem 1.5, we will deal with the solution to the heat equation with the form of $u(x)e^{-\lambda_1^D(B(p, r))t}$, where $u(x)$ is the eigenfunction corresponding to the first Dirichlet eigenvalue $\lambda_1^D(B(p, r))$ of the Laplacian operator, and then apply the Li-Yau's gradient estimate to the solution above and prove an integral estimate as follows.

Theorem 1.6. *Let (M^n, g) be a complete Riemannian manifold with Ricci curvature $Rc(g) \geq -K$, then for any geodesic ball $B(p, r)$ in M^n and arbitrary $\varepsilon > 0$, we have*

$$\begin{aligned} & \frac{\varepsilon}{t_2 - t_1} \int_{t_1}^{t_2} |\nabla \log u|^2 dt \\ & \geq \lambda_1^D(B(p, r)) - \left(\frac{4C(n)\alpha}{r^2} \left(\frac{\alpha^2}{\alpha^2 - 1} + \frac{\sqrt{K}}{2} r \right) + \frac{n\alpha K}{2(\alpha - 1)} \right) - \frac{n\alpha}{2t_1} \\ & \quad - \frac{1}{4} \left(\alpha + \frac{1}{\varepsilon} \right) \frac{d(x_1, x_2)^2}{(t_2 - t_1)^2} \end{aligned}$$

for arbitrary constant speed minimal geodesic $\gamma(t)$ joining points x_1 and x_2 such that $x_1, x_2 \in B(p, \frac{r}{2})$ and $x_1 = \gamma(t_1), x_2 = \gamma(t_2)$.

By using Theorem 1.6, we can derive the following upper and lower bound estimates for the first Dirichlet eigenvalue $\lambda_1^D(B(p, r))$ of Laplacian operator.

Theorem 1.7. *Let (M^n, g) be a complete Riemannian manifold with Ricci curvature $Rc(g) \geq -K$, then for any geodesic ball $B(p, r)$ in M^n , we have*

$$\frac{C_4}{d_{B(p, \frac{r}{2})}^2} + \frac{C_2}{d_{B(p, \frac{r}{2})}} + C_3 \leq \lambda_1^D(B(p, r)) \leq \frac{C_1}{d_{B(p, \frac{r}{2})}^2} + \frac{C_2}{d_{B(p, \frac{r}{2})}} + C_3, \quad (4)$$

where $d_{B(p, \frac{r}{2})}$ denotes the diameter of $B(p, \frac{r}{2})$ and C_i ($1 \leq i \leq 4$) are some constants only depending on the choice of p and r , which can be seen in the proof.

Remark 2. Recall that for the closed Riemannian manifold, the eigenfunction $u(x)$ corresponding to the first eigenvalue $\lambda_1(M^n)$ of the Laplacian operator satisfies

$$\int_{M^n} u(x) d\mu = 0,$$

which implies that $u(x)$ can not always be positive. But the Li-Yau's gradient estimate (3) only satisfies for the positive solution of the heat equation, thus in this paper, we deal with the upper and lower bound estimates for the first Dirichlet eigenvalue which can always be positive.

The paper is organized as follows: In section 2, we prove Theorem 1.6 by using the Li-Yau's gradient estimate for heat equations. Based on this, in section 3, we prove Theorem 1.7.

2 Proof of Theorem 1.6

Proof of Theorem 1.6. To prove Theorem 1.6, we consider the function in the form of

$$\tilde{u}(x, t) = u(x)e^{-\lambda_1^D(B(p,r))t},$$

where $u(x)$ is the eigenfunction corresponding to the first Dirichlet eigenvalue $\lambda_1^D(B(p,r))$ of the Laplacian operator. It is clear that $\tilde{u}(x, t)$ is actually a solution to the heat equation, in fact

$$\begin{aligned} \frac{\partial}{\partial t} \tilde{u}(x, t) &= -\lambda_1^D(B(p,r)) u(x)e^{-\lambda_1^D(B(p,r))t} \\ &= \Delta(u(x)e^{-\lambda_1^D(B(p,r))t}) \\ &= \Delta(\tilde{u}(x, t)). \end{aligned}$$

On the other hand, let φ be the eigenfunction corresponding to the first Dirichlet eigenvalue $\lambda_1^D(B(p,r))$, since $|\nabla|\varphi|| = |\nabla\sqrt{\varphi^2}| = \left| \frac{2\varphi \cdot \nabla\varphi}{2\sqrt{\varphi^2}} \right| \leq |\nabla\varphi|$, we have

$$\begin{aligned} \lambda_1^D(B(p,r)) &= \inf_{\phi \in H_0^1(B(p,r))} \frac{\int_{B(p,r)} |\nabla\phi|^2 d\mu}{\int_{B(p,r)} \phi^2 d\mu} \\ &\leq \frac{\int_{B(p,r)} |\nabla|\varphi||^2 d\mu}{\int_{B(p,r)} |\varphi|^2 d\mu} \\ &\leq \frac{\int_{B(p,r)} |\nabla\varphi|^2 d\mu}{\int_{B(p,r)} \varphi^2 d\mu} \\ &= \lambda_1^D(B(p,r)), \end{aligned}$$

which implies that $|\varphi|$ is also an eigenfunction corresponding to the first Dirichlet eigenvalue $\lambda_1^D(B(p,r))$. Thus without loss of generality, we would suppose that the eigenfunction corresponding to the first Dirichlet eigenvalue of Laplacian operator is nonnegative. Then by the strong maximum principle for the Laplacian operator, we can assume that the eigenfunction is positive. Thus $\tilde{u}(x, t) = u(x)e^{-\lambda_1^D(B(p,r))t}$ is actually a positive solution to the heat equation.

Then by using the Li-Yau's estimate (3) to $\tilde{u}(x, t)$ we have

$$\frac{|\nabla\tilde{u}|^2}{\tilde{u}^2} - \alpha \frac{\tilde{u}_t}{\tilde{u}} \leq \frac{4C(n)\alpha^2}{r^2} \left(\frac{\alpha^2}{\alpha^2 - 1} + \frac{\sqrt{K}}{2} r \right) + \frac{n\alpha^2 K}{2(\alpha - 1)} + \frac{n\alpha^2}{2t}$$

satisfies in $B(p, \frac{r}{2})$. Then for a constant speed minimal geodesic $\gamma(t)$ joining points x_1 and x_2 such that $x_1, x_2 \in B(p, \frac{r}{2})$ and $x_1 = \gamma(t_1), x_2 = \gamma(t_2)$, we

have

$$\begin{aligned}
\log \frac{\tilde{u}(x_2, t_2)}{\tilde{u}(x_1, t_1)} &= \int_{t_1}^{t_2} \frac{d}{dt} \log \tilde{u}(\gamma(t), t) dt \\
&= \int_{t_1}^{t_2} \left(\frac{\tilde{u}_t}{\tilde{u}} + \left\langle \frac{\nabla \tilde{u}}{\tilde{u}}, \frac{d\gamma}{dt} \right\rangle \right) dt \\
&\geq \int_{t_1}^{t_2} \left(\frac{\tilde{u}_t}{\tilde{u}} - \frac{1}{\alpha} \frac{|\nabla \tilde{u}|^2}{\tilde{u}^2} - \frac{\alpha}{4} \left| \frac{d\gamma}{dt}(t) \right|^2 \right) dt \\
&= -\frac{1}{\alpha} \int_{t_1}^{t_2} \left(\frac{|\nabla \tilde{u}|^2}{\tilde{u}^2} - \alpha \frac{\tilde{u}_t}{\tilde{u}} \right) dt - \frac{\alpha}{4} \int_{t_1}^{t_2} \left| \frac{d\gamma}{dt}(t) \right|^2 dt \\
&\geq -\int_{t_1}^{t_2} \left(\frac{4C(n)\alpha}{r^2} \left(\frac{\alpha^2}{\alpha^2 - 1} + \frac{\sqrt{K}}{2} r \right) + \frac{n\alpha K}{2(\alpha - 1)} + \frac{n\alpha}{2t} \right) dt \\
&\quad - \frac{\alpha}{4} \int_{t_1}^{t_2} \left| \frac{d\gamma}{dt}(t) \right|^2 dt
\end{aligned}$$

The second inequality we use the Li-Yau's gradient estimate, and it follows from $\gamma(t)$ being a constant speed minimal geodesic that

$$\left| \frac{d\gamma}{dt}(t) \right| \equiv \frac{d(x_1, x_2)}{t_2 - t_1}.$$

Thus we have

$$\begin{aligned}
&\log u(x_2) - \lambda_1^D(B(p, r)) t_2 - \log u(x_1) + \lambda_1^D(B(p, r)) t_1 \\
&\geq - \left(\frac{4C(n)\alpha}{r^2} \left(\frac{\alpha^2}{\alpha^2 - 1} + \frac{\sqrt{K}}{2} r \right) + \frac{n\alpha K}{2(\alpha - 1)} \right) (t_2 - t_1) \\
&\quad - \frac{n\alpha}{2} (\log t_2 - \log t_1) - \frac{\alpha}{4} \frac{d(x_1, x_2)^2}{t_2 - t_1}
\end{aligned}$$

Then for the path $\gamma(t)$ which is a constant speed minimal geodesic jointing points x_1, x_2 such that $\gamma(t_1) = x_1$ and $\gamma(t_2) = x_2$ we have

$$\begin{aligned}
\log u(x_2) - \log u(x_1) &= \log u(\gamma(t_2)) - \log u(\gamma(t_1)) \\
&= \int_{t_1}^{t_2} \left\langle \nabla \log u, \frac{d\gamma}{dt} \right\rangle dt \\
&\leq \int_{t_1}^{t_2} \varepsilon |\nabla \log u|^2 dt + \int_{t_1}^{t_2} \frac{1}{4\varepsilon} \left| \frac{d\gamma}{dt} \right|^2 dt \\
&= \int_{t_1}^{t_2} \varepsilon |\nabla \log u|^2 dt + \frac{d(x_1, x_2)^2}{4\varepsilon(t_2 - t_1)}.
\end{aligned}$$

Note that

$$\log t_2 - \log t_1 = \frac{1}{\xi}(t_2 - t_1) \leq \frac{1}{t_1}(t_2 - t_1), \quad (5)$$

where $\xi \in (t_1, t_2)$. Then by using these two inequalities we have

$$\begin{aligned} & \int_{t_1}^{t_2} \varepsilon |\nabla \log u|^2 dt + \frac{d(x_1, x_2)^2}{4\varepsilon(t_2 - t_1)} - \lambda_1^D(B(p, r))(t_2 - t_1) \\ & \geq - \left(\frac{4C(n)\alpha}{r^2} \left(\frac{\alpha^2}{\alpha^2 - 1} + \frac{\sqrt{K}}{2}r \right) + \frac{n\alpha K}{2(\alpha - 1)} \right) (t_2 - t_1) - \frac{n\alpha}{2t_1}(t_2 - t_1) \\ & \quad - \frac{\alpha d(x_1, x_2)^2}{4(t_2 - t_1)} \end{aligned}$$

which implies that

$$\begin{aligned} & \frac{\varepsilon}{t_2 - t_1} \int_{t_1}^{t_2} |\nabla \log u|^2 dt \\ & \geq \lambda_1^D(B(p, r)) - \left(\frac{4C(n)\alpha}{r^2} \left(\frac{\alpha^2}{\alpha^2 - 1} + \frac{\sqrt{K}}{2}r \right) + \frac{n\alpha K}{2(\alpha - 1)} \right) - \frac{n\alpha}{2t_1} \\ & \quad - \frac{1}{4} \left(\alpha + \frac{1}{\varepsilon} \right) \frac{d(x_1, x_2)^2}{(t_2 - t_1)^2} \end{aligned}$$

for arbitrary $\varepsilon > 0$. □

3 Proof of Theorem 1.7

In this section, we prove Theorem 1.7 by using Theorem 1.6.

Proof. For the geodesic ball $B(p, \frac{r}{2})$, by using the strong maximum principle for the Laplacian operator, we have

$$u^2 \geq C_{B(p, \frac{r}{2})} \left(\sup_{x \in B(p, \frac{r}{2})} u - u \right) \left(u - \inf_{x \in B(p, \frac{r}{2})} u \right), \quad (6)$$

where $C_{B(p, \frac{r}{2})}$ is a positive constant only depending on the choice of p and $\frac{r}{2}$.

Then for arbitrary $x, y \in B(p, \frac{r}{2})$, we can select the appropriate s and t such that

$$u(x) = u(\gamma(s)) = \inf_{x \in B(p, \frac{r}{2})} u$$

and

$$u(y) = u(\gamma(t)) = \sup_{x \in B(p, \frac{r}{2})} u.$$

Then it follows from (6) that

$$\begin{aligned} & \log u(y) - \log u(x) \\ &= \int_s^t \left\langle \nabla \log u, \frac{d\gamma}{dt} \right\rangle dt \\ &\leq \int_{u(x)}^{u(y)} \frac{du}{|u|} \\ &\leq \int_{\inf_{x \in B(p, \frac{r}{2})} u}^{\sup_{x \in B(p, \frac{r}{2})} u} \frac{du}{\sqrt{C_{B(p, \frac{r}{2})} \left(\sup_{x \in B(p, \frac{r}{2})} u - u \right) \left(u - \inf_{x \in B(p, \frac{r}{2})} u \right)}} \\ &= \frac{\pi}{\sqrt{C_{B(p, \frac{r}{2})}}}. \end{aligned}$$

Then substituting this and (5) into

$$\begin{aligned} & \log u(x_2) - \lambda_1^D(B(p, r)) t_2 - \log u(x_1) + \lambda_1^D(B(p, r)) t_1 \\ &\geq - \left(\frac{4C(n)\alpha}{r^2} \left(\frac{\alpha^2}{\alpha^2 - 1} + \frac{\sqrt{K}}{2} r \right) + \frac{n\alpha K}{2(\alpha - 1)} \right) (t_2 - t_1) \\ &\quad - \frac{n\alpha}{2} (\log t_2 - \log t_1) - \frac{\alpha}{4} \frac{d(x_1, x_2)^2}{t_2 - t_1}, \end{aligned}$$

we have

$$\begin{aligned} & \lambda_1^D(B(p, r)) (t_2 - t_1) - \frac{\pi}{\sqrt{C_{B(p, \frac{r}{2})}}} - \frac{n\alpha}{2t_1} (t_2 - t_1) \\ &\leq \left(\frac{4C(n)\alpha}{d_{B(p, \frac{r}{2})}^2} \left(\frac{\alpha^2}{\alpha^2 - 1} + \frac{\sqrt{K}}{2} d_{B(p, \frac{r}{2})} \right) + \frac{n\alpha K}{2(\alpha - 1)} \right) (t_2 - t_1) \\ &\quad + \frac{\alpha}{4} \frac{d_{B(p, \frac{r}{2})}^2}{t_2 - t_1}, \end{aligned}$$

where $d_{B(p, \frac{r}{2})}$ denotes the diameter of $B(p, \frac{r}{2})$. This implies that

$$\begin{aligned}
& \lambda_1^D(B(p, r)) \\
& \leq \left(\frac{4C(n)\alpha}{d_{B(p, \frac{r}{2})}^2} \left(\frac{\alpha^2}{\alpha^2 - 1} + \frac{\sqrt{K}}{2} d_{B(p, \frac{r}{2})} \right) + \frac{n\alpha K}{2(\alpha - 1)} \right) + \frac{n\alpha}{2t_1} \\
& \quad + \frac{\pi}{\sqrt{C_{B(p, \frac{r}{2})}}(t_2 - t_1)} + \frac{\alpha}{4} \frac{d_{B(p, \frac{r}{2})}^2}{(t_2 - t_1)^2}
\end{aligned}$$

for any $t_1 < t_2$, and

$$\begin{aligned}
& \lambda_1^D(B(p, r)) \\
& \geq \left(\frac{4C(n)\alpha}{d_{B(p, \frac{r}{2})}^2} \left(\frac{\alpha^2}{\alpha^2 - 1} + \frac{\sqrt{K}}{2} d_{B(p, \frac{r}{2})} \right) + \frac{n\alpha K}{2(\alpha - 1)} \right) + \frac{n\alpha}{2t_1} \\
& \quad - \frac{\pi}{\sqrt{C_{B(p, \frac{r}{2})}}(t_1 - t_2)} + \frac{\alpha}{4} \frac{d_{B(p, \frac{r}{2})}^2}{(t_1 - t_2)^2}
\end{aligned}$$

for any $t_1 > t_2$.

Furthermore, by selecting the parameters

$$t_1 = d_{B(p, \frac{r}{2})}^2$$

and

$$t_2 = \left(1 + \frac{1}{\sqrt{C_{B(p, \frac{r}{2})}}} \right) d_{B(p, \frac{r}{2})}^2,$$

and substituting them into above, we have

$$\begin{aligned}
& \lambda_1^D(B(p, r)) \\
& \leq \left(\frac{4C(n)\alpha}{d_{B(p, \frac{r}{2})}^2} \left(\frac{\alpha^2}{\alpha^2 - 1} + \frac{\sqrt{K}}{2} d_{B(p, \frac{r}{2})} \right) + \frac{n\alpha K}{2(\alpha - 1)} \right) + \frac{n\alpha}{2d_{B(p, \frac{r}{2})}^2} \\
& \quad + \frac{\pi}{d_{B(p, \frac{r}{2})}^2} + \frac{\alpha C_{B(p, \frac{r}{2})}}{4d_{B(p, \frac{r}{2})}^2} \\
& = \frac{C_1}{d_{B(p, \frac{r}{2})}^2} + \frac{C_2}{d_{B(p, \frac{r}{2})}} + C_3.
\end{aligned}$$

For the same reason, we can substitute the parameters

$$t_1 = \left(1 + \frac{1}{\sqrt{C_{B(p, \frac{r}{2})}}}\right) d_{B(p, \frac{r}{2})}^2$$

and

$$t_2 = d_{B(p, \frac{r}{2})}^2$$

into the same inequality to obtain that

$$\begin{aligned} & \lambda_1^D(B(p, r)) \\ & \geq \left(\frac{4C(n)\alpha}{d_{B(p, \frac{r}{2})}^2} \left(\frac{\alpha^2}{\alpha^2 - 1} + \frac{\sqrt{K}}{2} d_{B(p, \frac{r}{2})} \right) + \frac{n\alpha K}{2(\alpha - 1)} \right) \\ & \quad + \frac{n\alpha \sqrt{C_{B(p, \frac{r}{2})}}}{2 \left(1 + \sqrt{C_{B(p, \frac{r}{2})}}\right) d_{B(p, \frac{r}{2})}^2} - \frac{\pi}{d_{B(p, \frac{r}{2})}^2} + \frac{\alpha C_{B(p, \frac{r}{2})}}{4d_{B(p, \frac{r}{2})}^2} \\ & = \frac{C_4}{d_{B(p, \frac{r}{2})}^2} + \frac{C_2}{d_{B(p, \frac{r}{2})}} + C_3. \end{aligned}$$

□

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